The arithmetic & modularity of black holes in string theory - Part 1

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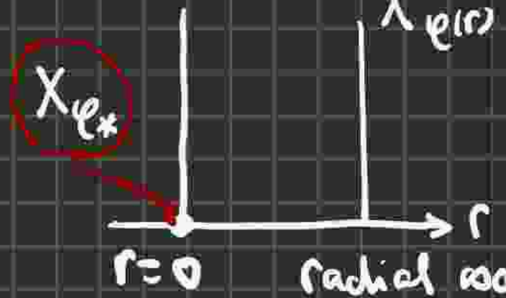
geometry of Calabi-Yau varieties and that of their moduli spaces ⟷ arithmetic of families of CY manifolds and modularity properties

periods

string theory

focus today: physics of Black hole solutions in string theory

entropy
↓
L-values



radial coordinate in space-time

attractor mechanism

AIM: an instance of the rich connections between theoretical physics and mathmatics (geometry & number theory)

At the end of these talks:

L-function associated to a modular form

$$A(\varphi_*)_{ke} = 14\pi \left\{ k \, \frac{7}{\pi} \frac{L_4(2)}{L_4(1)} + \left(e - \frac{5k}{2}\right)^2 \left(\frac{7}{\pi} \frac{L_4(2)}{L_4(1)}\right)^{-1} \right\}$$

proportional to BH entropy, so a certain counting of BH states given in terms of the arithmetic of an attractor CY manifold X_{φ_*}

work with

P. Candelas, M. Elmi,
D. van Straten

Dec 2019

P Candelas, XD, D van Straten April 2021

(P Candelas, XD, J McGovern, P Kuusela 2021 & Feb 2023)

PLAN

Part 1 (Xenia)

1. CY varieties ← review
2. Attractor mechanism ← BHs in Type II

Part 2 (Philip)

3. Arithmetic of families of CY varieties
4. Arithmetic of attractor varieties

CY manifolds: compact Kähler with $c_1 = 0$

(this talk is concerned with $d = 3$)

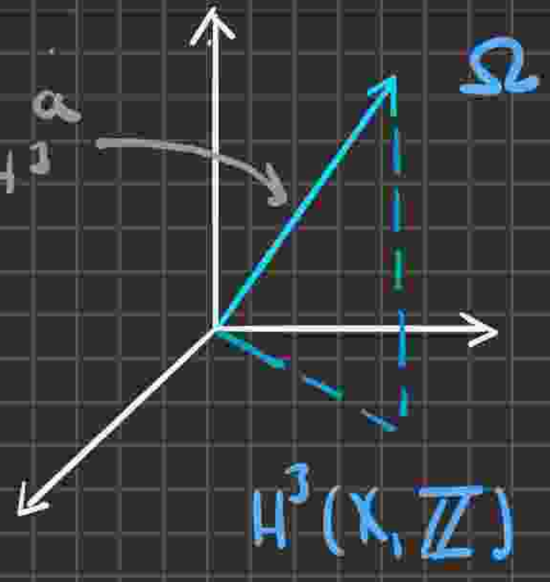
- It is a theorem that $\exists!$ (up to a constant) $(3,0)$-form Ω which is holomorphic $(d\Omega = 0)$

