Asymptotic behaviour of large-scale solutions of Hitchin's equations in higher rank

(joint works with Qiongling Li and Szilard Szabo)

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- The asymptotic behaviour of large-scale solutions of the Hitchin equation.
- The existence of harmonic metrics for generic Higgs bundles equipped with a non-degenerate symmetric pairing

Both problems are related with some easy estimates for harmonic metrics satisfying the compatibility condition with non-degenerate symmetric pairing.

Introduction

We consider a stable Higgs bundle (E,θ) of degree 0 and rank n on a compact Riemann surface X. The spectral curve is denoted by Σ_θ We consider the case Σ_θ is smooth. (The general fiber of the projection $\Sigma_\theta \to X$ consists of n-points.) We fix a flat metric $h_{\det(E)}$ of $\det(E)$.

Theorem (Hitchin, Simpson) There exists a unique harmonic metric h of (E,θ) such that $\det(h) = h_{\det(E)}$, i.e., h is a Hermitian metric of E satisfying the Hitchin equation:

$$R(h) + [\theta, \theta_h^\dagger] = 0.$$

($R(h)$: the curvature of the Chern connection of (E,h), $\theta_h^\dagger$: the adjoint of θ.)

For each $t > 0$, because $(E,t\theta)$ is stable, there exists a harmonic metric h_t of $(E,t\theta)$ (large-scale solution, large solution) such that $\det(h_t) = h_{\det(E)}$:

$$R(h_t) + t^2[\theta, \theta_{h_t}^\dagger] = 0.$$

Issue Study the asymptotic behaviour of h_t as $t \to \infty$.

Motivation

Let $\mathscr{M}_H$ denote the moduli space of Higgs bundles.

Theorem (Hitchin) $\mathscr{M}_H$ is naturally equipped with the hyperkähler metric g_H.

The tangent space of $\mathscr{M}_H$ at (E,θ) is identified with

$$\mathscr{H}^1(E,\theta,h) = \left\{\text{harmonic 1-forms of } (\mathrm{End}(E),\theta,h)\right\}.$$

The metric $(g_H)_{(E,\theta)}$ is identified with the natural metric on $\mathscr{H}^1(E,\theta,h)$.

We are interested in the behaviour of g_H around ∞ of $\mathscr{M}_H$.

Let $\mathscr{M}'_H$ denote the moduli space of Higgs bundles whose spectral curves are smooth. There exists the Hitchin fibration

$$\Phi_H : \mathscr{M}'_H \to A'_H \subset \bigoplus_{j=2}^{n} H^0(X, K_X^j).$$

There exists a hyperkähler metric g_{sf} of $\mathscr{M}'_H$ called the semi-flat metric determined by the structure of the integrable systems.

According to a conjectural description of g_H by Gaiotto-Moore-Neitzke

$$g_H - g_{sf} = e^{-\beta t} \quad (\beta > 0) \quad \textit{along the ray } (E, t\theta)\ (t \to \infty)$$

Known results in the rank 2 case

Suppose $\operatorname{rank} E = 2$. For simplicity, we assume $\operatorname{tr}\theta = 0$. We set $D := \{\det\theta = 0\} \subset X$ (the set of the critical value of $\Sigma_\theta \to X$).

Theorem (Mazzeo-Swoboda-Weiss-Witt)

$\exists$ a decoupled harmonic metric h_∞ ($R(h_\infty) = [\theta, \theta^\dagger_{h_\infty}] = 0$) of $(E,\theta)_{|X\setminus D}$, such that $\det(h_\infty) = h_{\det(E)}$ and

$$h_t - h_\infty = O(e^{-\beta(K)t}) \quad \textit{on } \forall \textit{ compact } K \subset X \setminus D \textit{ for } \exists \beta(K) > 0$$

More precisely, they constructed a family of metrics $\widetilde{h}_t$ of E such that $\det(\widetilde{h}_t) = h_{\det(E)}$ and

$$\widetilde{h}_t - h_\infty = O(e^{-\beta(K)t}) \quad (\text{on } \forall \text{ compact } K \subset X \setminus D \text{ for } \exists \beta(K) > 0)$$

$$R(\widetilde{h}_t) + t^2[\theta, \theta^\dagger_{\widetilde{h}_t}] = O(e^{-\varepsilon t}) \quad (\text{on } X \text{ for } \exists \varepsilon > 0)$$

