Unbounded Order convergence on infinitely distributive lattices

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Definition 1 (Order convergence)

Let $(x_\gamma)_{\gamma\in\Gamma}$ be a net and x a point in a poset $\mathcal{P}$. Then $(x_\gamma)_{\gamma\in\Gamma}$ is said to *O-converge* to x in $\mathcal{P}$ if there exists a directed subset $\mathcal{M} \subset \mathcal{P}$, and a filtered subset $\mathcal{N} \subset \mathcal{P}$, such that $\sup \mathcal{M} = \inf \mathcal{N} = x$, and for every $(m, n) \in \mathcal{M} \times \mathcal{N}$ the net is eventually contained in $[m, n]$.

Definition 2 (Unbounded order convergence)

A net $(x_\gamma)_{\gamma\in\Gamma}$ in a lattice $\mathcal{L}$ is said to unbounded order converge ($\mathfrak{u}O$-converge) to $x \in \mathcal{L}$, if $f_{s,t}(x_\gamma) \xrightarrow{\text{O}} f_{s,t}(x)$ for every $s, t \in \mathcal{L}$ and $s \leq t$ where $f_{s,t}(a) = (a \wedge t) \vee s$.

Proposition 3

Let X be a Riesz space. Then, $(x_\gamma)_{\gamma\in\Gamma}$ $\mathfrak{u}O$-converges to x if and only if $|x_\gamma - x| \wedge u \xrightarrow{O} 0$ for every $u \in X_+$.

The definition of unbounded order convergence coincides with $\mathfrak{u}O$-convergence on Riesz spaces. However, on Riesz spaces, $\mathfrak{u}O$-convergence is order continuous, the following example shows that this is not necessarily true for distributive lattices.

Example 4

Let $\mathcal{L}$ be the collection of closed subsets of $[0,1]$, ordered by inclusion. Clearly this is a distributive lattice. However, from the increasing sequence $\left(\left[\frac{1}{2^n},1\right]\right)_{n\in\mathbb{N}}$ we have that $\left[\frac{1}{2^n},1\right] \xrightarrow{O} [0,1]$ but $\left[\frac{1}{2^n},1\right] \wedge \{0\} \not\to^{O} \{0\}$.

When the lattice $\mathcal{L}$ is infinitely distributive, we have that $\mathfrak{u}O$-convergence is order continuous.

Proposition 5

Let $\mathcal{L}$ be an infinitely distributive lattice. If $(x_\gamma)_{\gamma\in\Gamma} \xrightarrow{O} x$ and $(y_\omega)_{\omega\in\Omega} \xrightarrow{O} y$, then $(x_\gamma \vee y_\omega)_{\gamma\times\omega\in\Gamma\times\Omega} \xrightarrow{O} x \vee y$ and dually.

By the adherence of a subset $\mathcal{X}$, we mean all those points which have a net in $\mathcal{X}$ O-converging to them. We denote the first adherence $\overline{\mathcal{X}}^{O}$ by $\mathcal{X}_1$ and for an ordinal λ, we denote the λ-adherence $\overline{\bigcup_{\beta<\lambda} X_\beta}^{O}$ by $\mathcal{X}_\lambda$. The first $\mathfrak{u}$O-adherence $\overline{\mathcal{X}}^{\mathfrak{u}O}$ is denoted by $\mathcal{X}_1^{\mathfrak{u}O}$ and for an ordinal λ we denote the λ-$\mathfrak{u}$O-adherence by $\mathcal{X}_\lambda^{\mathfrak{u}O}$.

A subset $\mathcal{X}$ of $\mathcal{P}$ is said to be *O-closed* ($\mathfrak{u}O$-closed) if there is no net in $\mathcal{X}$ that is O-converging ($\mathfrak{u}$O-converging) to a point outside of $\mathcal{X}$.

Lemma 6

Let Y be a sublattice of an infinitely distributive lattice $\mathcal{L}$. Then both Y_1 and $Y_1^{\mathfrak{u}O}$ are sublattices.