$$\dim H^{(3,0)} = \dim H^{(0,3)} = 1$$

$$H^3 = H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$$

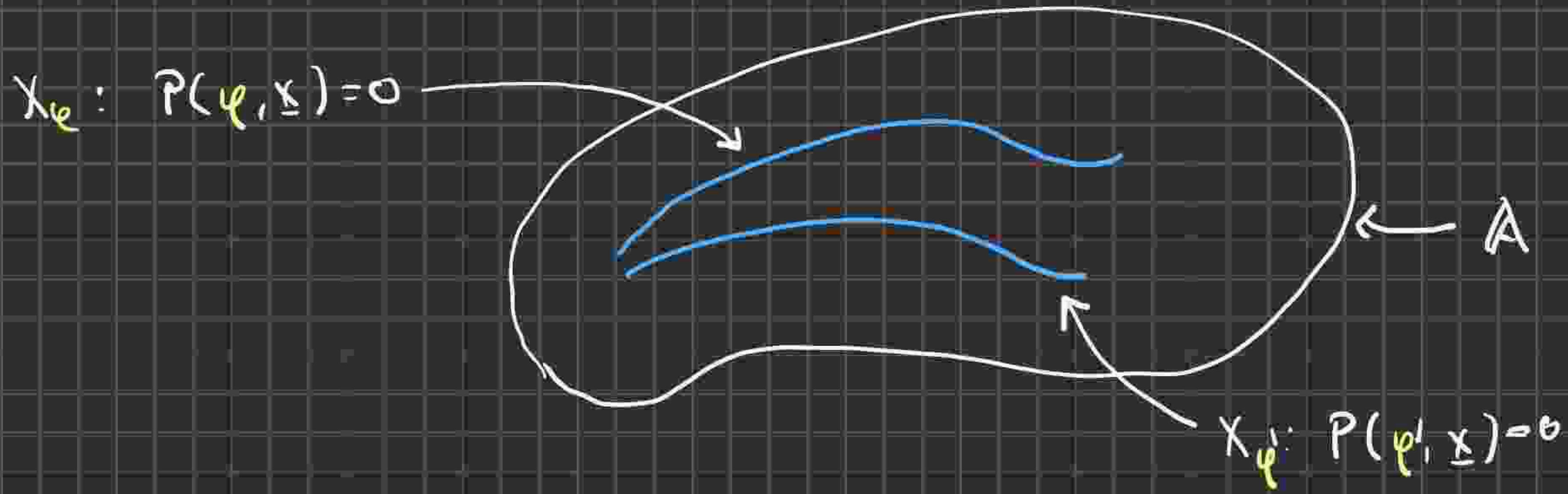
$$b_3 = 1 + h^{(2,1)} + h^{(1,2)} + 1 = 2(1 + h^{(2,1)})$$

Ω defines a line in H^3



- CY varieties have parameters X_φ

complex structure parameters ($h^{2,1}$ counts the number of deformations)



Kähler structure: "size" with respect to the metric

$\rightarrow$ ($h^{1,1}$ counts the number of deformations)

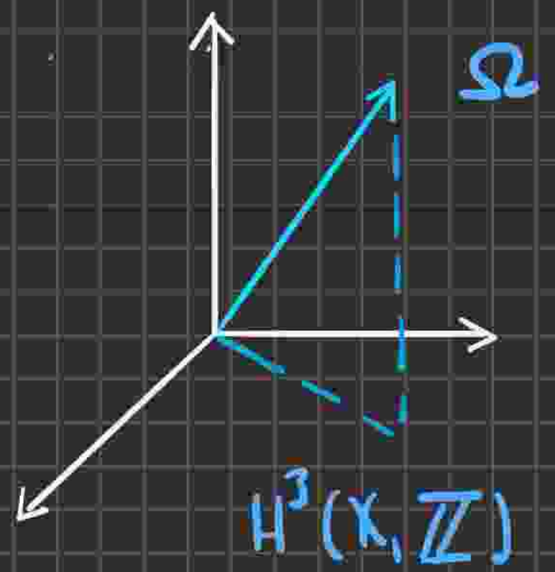
- moduli spaces have interesting geometry.

Periods and the complex structure

↳ there is a canonical way to give coordinates on the space of complex structures

Recall: Ω is defined up to a scale but is otherwise unique $\Rightarrow$ defines a line in H^3

- study the variations of the complex structure by studying how Ω varies in H^3



- the coordinates of this line are the periods (which then vary as we vary the complex structure)

$a, b = 0, \dots, h^{2,1}$

That is,

$$\Omega(\varphi) = z^a(\varphi)\,\alpha_a - \mathcal{F}_a(\varphi)\,\beta^a$$

where

$\{\alpha^a, \beta_b\} \to$ symplectic basis of $H^3(X, \mathbb{Z})$ $\quad a, b = 0, 1, \dots, h^{2,1}$

$$\int_X \alpha_a \wedge \beta^b = -\int_X \beta^b \wedge \alpha_a = \delta_a^b \,, \qquad \int_X \alpha \wedge \alpha = 0 \,, \qquad \int_X \beta \wedge \beta = 0$$

let $\{A^a, B_b\}$ the (dual) symplectic basis of $H_3(X, \mathbb{Z})$

Then: $\quad z^a(\varphi) = \int_{A^a} \Omega(\varphi) \qquad \mathcal{F}_a(\varphi) = \int_{B_a} \Omega(\varphi)$

