Introduction to order theory

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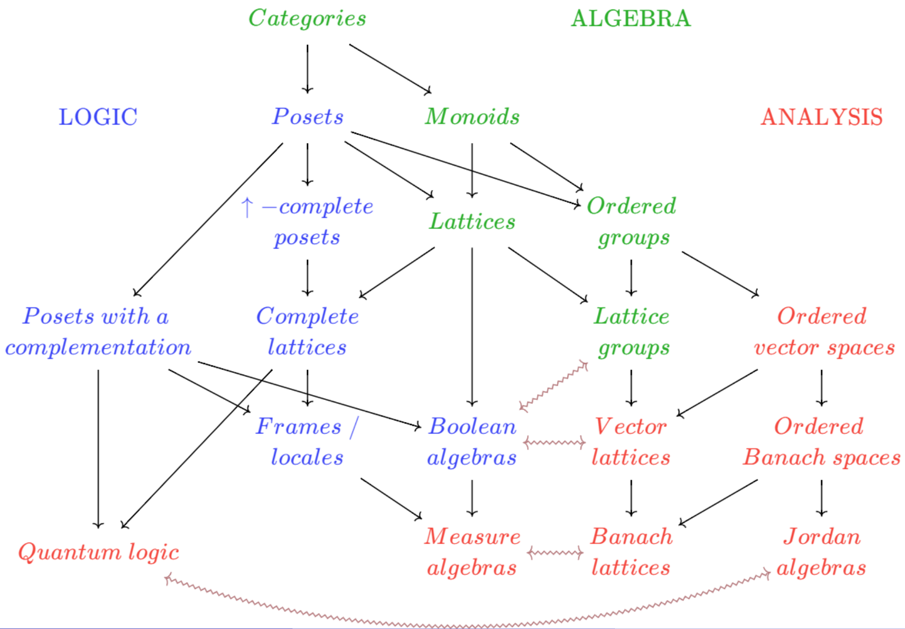
Map of order structures



Plan of the mini-course

- Basic definitions; continuity, closeness, density, completeness.
- Galois connections and polarities: properties and examples.
- Hull operators and hull structures: properties and examples.
- Distributivity; Boolean algebras; ordered vector spaces.

Not covered: Zorn's lemma; vector lattices; combinatorial order theory.

Disclaimer: the author is not an expert! Feedback is welcome!

Some general references:

- T.S. Blyth, *Lattices and Ordered Algebraic Structures*, 2005.
- B.A. Davey & H.A. Priestley, *Introduction to Lattices and Order*, 2002.
- P.T. Johnstone, *Stone spaces*, 1982.
- G. Birkhoff, *Lattice theory*, 1948 & 1967.
- S. Vickers, *Topology via logic*, 1989.

Relations

If X is a set, then $\mathcal{P}(X)$ denotes the collection of all subsets of X.

A *relation* from X into a set Y is a subset of $X \times Y$. Notations: $\varphi : X \to Y$; $\quad x\varphi y$ for $(x, y) \in \varphi$. Every map is a relation.

If $\psi : Y \to Z$, define $\varphi\psi : X \to Z$ by $x\varphi\psi z$ if $x\varphi y$ and $y\psi z$, for some $y \in Y$. Also, $\psi \circ \varphi := \varphi\psi$ agrees with composition of maps. The composition is associative.

The *converse* $\varphi^* : Y \to X$ of φ is defined by $y\varphi^* x$ if $x\varphi y$. Then, $\varphi^{**} = \varphi$ and $(\varphi\psi)^* = \psi^*\varphi^*$.

We say that φ is *stronger* than ψ if $\varphi \subset \psi$. Define the *negation* $\not\varphi := X \times Y \backslash \varphi$ of φ. Conversion commutes with negation.

The *domain* of φ is $\{x \in X, \ \exists y \in Y : x\varphi y\}$, the *image* is $dom(\varphi^*)$.

The domain of φ is X iff $Id_X \subset \varphi\varphi^*$.

$\varphi\varphi^* \subset Id_X$ iff φ is injective, i.e. if $x\varphi y$ and $z\varphi y \Rightarrow x = z$.

φ is a map iff $Id_X \subset \varphi\varphi^*$ and $\varphi^*\varphi \subset Id_Y$.

A relation on X is a relation from X into X. May be viewed as a directed graph on X.

Id_X, the identity map on X, is also the "diagonal" relation on X. It is neutral with respect to the composition. A *restriction* of φ to $Y \subset X$ is $\varphi \cap Y \times Y$.

φ is:

- *reflexive* if $Id_X \subset \varphi$;
- *symmetric* if $\varphi^* = \varphi$;
- *transitive* if $\varphi^2 \subset \varphi$;
- *equivalence relation* if it is reflexive, symmetric and transitive.

- *total* if $\varphi \cup \varphi^* = X \times X$;
- *anti-symmetric* if $\varphi \cap \varphi^* \subset Id_X$;

It is often convenient to view a reflexive symmetric relation as a measure of "closeness", i.e refer to $x\varphi y$ as "y is φ-close to x".

Equivalence relations correspond to *partitions* of X, i.e. collections $\{X_i\}_{i\in I}$ of disjoint sets with $X = \bigcup\limits_{i\in I} X_i$. Equivalence $\mapsto$ classes of equivalence; partition $\mapsto$ $x \sim y$ if they are in the same component.

$\sim$ is stronger than $\approx$ if every $\sim$-class is contained in a $\approx$-class.

Pre-orders

A relation $\leq$ on P is a *pre-order* if it is reflexive and transitive.

A *pre-ordered set* is a pair $(P, \leq)$.

Define $\geq := \leq^*$, and let $-P := (P, \geq)$. Clearly, $-(-P) = P$.

Any equivalence relation is a pre-order. The set inclusion is a pre-order on $\mathcal{P}(X)$. Divisibility is a pre-order on $\mathbb{Z}$.

$p \in P$ is an *upper bound* of $Q \subset P$ if $Q \leq p$. $Q^{\uparrow}$ is the set of all upper bounds of Q. If $Q^{\uparrow} \neq \varnothing$, Q is *bounded from above*. If $R \subset Q$, then $R^{\uparrow} \supset Q^{\uparrow}$. Same for $Q^{\downarrow}$. Q is *order bounded* if $Q^{\uparrow} \neq \varnothing \neq Q^{\downarrow}$.

P is *directed* (or *directed upward*) if any $\{p, q\}$ is bounded from above, for $p, q \in P$, and *co-directed* (or *directed downward* or *filtered*) if any $\{p, q\}$ is bounded from below, for $p, q \in P$.

If $p, q \in P$, then the *order interval* $[p, q]_P := \{r \in P,\ p \leq r \leq q\}$ (it is nonempty iff $p \leq q$). We will also denote $[p)_P := \{r \in P,\ p \leq r\}$ and same for $(p]_P$. If $Q \subset P$ denote $[p)_Q := [p) \cap Q$, etc (even if $p \notin Q$).

Also, $[Q) := \bigcup\limits_{q \in Q} [q)$, etc.

$Q \subset P$ a *lower set* if $(Q] \subset Q$; (same for *upper sets*). Upper sets are complements of lower sets.