Theorem 7

Let $\mathcal{L}$ be an infinitely distributive lattice and $Y \subseteq \mathcal{L}$ be a sublattice. Then Y_λ is a sublattice for every $\lambda \leq |\mathcal{L}|^+$. In particular the order closure of Y is a sublattice.

Remark 8

We have an example of a non-distributive lattice where the O-closure of a sublattice is not a sublattice. However, we do not yet have an example of a distributive lattice, where the O-closure of a sublattice is not a sublattice.

Now we look at several results on Riesz spaces and extend them to infinitely distributive lattices.

Proposition 9 (N. Gao & D. Leung (2017), Proposition 2.1)

Let Y be a sublattice of a Riesz space X. Then, $Y_1 \subseteq Y_1^{uO} \subseteq Y_2$. Moreover,

1. if Y_1 is order closed, then it is the smallest order closed sublattice of X containing Y, and $Y_1 = Y_1^{uO}$;
2. if Y_1^{uO} is order closed, then it is the smallest order closed sublattice of X containing Y, and $Y_1^{uO} = Y_2$.

Proposition 10

Let Y be a sublattice of an infinitely distributive lattice $\mathcal{L}$. Then, $Y_1 \subseteq Y_1^{uO} \subseteq Y_3$. In particular, it follows that

1. if Y_1 is order closed, then it is the smallest order closed sublattice of $\mathcal{L}$ containing Y, and $Y_1 = Y_1^{uO}$;
2. if Y_1^{uO} is order closed, then it is the smallest order closed sublattice of $\mathcal{L}$ containing Y, and $Y_1^{uO} = Y_3$.

Theorem 11

Let Y be a sublattice of an infinitely distributive lattice $\mathcal{L}$. Then, Y is O-closed if and only if Y is uO-closed.

Remark 12

We note that in our proposition, we have Y_3 not Y_2.

Question 1

Is it possible for Proposition 10 to be reduced from Y_3 to Y_2 or is there an example of an infinitely distributive lattice and a sublattice Y such that $Y_2 \neq Y_3$.

Question 2

Is there any example of a sublattice Y of an infinitely distributive lattice $\mathcal{L}$ such that $Y_1^{uO} \not\subseteq Y_2$.

Definition 13

A set $\mathcal{A}$ in a lattice $\mathcal{L}$ is a *down-set* if every $a \in \mathcal{A}$, $x \in \mathcal{L}$ with $x \leq a$ implies $x \in \mathcal{A}$. Furthermore, $\mathcal{A}$ is an *ideal* if it is a down-set and $a \vee b \in \mathcal{A}$ for every $a, b \in \mathcal{A}$.

Proposition 14

Let $\mathcal{L}$ be an infinite distributive lattice and $\mathcal{A} \subseteq \mathcal{L}$ be a down-set. Then $\mathcal{A}_1$ is a down-set. Furthermore, if $\mathcal{A}$ is an ideal then $\mathcal{A}_1$ and $\mathcal{A}_1^{\mathfrak{u}O}$ are ideals.

Theorem 15

Let $\mathcal{L}$ be an infinite distributive lattice and $\mathcal{A} \subseteq \mathcal{L}$ be an ideal. Then, $\mathcal{A}_1$ is O-closed and $\mathcal{A}_1 = \mathcal{A}_1^{\mathfrak{u}O}$.

The MacNeille completion has been thoroughly studied over the years. We shall now look at results that hold for the MacNeille completion of a Riesz space and see whether these hold when considering the MacNeille completion of lattices.

Let $\mathcal{P}$ be a poset and $\mathcal{D}$ a subset of $\mathcal{P}$.

1. The set of upper-bounds and lower-bounds of $\mathcal{D}$ are denoted by $\mathcal{D}^+$ and $\mathcal{D}^-$, respectively.
2. A set $\mathcal{D} \subseteq \mathcal{D}$ is said to be a cut if $\mathcal{D}^{+-} = \mathcal{D}$.
3. The MacNeille completion of a poset $\mathcal{P}$ denoted by $\mathcal{P}^\delta$ is the set of all cuts of $\mathcal{P}$.
4. For every $x \in \mathcal{P}$, the set $(\leftarrow, x]$ is a cut and the function $\varphi : \mathcal{P} \to \mathcal{P}^\delta$ defined by $\varphi(x) = (\leftarrow, x]$ is an order isomorphism from $\mathcal{P}$ onto the subspace $\varphi[\mathcal{P}]$ of $\mathcal{P}^\delta$.
5. $\varphi[\mathcal{P}]$ is join and meet dense in $\mathcal{P}^\delta$ and the set $\mathcal{P}^\delta$ is a complete lattice with respect to set-theoretic inclusion.