Bryant & Griffiths: the complex structure is completely determined by the periods $\{z^a\}$ (so $\mathcal{F}_a = \mathcal{F}_a(z)$). The set $\{z^a(\varphi)\}$ are projective coordinates on the moduli space and we have that $\mathbb{C}\dim \mathcal{M}_{cs} = h^{2,1}$

Periods determine the geometry of the moduli space

$\underbrace{\text{the geometry of the moduli space}}_{\text{special geometry}}$

$$g_{\varphi\bar{\varphi}} = \partial_\varphi \partial_{\bar{\varphi}} K, \quad e^{-K} = -i\int_X \Omega \wedge \bar{\Omega} = 2\,\mathrm{Im}(z^a \bar{\mathcal{F}}_a)$$

Part 2: periods also have arithmetic content!!

Periods are calculable: they satisfy a differential eq of degree b_3 (Picard-Fuchs equation)

To see this (intuitively) consider:

Ω and its variations wrt φ, Ω', Ω'', etc

$\Omega, \Omega', \Omega'', \ldots$ are all closed 3-forms

so at most b_3 of them are linearly independent.

Then there is a linear differential operator $\mathcal{L}$ with $\deg \mathcal{L} = b_3$ st

$$\mathcal{L}\,\Omega = \text{exact 3-form}$$

This implies the periods satisfy the same differential equation as the period integrals are taken over a fixed basis of homology

$$\boxed{\mathcal{L}\,\vec{z} = 0\,, \qquad \mathcal{L}\,\vec{\mathcal{F}} = 0}$$

Focus today
1-parameter examples
$(b_3 = 2(1+h_{11}) = 4)$

This system is known as the Picard-Fuchs equation

- The PF equation is Fuchsian, that is the singular points are regular singularities. Solutions are series around a singularity

$$\sum_{n=-r}^{\infty} a_n \varphi^n$$

series ends from below

and logarithmic solutions are allowed

- There is a prescription to obtain the PF eq

(Dwork and Griffiths; Gelfand, Kapranov and Zelevinskii)

Example

We have in mind a particular example:

Verrill 1996, Hulek & Verrill 2005

why this example?

exhibits interesting arithmetic properties which have an interpretation in BH solutions of string theory

[Simplest example $\mathbb{P}^4[5]$]

In fact

HV motivation: "to find further examples of modular CY varieties, i.e., of CY varieties which are defined over the rationals and whose L-series can be described in terms of modular forms"

they found modularity at conifold singularities

Part 2: modularity at values of φ^* where X_{φ^*} is smooth (attractor varieties)

This seminar : one parameter $h^{21} = 1$ ie $b_3 = 4$
(cs-structure)

Let : X_φ be a CY variety defined by

$$P(\underline{x}, \varphi) = 0$$

$$\mathcal{L} = S_4 \theta^4 + S_3 \theta^3 + S_2 \theta^2 + S_1 \theta + S_0 \, , \quad \theta = \varphi \frac{d}{d\varphi}$$

S_i = polynomials in φ

$S_4 \rightsquigarrow$ discriminant

roots $\rightsquigarrow$ values of φ for which X_φ is singular

Hulek + Verrill

$a_i = 1 \qquad h^{21} = 5$

$$P(X_\varphi) = \left(\sum_{i=1}^{5} X_i\right)\left(\sum_{i=1}^{5} \frac{a_i}{X_i}\right) - \varphi^{-1}$$

$\{X_1, \dots, X_5\} \in \mathbb{P}_4^*$

Take the quotient by:

$G_1 : X_i \longrightarrow X_{i+1}$, $\qquad G_2 : X_i \longrightarrow \frac{1}{X_i}$

Obtain a smooth CY

$h^{21} = 1$, $\qquad h^{11} = \begin{cases} 9 & \text{quotient by } G_1 \\ 5 & \text{quotient by } G_1 \times G_2 \end{cases}$