Theorem (Mazzeo-Swoboda-Weiss-Witt)

$$\widetilde{h}_t - h_t = O(e^{-\delta t}) \quad \text{on } X \text{ for } \exists \delta > 0$$

- On the basis of this result and some ideas of Dumas-Neitzke, MSWW and Fredrickson proved $g_{sf} - g_H = O(e^{-\beta t})$ along the ray $(E, t\theta)$.
- It was generalized to some parabolic case by Fredrickson-Mazzeo-Swoboda-Weiss.

Main result

We consider the case where $\mathrm{rank}(E)$ is arbitrary. Let $D \subset X$ denote the set of the critical values of $\Sigma_\theta \to X$.

Theorem (Fredrickson, M-Szabo)

$\exists$ a decoupled harmonic metric h_∞ of $(E,\theta)_{|X\setminus D}$, and

$$h_t - h_\infty = O(e^{-\beta(K)t}) \quad \textit{on } \forall \textit{ compact } K \subset X\setminus D \textit{ for } \exists\beta(K) > 0$$

A rough strategy is the same. We construct a family of metrics $\widetilde{h}_t$ of E such that $h_{\det(E)} = h_{\det(E)}$ and

$$\widetilde{h}_t - h_\infty = O(e^{-\beta(K)t}) \quad (\text{on } \forall \text{ compact } K \subset X\setminus D \text{ for } \exists\beta(K) > 0)$$

$$R(\widetilde{h}_t) + t^2[\theta, \theta^\dagger_{\widetilde{h}_t}] = O(e^{-\varepsilon t}) \quad (\text{on } X \text{ for } \exists\varepsilon > 0)$$

Theorem (Fredrickson, M-Szabo)

$$\widetilde{h}_t - h_t = O(e^{-\delta t}) \quad \text{on } X \text{ for } \exists\delta > 0$$

But, because the ramification of $\Sigma_\theta \to X$ is more complicated, we need new ideas for the proof.

Theorem (M-Szabo) It is generalized to the case of family of Higgs bundles (E_s, θ_s) $(s \in S)$ such that Σ_{θ_s} are smooth.

The fiducial solutions in the rank 2 case

Suppose $\operatorname{rank} E = 2$, $\det(E) = \mathscr{O}_X$ and $\operatorname{tr}\theta = 0$.

Let $P \in D = \{\det\theta = 0\}$. Because Σ_θ is smooth, P is a simple zero of $\det\theta$.

$\exists (X_P, z_P)$ a holomorphic coordinate neighbourhood around P such that

$$z_P(P) = 0, \quad \det(\theta) = z_P (dz_P)^2.$$

We identify $T^*X_P = X_P \times \mathbb{C}_\xi$, $(z_P, \xi\, dz_P) \longleftrightarrow (z_P, \xi)$

$$\Sigma_\theta \cap T^*X_P = \left\{ (z_P, \xi) \,\middle|\, \xi^2 - z_P = 0 \right\}.$$

$\exists$ a frame v_1, v_2 of $E_{|X_P}$ such that

$$\theta(v_1) = v_2\, dz_P, \quad \theta(v_2) = v_1\, z_P\, dz_P, \quad v_1 \wedge v_2 = 1.$$

There exists a family of harmonic metrics $h_{P,t}$ (fiducial solutions) of $(E, t\theta)_{|X_P}$ such that

$$h_{P,t}(v_1, v_2) = 0, \quad h_{P,t}(v_i, v_i)(z_P) = h_{P,t}(v_i, v_i)(|z_P|), \quad |v_1 \wedge v_2|_{h_{P,t}} = 1.$$

- It is explicitly described by using a solution of a special case of Painlevé III.
- It is also obtained by a rescaling of a wild harmonic bundle on $\mathbb{C}$.

We consider the Higgs bundle

$$E_0 = \mathscr{O}_{\mathbb{C}} e_1 \oplus \mathscr{O}_{\mathbb{C}} e_2, \quad \theta_0(e_1, e_2) = (e_1, e_2) \begin{pmatrix} 0 & w \\ 1 & 0 \end{pmatrix} dw.$$

This Higgs bundle has a unique harmonic metric h_0 such that $|e_1 \wedge e_2|_{\det(h)} = 1$.