There is a distinction between the MacNeille completion of a poset $\mathcal{P}$ and that of a Riesz space X.

1. The MacNeille completion $\mathcal{P}^\delta$ has a maximal element while X^δ does not.
2. The following example found in [5, p. 190] shows that removing the greatest or smallest elements of $\mathcal{P}^\delta$ might remove the lattice structure.

Example 16

Let $\mathcal{P} = \{x_1, x_2\}$ with the partial ordering $x_1 \leq x_2$ implies $x_1 = x_2$. For $\mathcal{A} \subseteq \mathcal{P}$, the set $\mathcal{A}^{ul}$ consists of either $\emptyset, \{x_1\}, \{x_2\}$ or $\mathcal{P}$. Thus, $\mathcal{P}^\delta = \{\emptyset, \{x_1\}, \{x_2\}, \mathcal{P}\}$.

Theorem 17 (N. Gao, V. Troitsky & F. Xanthos (2017), Corollary 2.9)

Let X be a Riesz space and $(x_\gamma)_{\gamma\in\Gamma}$ be a net in X. Then $x_\gamma \xrightarrow{o} x$ in X if and only if $x_\gamma \xrightarrow{o} x$ in X^δ.

Theorem 18 (K.A, E. Chetcuti, H. Weber (2022), Theorem 3 (ii))

Let $\mathcal{L}$ be a lattice and $(x_\gamma)_{\gamma\in\Gamma}$ be a net in $\mathcal{L}$. Then $x_\gamma \xrightarrow{o} x$ in X if and only if $x_\gamma \xrightarrow{o} x$ in X^δ.

1. If $\mathcal{P}$ is a lattice and not a poset, then removing the top or bottom elements does not affect the lattice structure.
2. It is also known that Riesz spaces are infinitely distributive, then the MacNeille completion X^δ is also infinitely distributive. The following example shows that this is not necessarily true when considering the MacNeille completion of a lattice $\mathcal{L}$.

Example 19

Take $\mathcal{L} = \{(0,b) : 0 \leq b \leq 1\} \cup \{(1,b) : 0 \leq b \leq 1\}$ and $\mathcal{L}_0 = \{(0,b) : 0 \leq b < 1\} \cup \{(1,b) : 0 \leq b < 1\}$. Then the MacNeille completion $\mathcal{L}_0^\delta$ is $\{(0,b) : 0 \leq b < 1\} \cup \{(1,b) : 0 \leq b \leq 1\}$. This is a sublattice but not infinitely distributive. Indeed let $x_n = (0, 1 - \frac{1}{n})$. Then $\sup_{\mathcal{L}_0^\delta} x_n = (1,1)$ and $(\sup_{\mathcal{L}_0^\delta} x_n) \wedge (1,0) = (1,0)$. On the other-hand, $\sup_{\mathcal{L}_0^\delta}(x_n \wedge (1,0)) = (0,0)$.

Theorem 20 (N. Gao, V. Troitsky & F. Xanthos (2017), Theorem 2.10)

Let Y be a regular sublattice of a Riesz space X. Then Y^δ is regular in X^δ.

The same example above shows that Theorem 20 above is not true when working with lattices.

We shall now give a property under which Theorem 20 remains true for lattices. First we answer the following questions. Given a sublattice $\mathcal{L}_0 \subseteq \mathcal{L}$, is it true that $\mathcal{L}_0^\delta \subseteq \mathcal{L}^\delta$? Moreover, is this still a sublattice?

Proposition 21

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}_0$ such that $\mathcal{A}^{+-} \subseteq \mathcal{B}^{+-}$. Then, $\mathcal{A}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}} \subseteq \mathcal{B}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}}$.