HV is the mirror of a CICY

(x_1, y_1)

(x_2, y_2)

$\vdots$

$$\begin{array}{cc} \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \\ \mathbb{P}^1 \end{array} & \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{array}\right] \\ & \begin{array}{cc} P^1 & P^2 \end{array} \end{array}$$

$h^{11} = 5$

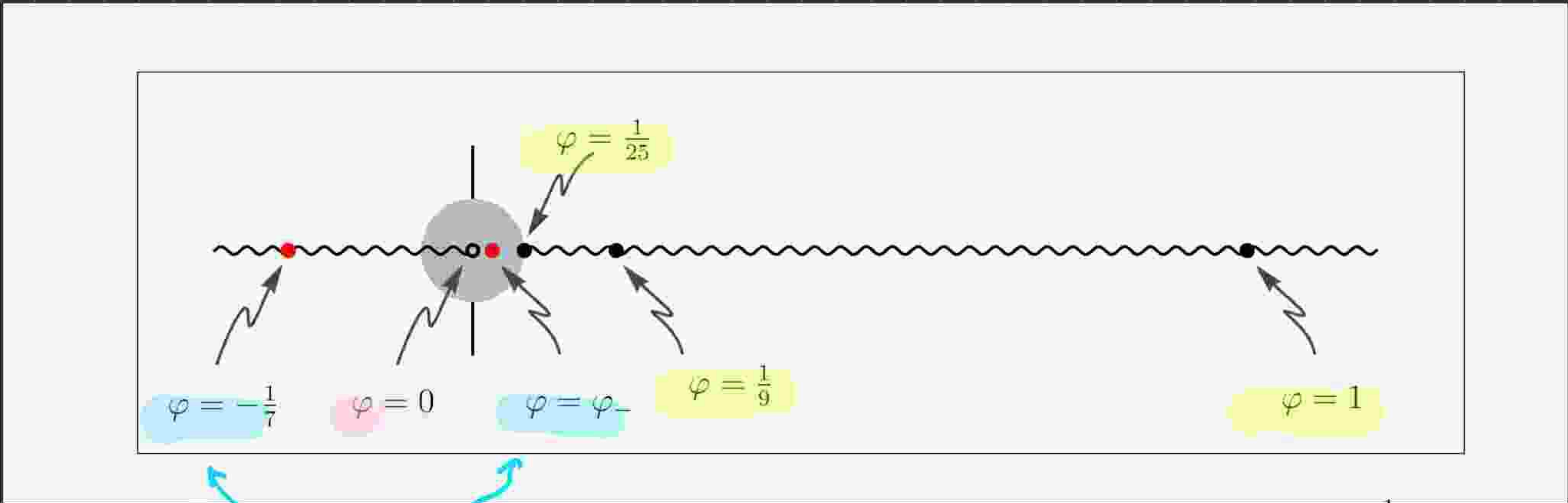
HV : X_φ is $\boxed{\text{smooth}}$ for $\varphi \neq 1, 1/9, 1/25, 0, \infty$

5 singular cases (not hypergeometric !)

$\varphi = 1, 1/9, 1/25$ conifold type

$\varphi = 0$ LCSL (maximal unipotent monodromy)

$\varphi = \infty$



Part 2 : attractor points

2 The attractor mechanism

(Ferrara, Kallosh, Strominger 95, ... Greg Moore 98 ...)

Physics : supersymmetric black hole solutions of type IIB supergravity

→ 10 dimensional generalisations of Einstein's equations for gravity + Maxwell equations for electromagnetism

We consider

10 dim space-time $\left\{ \begin{array}{l} \text{4 dim spherically symmetric, asymptotically flat, charged BH parametrised by a radial coordinate } r \\ \times \\ \text{6 dim CY } X_{\varphi(r)} \text{ at each point of the BH} \end{array} \right.$

$$\text{10 dim metric} = \left(\begin{array}{c|c} \text{BH metric } (r) & 0 \\ \hline 0 & \text{CY metric which depends on } \varphi, r \end{array} \right)$$

metric of a 4 dim spherically symmetric BH

4 dim metric: $ds^2 = -e^{2u(r)}dt^2 + e^{-2u(r)}d\underline{x}^2$, $\underline{x} = (x, y, z)$

r = radial coordinate ($r = |x|$)