Q is *full* if $[Q, Q] \subset Q$, i.e. if whenever $q, r \in Q$ and $q \leq p \leq r$, then $p \in Q$. Lower and upper sets are full.

Any intersection of upper / lower / full sets and any union of upper / lower sets is a set of the same type.

The sets $Q^{\uparrow} = \bigcap_{q \in Q} [q)$ and $[Q)$ are upper, while $Q^{\downarrow}$ and $(Q]$ are lower.

$Q \subset P$ *majorates* $R \subset P$ if $R \subset (Q]$; the dual notion is *refinement* or *minorization*. These two relations are transitive.

$\varphi : P \to R$ is *isotone* (or *order preserving*) if $p \leq q \Rightarrow \varphi(p) \leq \varphi(q)$ (equivalently, if pre-images of [principal] lower / upper sets are lower / upper), and *antitone* (or *order reversing*) if $p \leq q \Rightarrow \varphi(p) \geq \varphi(q)$.

The composition of two isotone or antitone maps is isotone, the composition of an isotone and antitone maps (in any order) is antitone.

An isotone (antitone) bijection whose inverse is also isotone (antitone) is called an *(anti-)isomorphism*.

Call $\varphi : P \to P$ *expansive* if $\varphi(p) \geq p$, for $p \in P$ and *contractive* dually.

Partially ordered sets, a.k.a. posets

An *order* is an antisymmetric pre-order. A set with a specified order is called a *poset*. If $\leq$ is an order, define the *strict order* by $<:=\leq\setminus Id_P$.

Any pre-order $\preceq$ can be factorized by $\preceq\cap\succeq$ to get an order.

If $q\in Q\subset P$ is an upper bound for Q, call it the *maximum* or the *greatest element* of Q (is unique, if it exists). Same for *minimum* or the *smallest element*; we denote them by $\max Q$ and $\min Q$.

The *exact upper bound* or *supremum* $\bigvee Q$ of Q is $\min Q^{\uparrow}$. The *exact lower bound (infimum)* is defined similarly and is denoted by $\bigwedge Q$.

$q=\bigvee Q$ iff $Q^{\uparrow}=\{q\}^{\uparrow}=[q)$, and the same for $\bigwedge$ and lower bounds.

If $R\subset Q$, then $\bigvee R\leq\bigvee Q$ and $\bigwedge R\geq\bigwedge Q$ (if exist). $\bigvee\varnothing=\min P$ and $\bigwedge\varnothing=\max P$ (if exist).

Lemma 1

Let I be an index set, and let $\{Q_i,\ i\in I\}\subset\mathcal{P}(P)$ and $\{q_i,\ i\in I\}\subset P$ be such that $q_i=\bigvee Q_i$, for every $i\in I$. Then, $\bigvee\bigcup\limits_{i\in I}Q_i=\bigvee\limits_{i\in I}q_i$ (with existence of one implying existence of another).

If $\max Q$ exists, then $\bigvee Q = \max Q$; otherwise $\bigvee Q \notin Q$. Same for $\bigwedge$ and $\min$.

We will say that $p \in P$ is *maximal* if $p \leq q$ implies $p = q$, i.e. $[p) = \{p\}$. *Minimal* elements are defined similarly.

A set can have more than one (or none) maximal / minimal elements. Maximal / minimal elements are mutually incomparable. The greatest / least element of a set is maximal / minimal, but the converse is false.

If $p, q \in P$ denote $p \wedge q := \bigwedge \{p, q\}$ and $p \vee q := \bigvee \{p, q\}$ (if exist). The partial operations $\vee$ and $\wedge$ are commutative and associative, since e.g. $(p \vee q) \vee r = \bigvee \{p, q, r\} = p \vee (q \vee r)$, if either LHS or RHS exist.

Theorem 1 (Szpilrajn)

Any order is the intersection of total orders.

We say that $\varphi : P \to R$ *reflects order* if $p \leq q \Leftarrow \varphi(p) \geq \varphi(q)$.

Order reflecting maps are injective.

An order reflecting isotone map is called *order embedding*. The two classes are closed under composition.

Continuity and regularity

If $\varphi : P \to Q$ is isotone and $R \subset P$ is such that $\bigvee R$ and $\bigvee \varphi(R)$ exist, then $\bigvee \varphi(R) \leq \varphi(\bigvee R)$.

Say that φ is $\underset{2}{\vee}$-*isotone* if $p = q \vee r \Rightarrow \varphi(p) = \varphi(q) \vee \varphi(r)$. Such maps are isotone, since $p \leq q \Rightarrow q = p \vee q \Rightarrow \varphi(p) \leq \varphi(p) \vee \varphi(q) = \varphi(q)$.

Call φ $\vee$-*isotone*, if $r = \bigvee R \Rightarrow \varphi(r) = \bigvee \varphi(R)$, for every nonempty finite $R \subset P$, and $\bigvee$-isotone, if the same is true for any $R \neq \varnothing$.

EXAMPLE: Let $P = \{a, b, c, 1, 2, 3, \star\}$ ordered by $a, b \leq 3, \star$, $a, c \leq 2, \star$ and $b, c \leq 1, \star$. Let $Q = \{0, 1\}$ and let φ map $\star$ into 1, and everything else into 0. Such φ is $\underset{2}{\vee}$-, but not $\underset{3}{\vee}$-isotone.

The notions of $\underset{2}{\wedge}$-, $\wedge$- and $\bigwedge$-isotone maps are defined similarly.

Additional variations are $\bigvee\limits_{\mathbb{N}}$-isotone maps (sequences), and $\bigvee\limits_{\geq 0}$-isotone maps (all sets, including $\varnothing$). Every isomorphism is $\bigvee\limits_{\geq 0}$- and $\bigwedge\limits_{\geq 0}$- isotone.

All these classes are closed under composition.

If $R \subset Q \subset P$, the supremums of R within Q and within P will be denoted $\bigvee_P R$ and $\bigvee_Q R$ (and same for infimums).

- If $\bigvee_Q R$ and $\bigvee_P R$ exist, then $\bigvee_Q R \geq \bigvee_P R$.
- If $\bigvee_P R$ exists and is an element of Q, then $\bigvee_Q R = \bigvee_P R$ (in particular, it exists).
- If $P = \ell_\infty$, $Q = c_0$ and $R = \{e_n\}_{n\in\mathbb{N}}$, $\bigvee_P R$ exists, but R is not bounded from above in Q.
- If $P = \mathbb{R}$, $Q = \mathbb{Q}$ and $R = \mathbb{Q} \cap [0, \pi]$, $\bigvee_P R$ exists, R is bounded from above in Q, but $\bigvee_Q R$ does not exist.
- If $P = \mathcal{F}[0, 1]$, $Q = \mathcal{C}[0, 1]$, and $R = \left\{\sqrt[n]{t}\right\}_{n\in\mathbb{N}}$, then $\bigvee_Q R$ and $\bigvee_P R$ exist but are not equal.
- If $P = [-1, 0) \cup (0, 1]$, $Q = [-1, 0) \cup \{1\}$, and $R = [0, 1)$, then $\bigvee_Q R$ exists, but $\bigvee_P R$ does not.