Proposition 22

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Then for $\mathcal{A} \subseteq \mathcal{L}_0$, $(\mathcal{A}^{+-} \cap \mathcal{L}_0)^{+-} = \mathcal{A}^{+-}$.

Let $\mathcal{L}_0^* = \{\mathcal{A}^{+-} : \mathcal{A} \subseteq \mathcal{L}_0 \text{ and } \mathcal{A}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}} = \mathcal{A}\}$.

Theorem 23

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Then $\mathcal{L}_0^*$ is isomorphic to $\mathcal{L}_0^\delta$.

To show Theorem 23, we show the following propositions.

Proposition 24

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Then for $a \in \mathcal{L}_0$, $\{a\}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}} = \{a\}^{+-} \cap \mathcal{L}_0 = (\leftarrow, a] \cap \mathcal{L}_0$.

Corollary 25

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Then for every $a \in \mathcal{L}_0$, $(\{a\}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}})^{+-} = ((\leftarrow, a] \cap \mathcal{L}_0)^{+-} = (\leftarrow, a] = \{a\}^{+-}$.

Proposition 26

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a sublattice. Then $\mathcal{L}_0$ is join and meet-dense in $\mathcal{L}_0^*$.

From Corollary 25, $\mathcal{L}_0$ can be embedded in $\mathcal{L}_0^*$ by considering the map $\rho : \mathcal{L}_0 \to \mathcal{L}_0^*$ where $\rho(a) = (\leftarrow, a]$. Furthermore by Proposition 26, as $\mathcal{L}_0$ is join and meet-dense in $\mathcal{L}_0^*$, $\mathcal{L}_0^*$ and $\mathcal{L}_0^\delta$ are order-isomorphic. Thus $\mathcal{L}_0^\delta$ can be seen as a subset of $\mathcal{L}^\delta$.

Definition 27

Let $\mathcal{L}$ be a lattice and $\mathcal{A} \subseteq \mathcal{L}$. Then $\mathcal{A}$ is said to be *convex* if for $x, y \in \mathcal{A}$ with $x \leq y$ then $[x, y] \subseteq \mathcal{A}$.

Proposition 28

Let $\mathcal{L}$ be a lattice and $\mathcal{L}_0 \subseteq \mathcal{L}$ be a convex sublattice. Then $\mathcal{A}^{+_{\mathcal{L}_0} -_{\mathcal{L}_0}} = \mathcal{A}^{+-} \cap \mathcal{L}_0$ for every $\mathcal{A} \subseteq \mathcal{L}_0$ satisfying $\mathcal{A}^{+_{\mathcal{L}_0}} \neq \emptyset$.

In the next example we note that if $\mathcal{L}_0$ is a convex sublattice and $\mathcal{A} \subseteq \mathcal{L}_0$ such that $\mathcal{A}^{+\mathcal{L}_0} = \emptyset$, then Proposition 28 does not hold.

Example 29

Consider $\mathcal{L} = \{(0, a) : 0 \leq a \leq 1\} \cup \{(1, b) : 0 \leq b \leq 1\} \cup \{(\frac{1}{2}, 1)\}$ ordered by pointwise partial order. Let $\mathcal{L}_0 = \{(0, a) : 0 \leq a \leq 1\} \cup \{(1, b) : 0 \leq b < 1\}$ such that $\mathcal{L}_0$ is a convex sublattice of $\mathcal{L}$ with no greatest element. Let $\mathcal{A} = \{(0, a) : 0 \leq a < 1\}$, then $\mathcal{A}^{+\mathcal{L}_0 -\mathcal{L}_0} = \mathcal{L}_0$. However, $\mathcal{A}^{+} = \{(\frac{1}{2}, 1), (1, 1)\}$ and $\mathcal{A}^{+-} = \mathcal{A} \cup \{(\frac{1}{2}, 1)\}$. Thus, $\mathcal{A}^{+\mathcal{L}_0 -\mathcal{L}_0} \neq \mathcal{A}^{+-} \cap \mathcal{L}_0$.

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