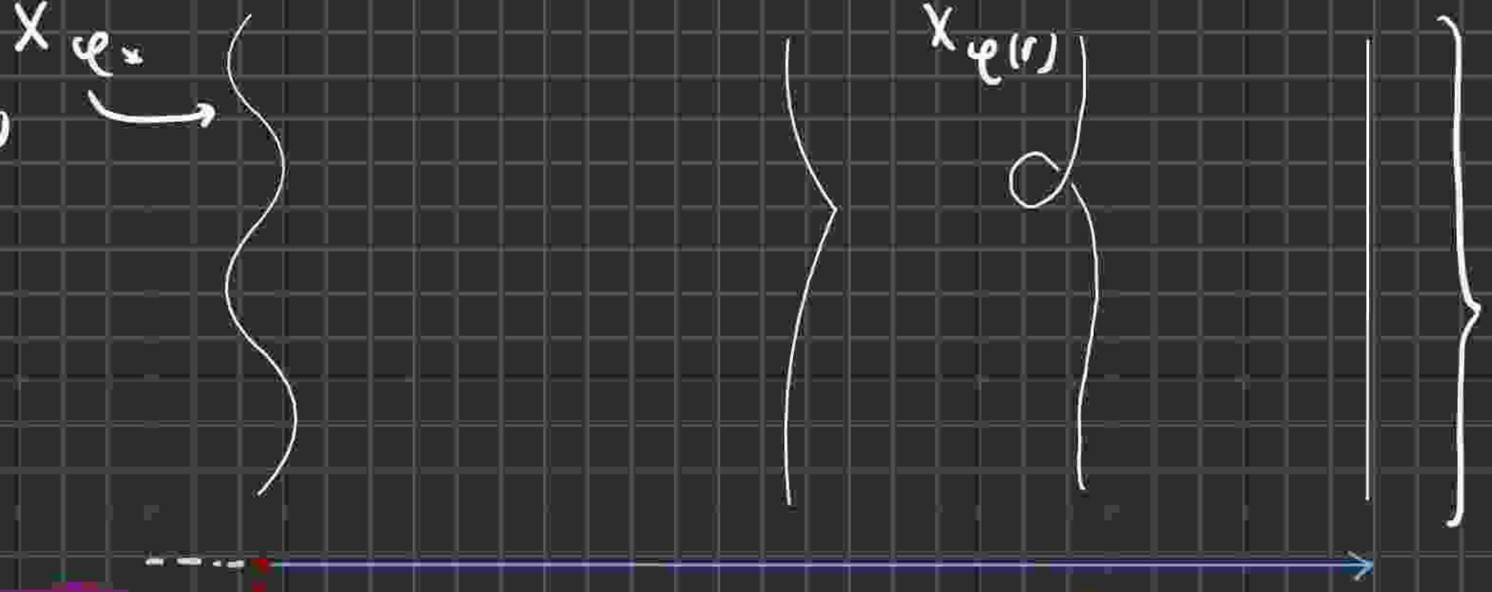
- asymptotically flat (Minkowski)
  $r \to \infty$ $u(r) \to 0$

coordinate singularity

- horizon at $r = 0$ $u(r) \to -\infty$
  - surface separating interior & exterior of BH
  - last surface visible from infinity

10 dim space: a CY, $X_{\varphi(r)}$, at each point of space-time (BH)

$\varphi_* = \varphi(r=0)$



CY with complex structure $\varphi(r)$ varies with r

horizon $(r=0)$

$U(r) \to -\infty$

(coordinate singularity)

flat 4-dim space-time $r \to \infty$

$(U(r) \to 0)$

4 dim spherically symmetric BH

$$ds^2 = e^{2U(r)} dt^2 + e^{-2U(r)} d\underline{x}^2$$

- surface separating interior & exterior of the BH
- last surface visible from infinity

Type IIB SUGRA is gravity together with $u(1)$ gauge fields (b_3 of them).