Call $Q \subset P$ $\bigvee$*-regular* if $r = \bigvee_Q R \Rightarrow r = \bigvee_P R$, for every $\varnothing \neq R \subset Q$. Equivalently, the inclusion map $Id_{Q,P} : Q \to P$ is $\bigvee$-isotone.

If $Q \subset P$ is $\bigvee$-regular, then for $q \in P$ and $\varnothing \neq R \subset Q$ we have $q = \bigvee_Q R \Leftrightarrow q = \bigvee_P R \;\&\; q \in Q$. Every upper set is $\bigvee$-regular.

Closure

Say that $Q \subset P$ is $\bigvee$*-closed* if $\varnothing \neq R \subset Q$ and $r = \bigvee_P R$ implies $r \in Q$ (and so $r = \bigvee_Q R$).

Upper sets and sets $Q^{\downarrow}$ are $\bigvee$-closed. In particular, $\bigvee Q^{\downarrow} = \bigwedge Q$ (if exists).

If $P = \mathcal{C}[-1,1]$, then $Q = \{f \in P,\ f(t) = f(0),\ -1 \leq t \leq 0\}$ is $\bigvee$-closed, but not $\bigvee$-regular. $R = \{f \in P,\ f(-1) = f(1)\}$ is $\bigvee$-regular, but not $\bigvee$-closed.

$$Q^{\bigwedge} := \left\{p \in P,\ \exists \varnothing \neq R \subset Q :\ p = \bigwedge R\right\} = \left\{p = \bigwedge [p)_Q,\ [p)_Q \neq \varnothing\right\}.$$

Q is $\bigwedge$-closed iff $Q = Q^{\bigwedge}$. Clearly, $Q \subset Q^{\bigwedge}$, and $Q \subset R \Rightarrow Q^{\bigwedge} \subset R^{\bigwedge}$.

Proposition 1

- *The operator $Q \mapsto Q^{\bigwedge}$ is idempotent, i.e. $Q^{\bigwedge\bigwedge} = Q^{\bigwedge}$, $\forall Q \subset P$.*
- *$Q^{\bigwedge}$ is the smallest $\bigwedge$-closed set, which contains Q.*
- *$\varphi : P \to R$ is $\bigvee$-isotone iff it is isotone and such that $\varphi\left(Q^{\bigvee}\right) \subset \varphi(Q)^{\bigvee}$, for every $\varnothing \neq Q \subset P$.*

Density

We say that $Q \subset P$ is $\bigvee$-*dense* if $P = Q^{\bigvee}$, i.e. Q refines P and $p = \bigvee (p]_Q$, for every $p \in P$.

For example, $\mathbb{Q}$ is $\bigvee$-dense in $\mathbb{R}$, singletons are $\bigvee$-dense in $\mathcal{P}(X)$, c_0 is $\bigvee$-dense in ℓ_∞, and and the collection of affine functions on $[0,1]$ is $\bigvee$-dense in the set of convex lower semi-continuous functions on $[0,1]$.

Along with $\bigvee$-regularity, closedness and density, we can define $\bigwedge$-, $\wedge$-etc regularity, closedness and density.

Proposition 2

- *$\bigwedge$-density implies $\bigvee$-regularity, and $\bigvee$-density implies $\bigwedge$-regularity.*
- *If $Q \subset P$ is $\bigwedge$-dense and $\bigwedge$-regular, and $R \subset Q$ is $\bigwedge$-dense in Q, then R is $\bigwedge$-dense in P.*
- *Being simultaneously $\bigvee$-dense and $\bigwedge$-dense is a transitive relation.*

Completeness

Similarly to regularity, closedness and density, we can talk about completeness.

A $\vee$-*semilattice* is a poset in which every pair of elements has a supremum, hence these are $\bigvee_2$-complete posets.

A $\bigvee_2$-isotone map between $\vee$-semilattices is called a $\vee$*-homomorphism*.

A *sub-semilattice* of a $\vee$-semi-lattice is a $\bigvee_2$-closed subset.

Any $\vee$-semilattice is directed. Any upper set is a $\vee$-semilattice. An *ideal* is a directed lower set. Any ideal is a subsemilattice.

- $\vee$-*semilattice* is $\vee$-complete. $\vee$-homomorphism is $\vee$-isotone.
- A maximal element of a $\vee$-semilattice is its greatest element.
- If P is $\vee$-semilattice, then every full set is $\bigwedge$-regular.
- Any injective $\vee$-homomorphism between $\vee$-semilattices is an order embedding (the converse is false, affine functions in $\mathcal{C}\,[0,1]$).
- $Q \subset P$ is a sub-semilattice iff it is $\vee$-closed iff it is $\vee$-regular and a $\vee$-semilattice in the induced order.

A $\wedge$-*semilattice* and $\wedge$-homomorphism is defined similarly.

A poset which is simultaneously a $\vee$- and $\wedge$-*semilattice* is called a *lattice*. A *lattice homomorphism* is a $\vee$- and $\wedge$-isotone map.

A *sublattice* of a lattice is a $\vee$- and $\wedge$-closed set. Any intersection of ideals / filters (=co-ideals) / sub-(semi)lattices is a set of the same type.

The set of all infinite subsets of $\mathbb{Z}$ is a $\vee$-semilattice, but not $\wedge$-semilattice; $\mathcal{C}^1(-1,1)$ is neither.

Any totally ordered set is a lattice. If P is a lattice, denote its smallest and largest elements by 0_P and 1_P (when they exist).

Proposition 3

A poset is $\bigvee\limits_{\geq 0}$-complete iff it is $\bigwedge\limits_{\geq 0}$-complete iff it is $\bigvee$-complete and has the least element and iff it is $\bigwedge$-complete and has the greatest element.

We will call such posets *complete lattices*. Among examples of complete lattices are $[0,1]$, $\mathcal{P}(X)$, $\mathbb{N}\cup\{\infty\}$, $\mathbb{N}\cup\{0\}$ with divisibility.

In a $\bigvee$-complete poset a $\bigvee$-closed lower set is a principal lower set of the form $(p]$.

Bounded completeness

Let us relax completeness a bit. The bare minimum necessary condition for a set to have a supremum is that it is bounded from above. The following property is called *bounded completeness*.

Proposition 4 (TFAE:)

- *Every nonempty subset of P which is bounded from above has supremum;*
- *Every nonempty subset of P which is bounded from below has infimum;*
- *For every nonempty $Q, R \subset P$ such that $Q \leq R$, there is $p \in P$ such that $Q \leq p \leq R$.*

A discrete order on at least two elements is a boundedly complete non-lattice; $\mathbb{Q}$ is a lattice which is not boundedly complete; $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, c_0, ℓ_p, L_p are boundedly complete lattices.

If P is boundedly complete, "add" 1_P and / or 0_P (whatever is missing). The result is a complete lattice.

Proposition 5

P is boundedly complete directed and co-directed iff it is a lattice such that the order intervals in P are complete lattices.

