Disjointly non-singular operators and dispersed subspaces

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UofA and PIMS

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This talk is a partial survey of the following sources:

- González, Martínez-Abejón, Martinón, *Dijointly non-singular operators on Banach lattices*, 2021.
- Bilokopytov, *Disjointly non-singular operators on order continuous Banach lattices complement the unbounded norm topology*, 2022.
- Freeman, Oikhberg, Pineau, Taylor, *Stable phase retrieval in function spaces*, 2023.
- González, Martinón, *Disjointly non-singular operators; Extensions and local variations*, 2024.
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Other sources:

- Flores, Hernández, Tradacete, *Strict Singularity; A Lattice Approach*, 2019.
- González, Martinón, *A quantitative approach to disjointly non-singular operators*, 2021.

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Call $T : F \to E$ *strictly DNS* if there is $r > 0$ such that $\liminf\limits_{n\to\infty} \rho_T(f_n) > r$, for every normalized disjoint $(f_n)_{n\in\mathbb{N}}$.

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Equivalently, there are no normalized disjoint $\{f_n\}_{n\in\mathbb{N}}$ such that $\rho_T(f_n) \le r$, for all n.

The set of strictly DNS operators is open in $\mathcal{L}(F, E)$.

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Theorem 3 (B., 2022)

E is reflexive iff for every USF $T : E \to H$ there is $\delta > 0$ such that no normalized basic sequence $(e_n)_{n\in\mathbb{N}} \subset E$ satisfies $\rho_T(e_n) < \delta$.

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Every normalized UN-null net contains an almost disjoint sequence, i.e. $(e_n)_{n\in\mathbb{N}}$ such that $\|e_n - f_n\| \to 0$, where $(f_n)_{n\in\mathbb{N}} \subset F$ is disjoint.

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Corollary 1

On every dispersed subspace UN=$\|\cdot\|$. If $T : F \to E$ is DNS, then T complements UN.

The *absolute weak (AW)* topology is the linear topology whose base at 0 consists of the sets

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where $\nu \in F^*_+$ and $\varepsilon > 0$. Note that $f_\alpha \xrightarrow{aw} 0$ iff $|f_\alpha| \xrightarrow{w} 0$.

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Theorem 7 (González, Martinón, 2024)

If F is an order continuous Banach lattice, then if $T : F \to E$ is DNS, it is tauberian.

Connection with phase retrieval

$e, f \in F$ are *r-disjoint* if $\||e| \wedge |f|\| \leq r$.

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In an order continuous Banach lattice (with a weak unit???) dispersed is equivalent to n-dispersed, for some n.

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Hence, SPR is equivalent to being 2-dispersed.

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Theorem 9 (Freeman, Oikhberg, Pineau, Taylor, 2023)

If $\dim F = \infty$, it contains a non-dispersed subspace with PR property.

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Question 4

Can we always find a dispersed $H \subset F$ with $\dim H = \infty$ and no PR? Or with PR but no SPR?

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If F is order continuous then for every closed dispersed subspace H with $\dim H = \infty$ there is a closed 2-dispersed $G \subset H$ with $\dim G = \infty$.

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THANK YOU!