So, the BH has electric & magnetic charges

$$\begin{pmatrix} q_a \\ p^b \end{pmatrix} \qquad a, b = 0, \ldots, h^{2,1}(X)$$

these are integers

Define $\Gamma := p^a \alpha_a - q_b \beta^b \in H^3(X, \mathbb{Z})$ charge vector

Poincaré dual $\gamma := q_a A^a - p^b B_b \in H_3(X, \mathbb{Z})$

Black hole solutions of type II SUGRA which preserve supersymmetry need to satisfy 1st order differential equations for $u(r)$ & $\varphi(r)$.

These are the attractor equations.

e^{-u} monotonically increasing as $r \to 0$

$$\boxed{\rho = \frac{1}{r}}$$

$$\boxed{\begin{aligned} \frac{du(\rho)}{d\rho} &= -e^{u} \, |Z_\gamma(\rho)| \\ \frac{d\varphi(\rho)}{d\rho} &= -2 e^{u} g^{\varphi\bar{\varphi}} \partial_{\bar{\varphi}} |Z_\gamma(\rho)| \end{aligned}}$$

non-linear dynamical system on the $\mathbb{C}$-structure moduli space with flow parameter $\rho = 1/r$

gradient flow eq for $|Z_\gamma|$

$$g_{\varphi\bar{\varphi}} = \partial_\varphi \partial_{\bar{\varphi}} K$$

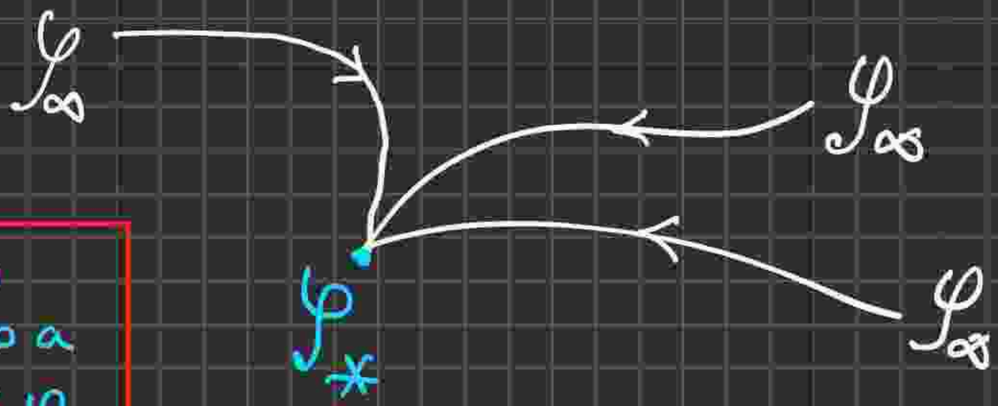
Kähler metric on $\mathbb{C}S$ moduli space

$$Z_\gamma(\rho) = e^{K/2} \int_\gamma \Omega$$

These equations determine the variation of the $\mathbb{C}$ structure of X_φ wrt r

Given a choice of $\Gamma \in H^3(X,\mathbb{Z})$, using the attractor equations, one can prove:

- the $\mathbb{C}$-structure parameters flow to a value $\varphi_* = \varphi(r=0)$ where $|Z_\Gamma|$ reaches a minimum, and it is independent of the starting value $\varphi_\infty = \varphi(r=\infty)$



attractor point in moduli space

φ evolves smoothly to a fixed point φ_* at $r=0$

95: Ferrara + Kallosh + Strominger

.....

98: G. Moore

conjectures on the arithmetic nature of attractor varieties X_{φ_*}

- The $\mathbb{C}$-structure at an attractor point $\varphi = \varphi_*$ is s.t.

$$\boxed{\begin{gathered}\Gamma = p^a \alpha_a - q_a \beta^a \in H^{(3,0)} \oplus H^{(0,3)} \\ \text{ie } \Gamma^{(2,1)} = \Gamma^{(1,2)} = 0\end{gathered}}$$