Proposition 6

- *In a boundedly complete poset every $\bigvee$-closed set is $\bigvee$-regular and boundedly complete.*
- *If $Q \subset P$ is $\bigvee$-regular, and either $\bigvee$-complete in the induced order, or boundedly complete and majorizing, then it is $\bigvee$-closed.*

If (X, τ) is a topological space, τ is a complete lattice, which is $\bigcup$- and $\cap$-closed in $\mathcal{P}(X)$, but not necessarily $\bigcap$-regular.

The collection of all convex sets in $\mathbb{R}^n$ is $\bigcap$-closed in $\mathcal{P}(\mathbb{R}^n)$, but not $\cup$-closed.

If P is a poset, then the collection $\mathcal{P}_{\downarrow}(P)$ of all lower subsets of P forms a complete lattice, which is $\bigcup$- and $\bigcap$-closed in $\mathcal{P}(P)$. P embeds into $\mathcal{P}_{\downarrow}(P)$ as a $\bigcup$-dense subset by $p \mapsto (p]$.

Galois connections

The pair $(\triangleright, \triangleleft)$ is a *Galois connection* from P to Q if $p^{\triangleright} \leq q \Leftrightarrow p \leq q^{\triangleleft}$.

In other words, the epigraph of $\triangleright$ is the converse to the hypograph of $\triangleleft$.

Then, $\triangleleft$ is the *left adjoint* of $\triangleright$, and $\triangleright$ is the *right adjoint* of $\triangleleft$.

Note that P and Q are not in entirely symmetric roles in this construction. $(\triangleright, \triangleleft)$ is a Galois connection from P to Q iff $(\triangleleft, \triangleright)$ is a Galois connection from $-Q$ to $-P$.

The pair of mutually inverse order isomorphisms forms a Galois connection.

If $(\triangleright, \triangleleft)$ and $(\trianglerighteq, \trianglelefteq)$ are Galois connections from P to Q and from Q to R, then $(\triangleright\trianglerighteq, \trianglelefteq\triangleleft)$ is a Galois connection from P to R.

Some references:

- M. Erné, J. Koslowski, A. Melton, G. Strecker, *A primer on Galois connections*, 1993.
- M. Erné, *Adjunctions and Galois connections: origins, history and development*, 2004.

Proposition 7

For $\triangleright : P \to Q$ *and* $\triangleleft : Q \to P$ *TFAE:*

- $(\triangleright, \triangleleft)$ *is a Galois connection;*
- $\triangleleft$ *and* $\triangleright$ *are isotone,* $\triangleright\triangleleft$ *is expansive, and* $\triangleleft\triangleright$ *is contractive;*
- $\triangleleft$ *and* $\triangleright$ *are isotone, and there are a* $\bigvee$*-dense* $P' \subset P$*, and* $\bigwedge$*-dense* $Q' \subset Q$ *such that* $\triangleright\triangleleft|_{P'}$ *is expansive, and* $\triangleleft\triangleright|_{Q'}$ *is contractive.*

Proposition 8

- $\triangleleft\triangleright\triangleleft = \triangleleft$ *and* $\triangleright\triangleleft\triangleright = \triangleright$*, and hence* $\triangleright\triangleleft$ *and* $\triangleleft\triangleright$ *are idempotent.*
- $Q^{\triangleleft} = P^{\triangleright\triangleleft}$ *is majorizing and* $\bigwedge$*-closed and* $P^{\triangleright} = Q^{\triangleleft\triangleright}$ *is refining and* $\bigvee$*-closed, and the restrictions of* $\triangleright$ *and* $\triangleleft$ *are mutually inverse.*
- *For* $p, r \in P$ *we have* $p^{\triangleright} \leq r^{\triangleright}$ *iff* $p^{\triangleright\triangleleft} \leq r^{\triangleright\triangleleft}$ *and iff* $p \leq r^{\triangleright\triangleleft}$*.*
- $p^{\triangleright} = \min\{q \in Q,\ p \leq q^{\triangleleft}\}$ *and* $q^{\triangleleft} = \max\{p \in P,\ p^{\triangleright} \leq q\}$*. Hence, the adjoints are unique.*

Proposition 9

For a map $\triangleright : P \to Q$ the following is true:

- *$\triangleright$ has a left adjoint iff it is isotone and preserves principal lower sets under the pre-image.*
- *If $\triangleright$ has a left adjoint, then it is $\bigvee_{\geq 0}$-isotone. The converse holds if P is a complete lattice.*

We call maps with left / right adjoint *residuated* / *residual*.

If $P = Q = [0,1]$ residuated = $\bigvee_{\geq 0}$-isotone maps are increasing, continuous from the left (equivalently, lower semi-continuous), which fix $0, 1$.

Residual = $\bigwedge_{\geq 0}$-isotone maps are increasing, continuous from the right (equivalently, upper semi-continuous), which fix $0, 1$.

The places where the $\triangleright$ "stalls" correspond to the places where $\triangleleft$ "jumps", and vice versa.

If $\varphi : X \to Y$ is a relation, define $\varphi_{\mathcal{P}} : \mathcal{P}(X) \to \mathcal{P}(Y)$ and ${}_{\mathcal{P}}\varphi : \mathcal{P}(Y) \to \mathcal{P}(X)$ by $\varphi_{\mathcal{P}}(A) = \{y \in Y,\ \exists x \in A :\ x\varphi y\}$ and

$$\begin{aligned} {}_{\mathcal{P}}\varphi(B) &= \{x \in X :\ y \in Y,\ x\varphi y \ \Rightarrow\ y \in B\} = X \backslash \varphi^*_{\mathcal{P}}(Y \backslash B) \\ &= \{x \in X :\ x \not\varphi y,\ \forall y \in Y \backslash B\} = \{x \in X :\ y \in Y,\ \varphi_{\mathcal{P}}(\{x\}) \subset B\}, \end{aligned}$$

for $A \subset X$ and $B \subset Y$.

If φ is a map, then $\varphi_{\mathcal{P}}(A) = \varphi(A)$ and ${}_{\mathcal{P}}\varphi(B) = \varphi^{-1}(B)$, for for $A \subset X$ and $B \subset Y$.

Note that $\varphi \mapsto \varphi_{\mathcal{P}}$ and $\varphi \mapsto_{\mathcal{P}} \varphi$ preserves / reverses compositions.

If φ : is a pre-order, $\varphi_{\mathcal{P}}(A) = [A)$, and ${}_{\mathcal{P}}\varphi(A) = \{x,\ [x) \subset A\}$.

Proposition 10

The collection of all relations from X to Y (ordered by inclusion) is isomorphic to the collection of all residuated maps from $\mathcal{P}(X)$ into $\mathcal{P}(Y)$. The isomorphism is implemented by $\varphi \mapsto \varphi_{\mathcal{P}}$ and $\triangleright \mapsto \{(x, y) \in X \times Y,\ y \in \{x\}^{\triangleright}\}$. Compositions of relations correspond to compositions of Galois connections.

Polarities

We will call $(\triangleright, \triangleleft)$ a *(co)polarity* between P and Q if it is a Galois connection from P to $-Q$ (from $-P$ to Q).

