The PCP in some Lipschitz-free spaces

LSAA workshop: Banach spaces and Banach lattices

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Throughout the presentation, X, Y will stand for real Banach spaces and M, N for metric spaces.

Definition

X is said to have the ***Point of Continuity Property*** *(PCP) if every closed bounded subspace F of X has a* ***weak point of continuity****, that is, a point at which the identity map*

$$id\colon (F, w) \to (F, \|.\|_X)$$

is continuous.

Geometric characterization of the PCP

Reminder

X has the PCP if every closed bounded subset F of X has a **weak point of continuity**, that is, a point at which the identity map

$$id\colon (F, w) \to (F, \|.\|_X)$$

is continuous.

Proposition

X has the PCP iff each bounded subset F of X is ***w-fragmentable****, i.e., has non-empty relatively w-open subsets of arbitrarily small diameter ($\forall \varepsilon > 0$, $\exists V \neq \varnothing$ w-open, $\operatorname{diam}(V \cap F) < \varepsilon$).*

Two closely related properties

X has the PCP iff each closed bounded subset of X has non-empty relatively weakly-open subsets of arbitrarily small diameter.

Proposition

X has the Radon-Nikodým property (RNP) iff each closed bounded subset of X has non-empty slices of arbitrarily small diameter.

Recall that a **slice** of F is any subset of F of the form

$$S(x^*, \alpha) = \{x \in F : x^*(x) > \sup_{y \in F} x^*(y) - \alpha\}$$

where $x^* \in X^*$ and $\alpha > 0$.

Implication

X has the RNP $\Longrightarrow$ X has the PCP

Examples

- Reflexive spaces, separable dual spaces, ℓ_1 have the RNP, and thus the PCP.
- c_0, ℓ_∞, $L_1[0,1]$, $C[0,1]$ do not have the PCP (their norms satisfy the diameter 2 property).
- The dual of the James tree space does not have the RNP but has the PCP.

Derivative sets

Given $F \subset X$ bounded and $\varepsilon > 0$, we define the **first derivative set of** F as

$$\sigma_\varepsilon(F) = \sigma_\varepsilon^1(F) := F \backslash \{V \subset X \ w-open : \mathrm{diam}(V \cap F) < \varepsilon\}.$$

Then, for every ordinal α, we define inductively $\sigma_\varepsilon^\alpha(F)$ by

$$\sigma_\varepsilon^{\alpha+1}(F) = \sigma_\varepsilon(\sigma_\varepsilon^\alpha(F))$$

and

$$\sigma_\varepsilon^\alpha(F) = \bigcap_{\beta<\alpha} \sigma_\varepsilon^\beta(F)$$

if α is a limit ordinal.

Consequently: $x \in PC(F)$ iff for every $\varepsilon > 0$, $x \notin \sigma_\varepsilon(F)$.

Weak-fragmentability index

Let $B_X = \{x \in X : \|x\| \leqslant 1\}$ denote the closed unit ball of X.

Definition

We define $\Phi(X, \varepsilon)$ as the smallest ordinal α such that $\sigma_\varepsilon^\alpha(B_X) = \varnothing$ if such an ordinal exists, and write $\Phi(X, \varepsilon) = \infty$ otherwise.

Definition

If $\Phi(X, \varepsilon)$ is well-defined for every $\varepsilon > 0$, we call ***w-fragmentability index of*** *X the ordinal $\Phi(X) := \sup_{\varepsilon > 0} \Phi(X, \varepsilon)$. Otherwise, we write $\Phi(X) = \infty$.*

$\longrightarrow$ same definition as the Szlenk index but with the w-topology instead of the w^*-topology

Example: if X is finite dimensional, $\Phi(X) = 1$.

Some properties of the *w*-fragmentability index

Proposition

1) *If* $Y \simeq X$ *then* $\Phi(Y) = \Phi(X)$.
2) *If* $F \subset X$ *then* $\Phi(F) \leqslant \Phi(X)$.
3) *If* $\Phi(X)$ *is well-defined, there exists an ordinal* α *such that* $\Phi(X) = \omega^{\alpha}$ *(where* ω *denotes the first infinite ordinal).*
4) *If* X *is separable,* X *has the PCP iff* $\Phi(X) < \omega_1$ *(where* ω_1 *denotes the first uncountable ordinal).*

$\longrightarrow$ The PCP is hereditary and invariant under isomorphisms.

A norm on the space of Lipschitz maps?