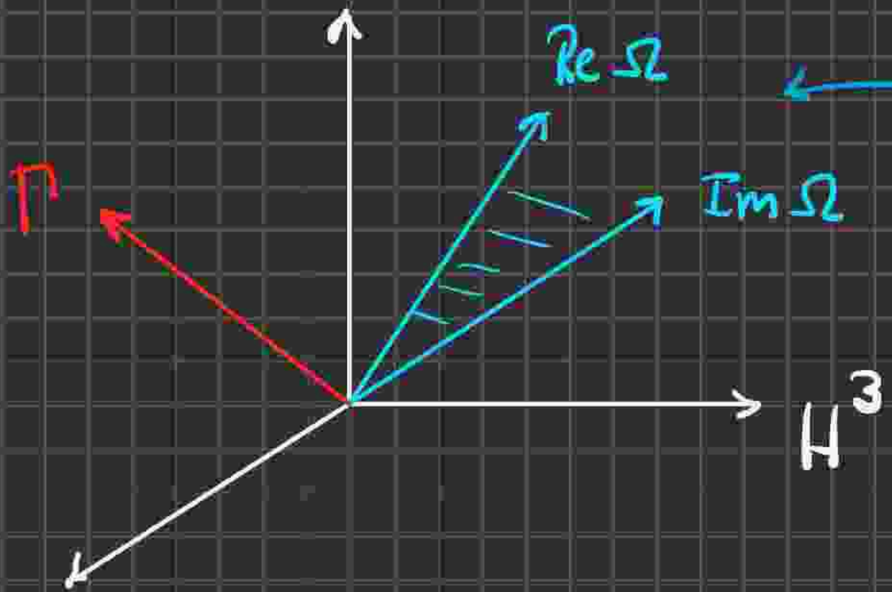
The proof of these facts is an exercise in special geometry.

one can solve the attractor eqs for charges Γ st φ_* is an attractor point but the result generically is that Γ is not integral

- φ_* and $X_* = X(\varphi_*)$ have special properties.

rank 1 attractors

Recall: Ω defines a line in H^3



Consider
$V_{\mathbb{R}}(\varphi)$ = plane spanned over $\mathbb{R}$ by Re Ω & Im Ω

$V_{\mathbb{R}}(\varphi)$ moves with φ

OTOH: Inside $H^3(X,\mathbb{R})$ we have a lattice of vectors

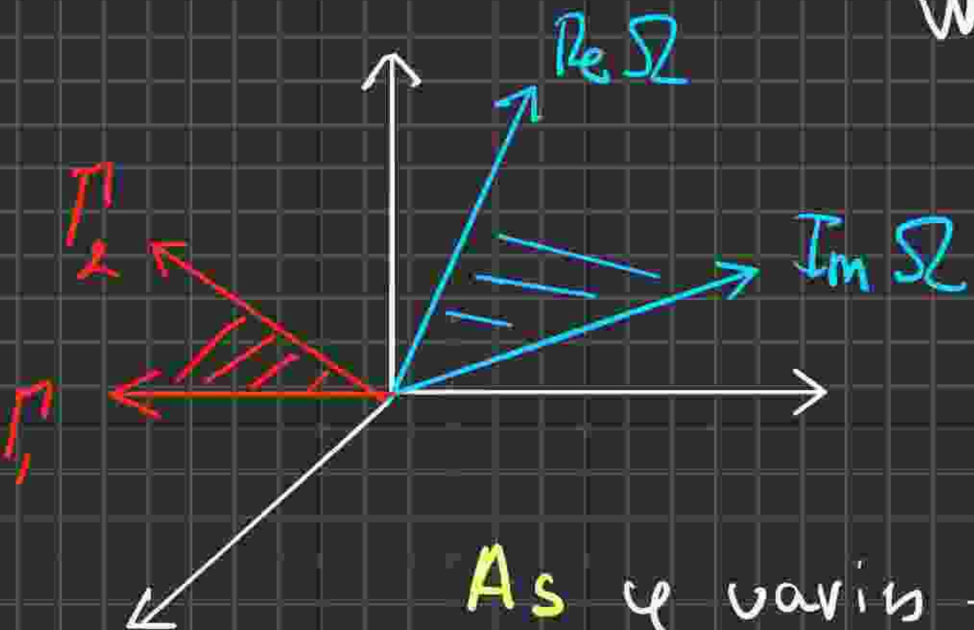
$\Gamma \in H^3(X,\mathbb{Z})$ & $\Gamma \in H^{(3,0)} \oplus H^{(0,3)}$