That is, $(\triangleright, \triangleleft)$ a polarity if for $p \in P$ and $q \in P$ we have $q \leq p^{\triangleright} \Leftrightarrow p \leq q^{\triangleleft}$, equivalently both $\triangleright$ and $\triangleleft$ are antitone, and both $\triangleright\triangleleft$ and $\triangleleft\triangleright$ are expansive. In these configurations P and Q are in symmetric roles.

A composition of a Galois connection and a polarity is a polarity. In particular, this works for self-anti-isomorphisms.

Corollary 1

If X and Y are sets, then the general form of a polarity from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is $A \mapsto A^{\perp}$, and $B \mapsto B_{\perp}$, where $\perp : X \to Y$ is a relation, and for $A \subset X$, $B \subset Y$ we have $A^{\perp} = \{y \in Y,\ \forall x \in A :\ x \perp y\}$ and $B_{\perp} = \{x \in X,\ \forall y \in B :\ x \perp y\}$ (we will call such sets polars of $\perp$).

- For an order $\leq$ on P the polars are $A^{\leq} = A^{\uparrow}$ and $A_{\leq} = A^{\downarrow}$.
- If P is a poset with the smallest element 0_P, define *disjointness* $\perp$ on P by $p \perp q$ if $p \wedge q = 0_P$. The polars are *disjoint complements*.

• For the usual orthogonality in a vector space with an inner product we have that $A^{\perp} = A_{\perp}$ is the orthogonal complement of $A \subset E$.

More generally, if E and F are vector spaces in duality, define $e \perp f$ if $\langle e, f \rangle = 0$, where $e \in E$ and $f \in F$. In this case, $A^{\perp}$ and $B_{\perp}$ are the annihilators of $A \subset E$ and $B \subset F$.

• If G is a semigroup that acts on a set X, define the *stability / invariance* relation $\perp : G \to X$ by $g \perp x$ if $gx = x$. For $A \subset G$, $A_{\perp}$ is the stabilizer of A, and is a sub-semi-group of G. For $F \subset G$, $F^{\perp}$ is the collection of common fixed points of F.

• Let $P := \mathcal{F}(E, [-\infty, +\infty])$ and $Q := \mathcal{F}(E^*, [-\infty, +\infty])$, where E is a locally convex space. Define the *Legendre transform* by

$$f^{\triangleright}(e^*) = \bigvee_{e \in E} (\langle e^*, e \rangle - f(e)) \text{ and } g^{\triangleleft}(e) = \bigvee_{e^* \in E^*} (\langle e^*, e \rangle - g(e)).$$

$f^{\triangleright} = \bigvee_{e \in E} (e - f(e))$, and $g^{\triangleleft} = \bigvee_{e^* \in E^*} (e^* - g(e^*))$, if we view e as a function on E^*, hence $f^{\triangleright}$ and $g^{\triangleleft}$ are both convex.

$(\triangleright, \triangleleft)$ is a **copolarity** since both $f^{\triangleright} \leq g$ and $f \geq g^{\triangleleft}$ are equivalent to $\langle e^*, e \rangle - f(e) \leq g(e)$, for all $e \in E$ and $e^* \in E^*$ (i.e. $\langle \cdot, \star \rangle \leq f \oplus g$).

Hulls

Let P be a poset. Fix a map $\triangle\colon P \to P$.

Call $\triangle$ a *pre-hull* if it is isotone and expansive. We then call $p \in P$ *hulled* if $p = p^{\triangle}$. Let $P_{\triangle}$ be the collection of all $\triangle$-hulled elements of P.

Proposition 11

- *If $\triangle$ is a pre-hull on P, then $P_{\triangle}$ is $\bigwedge\limits_{\geq 0}$-closed (in particular, if P has the greatest element 1_P, then $1_P \in P_{\triangle}$).*
- *The correspondence $\triangle \mapsto P_{\triangle}$ is antitone from the collection of all pre-hulls on P into $\mathcal{P}(P)$.*

A *hull* is an idempotent pre-hull. In this case p is hulled iff it is in the image $P^{\triangle}$ of $\triangle$; hence, $P_{\triangle} = P^{\triangle}$. $P^{\triangle}$ is always majorating.

Id_P is a hull on P. If there is 1_P, then $p \mapsto 1_P$ is a hull on P.

For $P = \mathbb{R}$ the upper integer part $p \mapsto \lceil p \rceil$ is a hull.

A *(pre-)hull operator* on a set X is a (pre-)hull on $\mathcal{P}(X)$.

The adherence / closure with respect to any convergence structure on X is a pre-hull / hull operator.

Order adherence on a poset, sequential weak adherence on a Banach space are examples of non-idempotent pre-hull operators.

If P is a pre-ordered set $Q \mapsto (Q]$ is a hull operator on P.

$Q \mapsto Q^{\vee}$ and other similar closures are hull operators on P.

Some references:

- M. Erné, *Closure*, 2009.
- I. Singer, *Abstract Convex Analysis*, 1997.
- M.L.J. van de Vel, *Theory of Convex Structures*, 1993.
- Á. Császár, *Generalized open sets*, 1997 [a parallel development, in which an attempt was made to work with hulled sets as with closed sets and get some results of topological nature, see also forward citations].

Hulls vs Galois connections vs polarities

Corollary 2

If $(\triangleright, \triangleleft)$ is a polarity between P and Q, then $\triangleright\triangleleft$ and $\triangleleft\triangleright$ are hulls on P and Q, respectively.

Every $\perp : X \to Y$ generates hulls on both $\mathcal{P}(X)$ and $\mathcal{P}(Y)$: $A \mapsto A^{\perp}{}_{\perp}$, for $A \subset X$, and $B \mapsto B_{\perp}{}^{\perp}$, for $B \subset Y$. The hulled elements are polars.

• Let X, Y be sets, let $Z \subset Y$, and let $F \subset \mathcal{F}(X, Y)$. Consider the relation $\perp : F \to X$ defined by $f \perp x$ if $f(x) \in Z$.

• Let X be a locally convex space, $Y = \mathbb{C}$ and $F = X^*$. If $Z = \{0\}$, we get the annihilators as polars, hence closed linear span as hull.
If $Z = \overline{\mathbb{D}}$, we get the standard polarity from the locally convex theory, hence closed absolute convex hulls as hulls.

• If $Y = \mathbb{R}$, F is the collection of affine or convex functions and $Z = (-\infty, 0]$, we get closed convex hulls as hulls.

• Let $X = \mathbb{C}^n$. Take $Y = \mathbb{C}$ and $F =$ polynomials, or $Y = \mathbb{R}$ and $F =$ pluri-sub-harmonic functions to get some hulls from Complex Analysis.

• Let $X = \mathbb{C}^n$, $Y = \mathbb{C}$, $F =$ polynomials, and $Z = \{0\}$. Then, the hull operator on X corresponds to the Zariski topology.

• If X – spectrum of a Banach algebra A, $F = A$, and $Z = \{0\}$, the hull operator on X corresponds to the hull-kernel topology.

Other polarities:

• If E is a vector lattice, then the disjointness $\perp$ generates a hull operator $A \mapsto A^{\perp\perp}$.