Definition

Given a Lipschitz map $f\colon M \to X$, we write $\|f\|_L$ the best Lipschitz constant of f, that is:

$$\|f\|_L := \sup_{x \neq y \in M} \left\{ \frac{\|f(x) - f(y)\|_X}{d(x,y)} \right\}.$$

Notice that if f is constant, $\|f\|_L = 0$...

Let us distinguish a point 0 of M, and denote $Lip_0(M, X)$ the set of Lipschitz maps $f\colon M \to X$ such that $f(0) = 0$.

Proposition

$\|.\|_L$ is a norm on $Lip_0(M, X)$, and $Lip_0(M, X)$ endowed with this norm is a Banach space.

Setting $Lip_0(M) := Lip_0(M, \mathbb{R})$, let δ be the following isometry:

$$\delta = \begin{cases} M & \to & Lip_0(M)^* \\ x & \mapsto & \delta(x) = (f \mapsto f(x)) \end{cases}.$$

Definition

The Lipschitz-free space over *M is the following subspace of $Lip_0(M)^*$:*

$$\mathcal{F}(M) := \overline{Vect(\delta(x),\ x \in M)}.$$

Universal extension property

Theorem

Given $f \in \mathrm{Lip}_0(M, X)$, there exists a unique bounded operator $\widehat{f} \colon \mathcal{F}(M) \to X$ with $\left\|\widehat{f}\right\| = \|f\|_L$ such that $\widehat{f} \circ \delta = f$.

In addition to being isometric, $f \mapsto \widehat{f}$ is also surjective. Thus:

$$\mathrm{Lip}_0(M, X) \cong \mathcal{L}(\mathcal{F}(M), X).$$

Picking $X = \mathbb{R}$, we get that $\mathrm{Lip}_0(M) \cong \mathcal{F}(M)^*$.

Lipschitz-free spaces are of significant interest when dealing with the PCP :

Theorem (R. J. Aliaga, C. Gartland, C. Petitjean, A. Procházka; 2022)

Let M be a pointed metric space. The following are equivalent:
1) $\mathcal{F}(M)$ has the PCP.
2) $\mathcal{F}(M)$ has the RNP.
3) $\mathcal{F}(M)$ has the Schur property.
4) $\mathcal{F}(M)$ has the Krein-Milman property.
5) $\mathcal{F}(M)$ does not contain any isomorphic copy of L^1.
6) The completion of M is purely 1-unrectifiable.

Definition

*M is said to be **purely 1-unrectifiable (p-1-u)** if for every $A \subset \mathbb{R}$ and for every Lipschitz map $f : A \to M$, the Hausdorff's measure of $f(A)$ is null.*

If M is a separable metric space, the **Hausdorff's measure** of M is the amount $\lim_{\delta \to 0} \mathcal{H}^1_\delta$ where

$$\mathcal{H}^1_\delta = \inf \left\{ \sum_{i=1}^{\infty} \operatorname{diam}(E_i) : M \subset \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) < \delta \right\}.$$

In the sequel, all the metric spaces considered will be countable, and thus purely 1-unrectifiable.

To keep in mind

If M is a countable complete metric space, $\mathcal{F}(M)$ have the PCP.

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Goal

Theorem

For every $\alpha \in (0, \omega_1)$, there exists a countable complete metric space D_α such that $\Phi(\mathcal{F}(D_\alpha)) > \alpha$.

Reminder: if X is separable, X has the PCP iff $\Phi(X) < \omega_1$ (first uncountable ordinal).

Uncountable family of pairwise non isomorphic Lipschitz-free spaces

Proposition

There exists an uncountable family $(M_i)_{i \in I}$ of countable complete metric spaces such that their Lipschitz-free spaces $(\mathcal{F}(M_i))_{i \in I}$ are pairwise non isomorphic.

[P. Hájek, G. Lancien, E. Pernecká; 2016] : there exists an uncountable family of pairwise non isomorphic Lipschitz-free spaces over separable Banach spaces.

Lipschitz-free spaces over compact metric spaces

Open question

Given a separable metric space M, does there exist a compact metric space K such that $\mathcal{F}(M) \simeq \mathcal{F}(K)$?
(same question with a separable Banach space)

- [P.L. Kaufmann, 2015] : $\mathcal{F}(X) \simeq \mathcal{F}(B_X)$ so YES if $\dim(X) < \infty$.
- [L. García-Lirola, A. Procházka; 2019] : YES for the Pełczyński universal space.