In general $\Gamma \not\subset V_{\mathbb{R}}(\varphi)$

A rank 1 A.P is a value φ st $V_{\mathbb{R}}(\varphi)$ contains the line Γ

rank 2 attractors

At an attractor point of rank 2 we take two vectors Γ_1, Γ_2 in $H^3(X,\mathbb{Z})$ and also $\Gamma_{1,2} \in H^{(3,0)} \oplus H^{(0,3)}$



As φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a rank 2 attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ coincides with the plane generated by Γ_1 & Γ_2

RARE, very difficult to find a CY which has rank 2 attractor points (attractor varieties)

A line which passes through the origin in general will __not__ pass through another lattice point unless the slope is rational.

Not too hard to find φ st $V_{\mathbb{R}}(\varphi)$ coincides with Γ

For rank 2 attractors we have a plane and it is then __much harder__ to find φ st it and $V_{\mathbb{R}}(\varphi)$ coincide.

(Part 2: some progress ... P. Candelas + XD + M. Elmi + D van Straten
K Bönish + A Klemm + Scheidegger + Zagier
P. Candelas + XD + J McGovern + P Kuusela in progress ...)

Geometrically

$V = H^{(3,0)} \oplus H^{(0,3)}$ is a lattice plane in $H^3(X_*, \mathbb{Z})$

Then $V^\perp = H^{(2,1)} \oplus H^{(1,2)}$

is orthogonal to V (under the natural symplectic product on 3-forms) and it is also a lattice plane in $H^3(X_*, \mathbb{Z})$

This amounts to a

$\therefore$ $\boxed{\text{splitting of the Hodge structure of } H^3(X_*, \mathbb{Z})}$

Hodge conjecture $\Rightarrow$ splitting of the Hodge structure has a geometrical origin

In Part 2:

HV for $\varphi = -1/7$ and $\varphi_{\pm} = 33 \pm 8\sqrt{17}$ are attractor varieties

We have not yet been able to find the geometrical explanation!

So

Q1: how do we find attractor varieties?

this is very hard

Q2: why do we care?

Q1: how do we find attractor varieties?

this is very hard

Part 2: the splitting becomes apparent in the arithmetic structure of X_φ

arithmetic strategy to find attractor varieties!

Part 2: Philip will explain that

$\boxed{\text{at } \varphi = \varphi_*}$ where X_* is an attractor variety

$\mathrm{Frob}^{-1} \rightsquigarrow$ 

$$R(T) = \det(\mathbb{1} - T\,\mathrm{Frob}_p^{-1})$$

$$= \underbrace{(1 - p\alpha T + p^3 T^2)}_{H^{1,2} \oplus H^{2,1}} \underbrace{(1 - \beta T + p^3 T^2)}_{H^{3,0} \oplus H^{0,3}}$$

$(1 - \alpha (pT) + p(pT)^2)$

$\boxed{\text{factors over } \mathbb{Z} \ \forall p}$

attached to modular forms of specific weight and conductor (Tate & Serre's conjectures) $\rightsquigarrow$ X_* is modular

Q2: why do we care?

mathematics:

attractor mechanism → splitting of the Hodge structure

moreover: X_{φ_*} are modular **(Part 2)**

BH physics: attractor mechanism, data of the BHs given in terms of arithmetic properties of the attractor variety **(Part 2)**

Part 2 But this strategy gives even more

↳ data of the BH in terms of arithmetic data

eg for $\varphi = -1/7$ we find a two parameter family of BHs

with charges $Q_{k\ell} = k(4K, -15K, -5, 0) + \ell(0, 0, 2, 1)$ $(K = 1, 2)$

These BHs have

$$S_{k\ell} = \frac{1}{4} A(\varphi_*)_{k\ell} = \frac{14\pi}{4} \left\{ k^2 V_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{V_*} \right\}$$

where $V_* = \frac{7}{\pi} \frac{L_4(2)}{L_4(1)}$ $L_4 \rightarrow$ L-function associated to f_4

THANKS!