• If P is a poset, then $Q \mapsto Q^{\uparrow\downarrow}$ is a hull operator. The hulled sets are of the form $Q^{\downarrow}$. Call them *cuts*, and denote the collection of all cuts by $\mathcal{P}_{\uparrow\downarrow}(P)$.

Proposition 12

Let $\triangle$ and $\triangledown$ be respectively a hull and a kernel on P. Then, $(\triangle, \triangledown)$ is a Galois connection from $P^{\triangledown}$ to $P^{\triangle}$. In particular, $\triangle\triangledown$ and $\triangledown\triangle$ are idempotent, and $\triangle, \triangledown$ are isomorphisms between $P^{\triangle\triangledown}$ to $P^{\triangledown\triangle}$.

Examples: regularly closed / open sets and normally lower / upper semi-continuous functions. Latter appear in order completion of $\mathcal{C}(X)$.

Proposition 13

A map $\triangle\colon P \to P$ is a hull iff $\left(\triangle, Id_{P^{\triangle},P}\right)$ is a Galois connection.

Corollary 3

Any hull is $\bigvee$-isotone. The image of a hull is majorating, $\bigwedge$-regular and $\bigwedge$-closed.
If $\triangle$ is a hull on P, and $r = \bigvee_P R$, for some $R \subset P^{\triangle}$, then $r^{\triangle} = \bigvee_{P^{\triangle}} R$.

Let P be a complete lattice. For $R \subset P$ let $p^{\triangle_R} := \bigwedge [p)_R$.

Proposition 14

$\triangle_R$ is a hull, $P^{\triangle_R} = R^{\bigwedge}$ is a complete lattice. If $Q \subset R$, then $\triangle_Q \geq \triangle_R$.

Examples: If $P = \mathcal{P}\left(\mathbb{R}^n\right)$, R – closed convex sets, or just closed half-spaces, then $\triangle_R$ is the closed convex hull.

F – vector lattice, $P = \mathcal{P}\left(F\right)$, $R =$ (prime) ideals, then $\triangle_R\left(A\right) = I\left(A\right)$.

Proposition 15

$R \mapsto \triangle_R$ and $\triangle \mapsto P_\triangle$ is a polarity between $\mathcal{P}(P)$ and the collection of all pre-hulls on P. We have $P_{\triangle_R} = R^{\wedge}$ and $\triangle_{P_\triangle} = \triangle^{\mathfrak{n}}$, for sufficiently large ordinal $\mathfrak{n}$.

Let X be a topological space and $P = \mathcal{F}(X, [-\infty, +\infty])$. If R is the set of continuous functions, then we get upper semi-continuous hull.

Assume that X is locally convex and $f \mapsto f^{\triangleright}$ and $f \mapsto f^{\triangleleft}$ is the Legendre transform. Then, $\triangleright\triangleleft$ is a *kernel* (i.e. a hull on $-P$). Also, $\triangleright\triangleleft = \triangledown_R$, where R is the collection of all convex (or affine) upper semi-continuous functions.

A *hull structure* on X is a $\bigcap$-closed subset of $\mathcal{P}(X)$.

Topologically closed, solid, convex sets, ideals of rings, lattices and vector lattices, sublattices, subgroups, lower subsets in a poset, ...

Kuratowski: A hull operator is a closure with respect to some topology iff it is $\cup$-isotone. However, such hulls are almost never $\bigcup$-isotone.

Algebraic hull operators

A hull operator on X is called *algebraic* if $A^{\triangle} = \bigcup \{B^{\triangle},\ B \subset A - \text{finite}\}$, for every $A \subset X$. Moreover, we will call it *n-algebraic*, for $n \in \mathbb{N}$, if $A^{\triangle} = \bigcup\limits_{x_1,...,x_n \in A} \{x_1, ..., x_n\}^{\triangle}$, for every $A \subset X$.

Caratheodory: convex sets on $\mathbb{R}^n$ form a $n+1$-algebraic hull structure.

Solid sets, ideals of rings and vector lattices form 1-algebraic hull structures.

Full sets on a poset form a 2-algebraic hull structure.

Closed convex sets is neither a topological nor algebraic hull structure.

(Algebraic / topological) hull structures form a hull structure on $\mathcal{P}(X)$.

If $\mathcal{Q}$ is a hull structure on X, then $\mathcal{Q}^{\triangle} := \{A \subset X,\ B^{\triangle},\ B \subset A - \text{finite}\}$ is the algebraic hull of $\mathcal{Q}$.

Let $\triangle$ and $\blacktriangle$ be hull operators on X and Y. Then, $\varphi : X \to Y$ is *continuous* if pre-images of $\blacktriangle$-hulled sets are $\triangle$-hulled. Equivalently, $\varphi(A^{\triangle}) \subset \varphi(A)^{\blacktriangle}$, for every$A \subset X$.

Examples from vector lattice theory: A linear map between Archimedean VL's is disjointness-preserving iff it is solid-continuous. A homomorphism is order continuous iff it is band-continuous.

PROJECT: A homomorphism of VL's induce homomorphisms between the lattices of all ideals (forward and backward). Same for order continuous homomorphisms and bands. What information can be recovered? What information about VL is contained in its ideal lattice?

Proposition 16

Let $\mathcal{P}_{\triangle}$ be the set of all $\triangle$-hulled sets. TFAE:

- *The hull is algebraic;*
- *If $A \subset X$ is such that $B^{\triangle} \subset A$, for every finite $B \subset A$, then $A \in \mathcal{P}_{\triangle}$;*
- *$\mathcal{P}_{\triangle}$ is $\uparrow$-closed, i.e. if $(A_i)_{i \in I} \subset \mathcal{P}_{\triangle}$ is increasing, then $\bigcup\limits_{i \in I} A_i \in \mathcal{P}_{\triangle}$;*
- *$\triangle$ is $\uparrow$-isotone, i.e. For every increasing net $(A_i)_{i \in I} \subset \mathcal{P}(X)$ we have $\left(\bigcup\limits_{i \in I} A_i\right)^{\triangle} = \bigcup\limits_{i \in I} A_i^{\triangle}$.*

1-algebraic hull structures

1-algebraic hull operators = algebraic & topological hull operators.

In the proposition we can also replace "$\uparrow$" with "$\bigcup$", and "finite" with "1". No analogue for 2-algebraic hulls (convex sets).

If $\leq$ is a pre-order on X, then upper sets in $(X, \leq)$ form a topology $\tau_{\leq}$ with the property that any intersection of open sets is open. This property is called *Alexandrov* discreteness. Isotone maps = continuous.

Conversely, given a hull operator $\triangle$ on X define $x \leq_{\triangle} y$ if $x \in \{y\}^{\triangle}$. This is a pre-order. Continuous maps are isotone.

Then, $\leq_{\tau_{\leq}} = \leq$, and each $\triangle$-hulled set is $\tau_{\leq_{\triangle}}$-closed. Thus, $\triangle \mapsto \leq_{\triangle}$ is the right adjoint to $\leq \mapsto \tau_{\leq}$. In fact, $\tau_{\leq_{\triangle}}$ is the 1-algebraic hull of $\triangle$.

1-algebraic hull operators = closures in Alexandrov-discrete topologies = lower set hulls with respect to pre-orders on X.

