Yano's extrapolation theory
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The starting point of the theory we are going to study was a mathematical discussion between A. Zygmund and E. Titchmarsh, when the second author proved that $\mathcal{C}f$ is integrable whenever $f\log(2+|f|)^{1+\varepsilon}$ is integrable for some $\varepsilon > 0$, where $\mathcal{C}$ is the conjugate operator on the torus. Almost at the same time, Zygmund announced that the above result is true for $\varepsilon = 0$. One year later, Titchmarsh published a second paper where, using the fact that, for every $1 < p \leq p_1$,

$$\mathcal{C} : L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{T}), \qquad \|\mathcal{C}\| \lesssim \frac{1}{p-1},$$

he was able to "extrapolate" the result of Zygmund. The same year, and using a different argument, Zygmund proved his announced result. However, it was not until 1951 that, inspired by the ideas of Titchmarsh, Yano published the theorem that would later give the theory his name. The starting result is the following theorem.

Theorem (S. Yano, 1951). *Let (X,μ) and (Y,ν) be finite measure spaces. If T is a sublinear operator such that, for every $1 < p \leq 2$ and for some fixed $\alpha > 0$,*

$$T : L^p_\mu \longrightarrow L^p_\nu, \qquad \|T\| \lesssim \frac{1}{(p-1)^\alpha},$$

then

$$T : L(\log L)^\alpha(\mu) \longrightarrow L^1_\nu.$$

Since then, this theorem has had multiple extensions with very interesting applications such as, for example, in the problem of the almost everywhere convergence of the Fourier series, among many others.

In this course of 6 hours, we shall present the easiest proof of the above result and analyze several extensions to the context of weak-type estimates, bilinear operators and compatible couples of Banach spaces. We shall also see some connections with the extrapolation theory of Rubio de Francia.

The structure of the course will be approximately 8 talks of 45 minutes as follows:

1. History of the problem and motivation.
2. Preliminaries and some technical results.
3. Several proofs Yano's theorem.
4. The restricted weak-type case with application to Fourier series.
5. Endpoint estimates whenever

$$T : L^p_\mu \longrightarrow L^p_\nu, \qquad \|T\| \lesssim \frac{1}{(p-p_0)^\alpha},\ p_0 > 1.$$

6. The case of multilinear operators.
7. The connection with Rubio de Francia extrapolation theory
8. Extension to compatible couples of Banach spaces.

References

[1] S. Yano, *Notes on Fourier analysis. An extrapolation theorem*, J. Math. Soc. Japan **3** (1951), 296–305.

[2] E. C. Titchmarsh, *On Conjugate Functions*, Proc. London Math. Soc. **29** (1928), no. 1, 49–80.

[3] A. Zygmund, *A Remark on Conjugate Series*, Proc. London Math. Soc. **34** (1932), no. 5, 392–400.