Proposition

Let $\alpha \in [\omega,\ \omega_1)$ and K be any compact metric space. Then: $\mathcal{F}(D_\alpha)$ and $\mathcal{F}(K)$ are not isomorphic.

Universal space for countable complete metric spaces

Theorem

Let U be a separable complete metric space such that for every M countable complete, $M \underset{L}{\subset} U$. Then: U is not purely 1-unrectifiable.

Reminder: if X is separable, X has the PCP iff $\Phi(X) < \omega_1$.

[Szlenk, 1968] : a separable reflexive Banach space cannot be universal for separable reflexive Banach spaces.

Definition

A Banach space property $\mathcal{P}$ is said to be ***compactly determined*** *if a Lipschitz-free space $\mathcal{F}(M)$ has $\mathcal{P}$ whenever the subspace $\mathcal{F}(K)$ has $\mathcal{P}$ for each compact $K \subset M$.*

Examples: the Schur property, the RNP, the approximation property, weak sequential completeness, ...

Proposition

The following Banach space properties are not compactly determined:

- *Being AUC;*
- *Being AUC renormable;*
- *Having a weak-fragmentability index lower than ω.*

X is said to be **asymptotically uniformly convex** (AUC) if $\inf_{x\in S_X} \overline{\delta}_X(t,x) > 0$ for every $t > 0$, where

$$\overline{\delta}_X(t,x) = \sup_{\dim(X/Y)<\infty} \inf_{y\in S_Y} \|x + ty\| - 1.$$

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Reminder

Let $\alpha \in (0, \omega_1)$. We want to show that there exists a countable complete metric space D_α such that $\Phi(\mathcal{F}(D_\alpha)) > \alpha$.

To this end, we must find $\varepsilon > 0$ such that $\Phi(\mathcal{F}(D_\alpha), \varepsilon) > \alpha$
i.e. $\sigma_\varepsilon^\alpha(B_{\mathcal{F}(D_\alpha)}) \neq \varnothing$.

Let us fix $\varepsilon := 1$.

Definition

*A **molecule** is an element of $\mathcal{F}(M)$ of the form*

$$m_{x,y} := \frac{\delta(x) - \delta(y)}{d(x,y)}$$

with $x \neq y \in M$.

Molecules are of norm one.

First derivative set

D_1 consists of two poles t_1 and b_1 at a distance 2 from each other, and of a sequence $(x_n)_{n\in\mathbb{N}}$ of points at a distance 1 from each pole. For every $n \neq m$, the distance between x_n and x_m is also 2.

The distance on D_1 correspond to the shortest metric path in a connex graph.

Proposition

For $\varepsilon = 1$, $m_{t_1,b_1} \in \sigma_\varepsilon(B_{\mathcal{F}(D_1)})$ and thus $\sigma_\varepsilon(B_{\mathcal{F}(D_1)}) \neq \varnothing$.

Idea of the proof

Let V be a w-neighborhood of m_{t_1,b_1} in $\mathcal{F}(D_1)$. We must show that $\operatorname{diam}(V \cap B_{\mathcal{F}(D_1)}) \geqslant 1$.

- We show that there exist $j > i \in \mathbb{N}$ such that $\mu_V := \frac{1}{2}(m_{t_1,x_j} + m_{x_i,b_1}) \in V$.

-We compute :

$$\begin{aligned}\|\mu_V - m_{t_1,b_1}\|_{\mathcal{F}(D_1)} &= \frac{1}{2}\|\delta(t_1) - \delta(x_j) + \delta(x_i) - \delta(b_1) - \delta(t_1) + \delta(b_1)\| \\ &= \left\|\frac{\delta(x_i) - \delta(x_j)}{2}\right\| = \|m_{x_i,x_j}\| = 1.\end{aligned}$$

Next step

$\sigma_\varepsilon^2(B_{\mathcal{F}(D_2)})$ is the set
$\sigma_\varepsilon(B_{\mathcal{F}(D_2)}) \setminus \left\{V \subset X \ w-open : \mathrm{diam}(V \cap \sigma_\varepsilon(B_{\mathcal{F}(D_2)})) < \varepsilon\right\}$.

Is it possible to build a bigger metric space D_2 such that :
- we still have $m_{t,b} \in \sigma_\varepsilon^1(B_{\mathcal{F}(D_2)})$ (where $\varepsilon = 1$),
- but also $\mu_V \in \sigma_\varepsilon^1(B_{\mathcal{F}(D_2)})$, so that $m_{t,b} \in \sigma_\varepsilon^2(B_{\mathcal{F}(D_2)})$?