If $\triangle, \blacktriangle$ are (1-) algebraic hull operators on X, Y. Then, $\varphi : X \to Y$ is continuous iff $\varphi(A^{\triangle}) \subset \varphi(A)^{\blacktriangle}$, for any finite (singleton) $A \subset X$.

MacNeille completion

The collection $\mathcal{P}_{\downarrow}(P)$ of all lower sets on a poset P is a complete lattice, such that $p \mapsto (p]$ is a an embedding whose image R is $\bigvee$-dense. Then, $R^{\bigwedge} = \mathcal{P}_{\uparrow\downarrow}(P)$ – the collection of all cuts on P, i.e. sets of the form $Q^{\downarrow}$.

- R is $\bigwedge$-dense in $\mathcal{P}_{\uparrow\downarrow}(P)$.
- $\mathcal{P}_{\uparrow\downarrow}(P)$ is a complete lattice which verifies the universal property for isotone maps and order embeddings from P into complete lattices, BUT in a non-unique way.
- Nevertheless, even this weak universal property for order embeddings determines $\mathcal{P}_{\uparrow\downarrow}(P)$ up to an isomorphism.
- Even though the embedding of P into $\mathcal{P}_{\uparrow\downarrow}(P)$ is $\bigvee\limits_{\geq 0}$-isotone, $\mathcal{P}_{\uparrow\downarrow}(P)$ does NOT verify the universal property for $\bigvee\limits_{\geq 0}$-isotone maps from P into complete lattices.

The collection of all $\bigvee$-closed lower subsets of P verifies the universal property for $\bigvee\limits_{\geq 0}$-isotone maps from P into complete lattices.

Distributivity

Let P be a lattice.

P is *distributive* if $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$, for any $p, q, r \in P$, i.e. $q \mapsto p \wedge q$ is $\vee$-isotone. Note that the inequality $\geq$ is always satisfied.

- P is distributive iff $-P$ is distributive, i.e. $q \mapsto p \vee q$ is $\wedge$-isotone.
- Another equivalent condition is $r \leq p \vee q$ implies existence of $p' \leq p$ and $q' \leq q$ with $r = p' \vee q'$. In other words, $(p \vee q] = (p] \vee (q]$, for every $p, q \in P$ (and dually).
- Yet another one: for any $p, q, r \in P$ we have $(p \vee q) \wedge (q \vee r) \wedge (r \vee p) = (p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$.
- P is distributive iff it does NOT contain non-distributive sublattices with 5 elements (can be specified further).

A sublattice of a distributive lattice is distributive.

Any totally ordered set is a distributive lattice. $\mathcal{P}(X)$ is distributive.

Subspaces of a vector space or sublattices of a lattice form non-distributive complete lattices.

Theorem 2 (Stone representation theorem)

P is distributive iff it isomorphic to a sublattice of $\mathcal{P}(X)$.

We will say that P is $\bigvee$*-distributive*, if $q \mapsto p \wedge q$ is $\bigvee$-isotone. In this case $\bigvee Q \wedge \bigvee R = \bigvee\limits_{q \in Q,\ r \in R} p \wedge q$, for any $Q, R \subset P$ with supremums.

Any vector lattice is $\bigvee$- and $\bigwedge$-distributive.

A $\bigvee$-regular sublattice of a $\bigvee$-distributive lattice is $\bigvee$-distributive.

If τ is a topology on X, it is $\bigvee$-distributive, but not always $\bigwedge$-distributive.

Denote the set of all ideals (= $\vee$-closed lower sets) of P by $\mathcal{J}_P$.

Proposition 17 (Let P be distributive. Then:)

- *$\mathcal{J}_P$ is a $\bigvee$-distributive lattice. It is complete iff P has the least element. Otherwise, $\mathcal{J}_P \cup \{\varnothing\}$ is complete.*
- *$J \vee_{\mathcal{J}_P} H = J \vee H := \{j \vee h,\ j \in J,\ h \in H\}$, for any $J, H \in \mathcal{J}_P$.*
- *If $\mathcal{I} \subset \mathcal{J}_P$, then $\bigvee_{\mathcal{J}_P} \mathcal{I} = \bigvee \mathcal{I} := \{j_1 \vee ... \vee j_n,\ j_k \in J_k \in \mathcal{I}\}$ and $\bigwedge_{\mathcal{J}_P} \mathcal{I} = \bigcap \mathcal{I}$. In particular, $\mathcal{J}_P \cup \{\varnothing\}$ is $\bigcap$-closed in $\mathcal{P}(P)$.*

Boolean algebras

Let P be a lattice with 0_P. Disjointness: $p \perp q$ if $p \wedge q = 0_P$.

P is called a *Boolean algebra* if it is distributive, there is 1_P, and for every $p \in P$ there is p^* such that $p^* \perp p$ and $p^* \perp_{-P} p$, i.e. $p \vee p^* = 1_P$.

Every Boolean algebra is $\bigvee$- and $\bigwedge$-distributive.

A map between BA's is a *Boolean homomorphism* if it preserves Boolean operations (including 0-nary operations 0 and 1).

(σ-)algebras of sets from measure theory are precisely ($\bigvee\limits_{n \in \mathbb{N}}$ -closed) subalgebras of $\mathcal{P}(X)$. $Clop(X)$ is a subalgebra of $\mathcal{P}(X)$.

A Boolean algebra is *(σ-)complete* if it is $\bigvee$-complete (or $\bigvee\limits_{n \in \mathbb{N}}$ -).

Theorem 3 (Stone Representation theorem)

For any Boolean algebra P there is a unique totally disconnected compact Hausdorff space X such that $Clop(X) \simeq P$. Hence, any BA is isomorphic to an algebra of sets. Homomorphisms between BA's correspond to continuous maps of these space via pre-image.

- P is (σ-)complete iff its Stone space is extremally (basically) disconnected, i.e. the closure of an open (F_σ) set is open.
- MacNeille completion of a BA is a BA (not true for distributive).
- A homomorphism between BA's is $\bigvee$-isotone iff the corresponding Stone map preserves sets with nonempty interior (*almost open*).
- (Loomis-Sikorsky) Any σ-complete BA is isomorphic to a factor of a σ-algebra of sets over a $\bigvee\limits_{n\in\mathbb{N}}$-closed ideal.
- $Q \subset P$ is $\bigvee\limits_{\geq 0}$-dense iff $Q \backslash \{0_P\}$ is refining, i.e. $(p]_Q \neq \varnothing$, for $p > 0_P$.

Isomorphisms of categories:

- Boolean Algebras;
- *Hyperarchimedean* (=every principal ideal is a projection band) vector lattices with selected strong units and unit-preserving homomorphisms.
- Compact totally disconnected spaces;

- Complete Boolean Algebras and $\bigvee$-isotone homomorphisms;
- Extremally disconnected spaces and almost open maps;
- Universally complete vector lattices with selected weak units and unit-preserving order continuous homomorphisms.

Theorem 4 (Funayama)

P is $\bigvee$- and $\bigwedge$-distributive iff it is isomorphic to a $\bigvee$- and $\bigwedge$-regular sublattice of a Boolean Algebra.

$0_P \neq p \in P$ is an *atom* iff $(p] = \{0_P, p\}$. P is *atomless* if there are no atoms, and *atomic* if the set of atoms is $\bigvee_{\geq 0}$-dense.

Atoms correspond to the isolated points of the Stone space of a BA. Atomless = no isolated point. Atomic = isolated points are dense.

If P is an atomic BA, then $P^\delta = \mathcal{P}(P_a)$, where P_a – atoms of P.

There is a unique countable atomless BA. Its Stone space is the Cantor space.

The set of fragments (=components) of any element of a VL is a BA.

For example, if K is totally disconnected, the characteristic functions of clopen sets form $\bigvee$- and $\bigwedge$-closed and regular subset of $\mathcal{C}(K)$.

Positive disjoint sets in a VL represent embedding of the Boolean *ring* ("local Boolean algebra") of finite sets.

Proposition 18

If P is a distributive lattice with 0_P and 1_P, then the set P_c of all complemented elements in P is a Boolean algebra, which is a sublattice of P, that contains $0_P, 1_P$. In particular, $(p \wedge q)^* = p^* \vee q^*$ and $(p \vee q)^* = p^* \wedge q^*$, for every $p, q \in P_c$.

The *pseudo-complement* of p is $p^* := \max \{p\}^{\perp}$ (if exists).

Theorem 5 (Glivenko Theorem)

Let P be a complete $\bigvee$-distributive lattice. Then:

- *p^* exists for every $p \in P$, $(*, *)$ is a polarity from P to P, and $p \mapsto p^{**}$ is a hull on P.*
- *$P^* = \{p^*,\ p \in P\} = \{p \in P,\ p = p^{**}\}$ is a complete Boolean algebra, which is also $\bigwedge\limits_{\geq 0}$-closed in P.*
- *$p \vee_{P^*} q = (p^* \wedge q^*)^* = (p \vee q)^{**}$, for every $p, q \in P^*$; we also have $(p \wedge q)^* = (p^{**} \wedge q)^*$ and $(p \wedge q)^{**} = p^{**} \wedge q^{**}$, for every $p, q \in P$.*

For example, regularly open sets form a complete Boolean algebra.

Let F be a VL. Then, $I \subset F$ is an ideal of F iff I_+ is an ideal in F_+, with $2I_+ \subset I_+$ and $I = \{f, \ |f| \in I_+\}$.

If E and H are ideals, then $E \cap H = \{0_F\} \Leftrightarrow e \perp h$, for all $e \in E$, $h \in H$; also $E \vee_{\mathcal{I}_F} H = E + H$. If $\mathcal{J} \subset \mathcal{I}_F$, then

$$\bigvee_{\mathcal{I}_F} \mathcal{J} = \sum \mathcal{J} = \{f_1 + ... + f_n, \ f_k \in E_k \in \mathcal{J}\}.$$

$I \mapsto I_+$ is $\bigvee$- and $\bigwedge$-isotone order embedding from $\mathcal{I}_F$ into $\mathcal{J}_{F_+}$. Hence, $\mathcal{I}_F$ is $\bigvee$-distributive.

Corollary 4

If F is Archimedean, then bands are the pseudo-complemented elements of $\mathcal{I}_F$; they form a Boolean algebra, which is a $\bigcap$-closed subset of $\mathcal{I}_F$. Projection bands are the complemented elements of $\mathcal{I}_F$; they form a Boolean algebra, which is a sublattice of $\mathcal{I}_F$.

Theorem 6 (B.)

$H \in \mathcal{I}_F$ is a PB iff $H + \bigcap \mathcal{J} = \bigcap \{H + E, \ E \in \mathcal{J}\}$, for any $\mathcal{J} \subset \mathcal{I}_F$.

Ordered vector spaces

An order $\leq$ on a vector space E is *linear* if $e \mapsto e + h$ and $e \mapsto \lambda e$ are isotone, for every $h \in E$ and $\lambda > 0$. E is directed iff $E_+ - E_+ = E$.

E is *Archimedean* if $\bigwedge_{n \in \mathbb{N}} \frac{1}{n} e = 0_E$, for every $e \geq 0_E$.

$\varnothing \neq A \subsetneq E$ is called a *Dedekind cut* iff $A = A^{\uparrow\downarrow}$.

A is called a *Frink ideal* if $B^{\uparrow\downarrow} \subset A$, for every finite $B \subset A$. A linear $T : E \to F$ between **VL's** is a homomorphism iff it is Frink-continuous.

Theorem 7 (Dedekind completion)

For every directed Archimedean vector space E there is an (essentially unique) OVS E^δ and a linear order embedding $j : E \to E^\delta$ such that:

- *E^δ is a boundedly complete vector lattice;*
- *jE is $\bigvee$- and $\bigwedge$-dense in E^δ;*
- *E^δ verifies the universal property for boundedly complete vector lattices and:*
  - *Frink-continuous linear maps;*
- *Positive operators;*
  - *$\bigvee$-isotone linear maps.*

Sketch of the proof for Dedekind completion

Fact: If $A \subset E$ is bounded from above, then $\bigwedge \left(A^{\uparrow} - A\right) = 0_E$.

Lemma: If a cone is a group, then it is a vector space.

- Every Dedkind cut is convex and bounded from above in E;
- E^{δ} is directed upward and downward;
- Addition is associative;
- $0_{E^{\delta}} := -E_+ = (0_E] = \{0_E\}^{\downarrow}$ is the neutral element for addition. Hint: If A is a Dedekind cut, then $A - E_+ = A$;
- Additive inversion is given by $-A^{\uparrow}$. Hint: use the fact;
- If $\lambda > 0$, and $A \subset E$, then $\lambda A^{\uparrow\downarrow} = (\lambda A)^{\uparrow\downarrow}$;
- Show that positive scalar multiplication defined this way turns E into a cone; then use the Lemma;
- The order on E^{δ} is a linear order.

If $F \subset E$ is a subspace, such that F_+ is refining in E_+ (i.e. for every $f > 0_F$ there is $e \in E \cap (0_F, f]$), then F is $\bigvee$-dense.

In general, even though E is $\bigvee$-dense in E^δ, it is not always true that E_+ is refining in E^δ_+.

Theorem 8 (B., Deng, Kalauch, Malinowski, van Gaans)

For a directed Archimedean OVS TFAE:

- *For every $e \not\leq 0$ there is $f > 0$ such that $g \geq e, 0 \Rightarrow g \geq f$;*
- *For every $e, f \in E$ such that $\{e, f\}^\uparrow \subsetneq E_+$ there is $g > 0_F$ such that $\{e, f\}^\uparrow \subset [g)$;*
- *If $e, f \in E_+$ with $e \not\leq f$ then there is $g > f$ such that $\{e, f\}^\uparrow \subset [g)$;*
- *E_+ is refining in E^δ_+;*
- *E_+ is $\uparrow$-dense in E^δ_+;*
- *E embeds into a vector lattice with E_+ refining in F_+;*
- *$\left(E^\delta\right)_+ \cup \{+\infty\}$ is the MacNeille completion of E_+.*

Examples: spaces of smooth functions.

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