Diamond graph D_α for α a successor ordinal

Definition

If $\alpha = \beta + 1$ is a successor ordinal, D_α is obtained by replacing each edge of D_1 by an isometric copy of D_β.

Diamond graph D_α for α a limit ordinal

Definition

If α is a limit ordinal, we define

$$D_\alpha := \{t_\alpha, b_\alpha\} \cup \bigcup_{\beta<\alpha} \{\beta\} \times D_\beta \backslash \{t_\beta, b_\beta\}$$

with the distance

- $d_\alpha(t_\alpha, b_\alpha) = 2$;
- $d_\alpha((\beta, x), (\beta, y)) = d_\beta(x, y)$;
- $d_\alpha((\beta, x), (\gamma, y)) = \min\left(d_\beta(x, t_\beta) + d_\gamma(t_\gamma, y),\ d_\beta(x, b_\beta) + d_\gamma(b_\gamma, y)\right)$ *if β, $\gamma < \alpha$ with $\beta \neq \gamma$.*

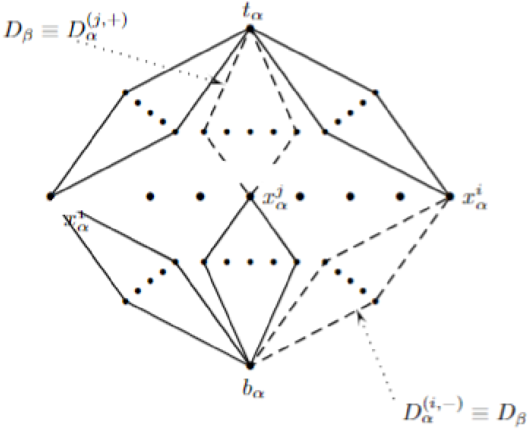
In D_2

- We can show again that in every w-neighborhood V of m_{t_2,b_2}, there is a $\mu_V \in V$ of the same form as before (so such that $\|m_{t_2,b_2} - \mu_V\| \geqslant 1$):

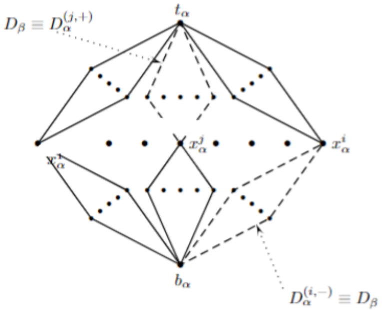
$$\mu_V = \frac{1}{2}(m_{t_2,x_2^j} + m_{x_2^i,b_2})$$

- Do we have $\mu_V \in \sigma_\varepsilon^1(B_{\mathcal{F}(D_2)})$?

Notation :



$$D_\alpha = D_{\beta+1}$$



$D_\alpha = D_{\beta+1}$

Lemma (for α being a successor ordinal)

Let $\alpha \in (0,\ \omega_1)$, let $i \neq j \in \mathbb{N}$. Let γ^+, $\gamma^- \in \sigma_\varepsilon^\alpha(B_{\mathcal{F}(D_\alpha)})$ such that $\gamma^+ \in \mathcal{F}(D_\alpha^{(j,+)})$ and $\gamma^- \in \mathcal{F}(D_\alpha^{(i,-)})$. Then: $\frac{\gamma^+ + \gamma^-}{2} \in \sigma_\varepsilon^\alpha(B_{\mathcal{F}(D_\alpha)})$.

Conclusion

Theorem

Given $\alpha \in (0,\ \omega_1)$, $m_{t_\alpha, b_\alpha} \in \sigma_\varepsilon^\alpha(B_{\mathcal{F}(D_\alpha)})$ and so $\sigma_\varepsilon^\alpha(B_{\mathcal{F}(D_\alpha)}) \neq \varnothing$.
Hence: $\Phi(\mathcal{F}(D_\alpha)) > \alpha$.

Theorem [B.M. Braga, G. Lancien, C. Petitjean, A. Procházka; 2019]

There exists M uniformly discrete such that each Banach space X verifying $\mathcal{F}(M) \underset{\sim}{\subset} X^*$ has a Szlenk index greater than ω^2.

Here: there exists D_α such that each Banach space X verifying $\mathcal{F}(D_\alpha) \underset{\sim}{\subset} X^*$ has a Szlenk index greater than α.