For $t > 0$, the map $\varphi_t : X_P \to \mathbb{C}$ is defined by $\varphi_t(z_P) = t^{2/3} z_P$.

$$\varphi_t^*(E_0, \theta_0) \simeq (E, t\theta)_{|X_P}, \quad t^{1/6} \varphi_t^*(e_1) \leftrightarrow v_1, \quad t^{-1/6} \varphi_t^*(e_2) \leftrightarrow v_2.$$

We obtain $h_{P,t} = \varphi_t^*(h_0)$.

In the rank 2 case, if the spectral curve is smooth,

- the local structure of Higgs bundle is uniquely determined, and highly symmetric,
- there exists a good family of harmonic metric (a solution of an ODE, a rescaling of wild harmonic bundle on $(\mathbb{P}^1, \infty)$)

A problem in the higher rank case

Suppose $\mathrm{rank}(E) = 3$, $\det(E) = \mathscr{O}_X$ and $\mathrm{tr}\,\theta = 0$.
Let P be a critical value of $\Sigma_\theta \to X$. Assume that $T_P^* X \cap \Sigma_\theta = \{0\}$.
Let (X_P, z_P) be a coordinate neighbourhood around P such that $z_P(P) = 0$.

$$T^* X_P \cap \Sigma_\theta = \left\{ (z_P, \xi) \,\middle|\, \xi^3 + \alpha_1(z_P)\xi + \alpha_0(z_P) = 0 \right\}.$$

- $\alpha_0(0) = 0$ and $\partial_{z_p}\alpha_0(0) \neq 0$. By changing z_P, we may normalize $\alpha_0(z_P) = -z_P$.
- $\alpha_1(0) = 0$. But, in general, there is no additional condition.

If α_1 is constantly 0, there is a direct generalization of the fiducial solution.

Indeed, $(E, \theta)_{|X_P}$ has a frame v_1, v_2, v_3 such that

$$\theta(v_1) = v_2\, dz_P, \quad \theta(v_2) = v_3\, dz_P, \quad \theta(v_3) = z_P\, v_3\, dz_P, \quad v_1 \wedge v_2 \wedge v_3 = 1.$$

By solving an ODE, or by rescaling a wild harmonic bundle, we can obtain a family of harmonic metrics $h_{P,t}$ of $(E, t\theta)_{|X_P}$ such that

$$h_{P,t}(v_i, v_j) = 0\ (i \neq j), \quad h_{P,t}(v_i, v_i)(z_P) = h_{P,t}(v_i, v_i)(|z_P|), \quad |v_1 \wedge v_2 \wedge v_3|_{h_{P,t}} = 1.$$

But, to study the case where α_1 is not constantly 0, we need new ideas.

Basic tools

Non-degenerate symmetric pairings

Let (E,θ) be a Higgs bundle on a Riemann surface X.

- A non-degenerate symmetric pairing C of (E,θ) is a non-degenerate symmetric pairing of E such that θ is self-dual with respect to C.

  $$C(\theta u, v) = C(u, \theta v) \quad \text{for any local sections } u, v \text{ of } E$$

  It induces an isomorphism $\Psi_C : E \simeq E^\vee$.

- A Hermitian metric h is compatible with C if Ψ_C is an isometry with respect to h and $h^\vee$.

Dirichlet problem for harmonic metrics

Let $Y \subset X$ be a relatively compact open subset such that ∂Y is smooth.

Let $h_{\partial Y}$ be a Hermitian metric of $E_{|\partial Y}$.

Theorem (Donaldson) There exists a unique harmonic metric h of $(E,\theta)_{|Y}$ such that $h_{|\partial Y} = h_{\partial Y}$.

The following is a minor complement.

Proposition (Li-M) If (E,θ) is equipped with a non-degenerate symmetric pairing C, and if $h_{\partial Y}$ is compatible with $C_{|\partial Y}$, then h is compatible with C.

Canonical decoupled harmonic metric in the regular semisimple case

Let (E,θ) be a Higgs bundle of rank n on X.

Definition (E,θ) is called regular semisimple if each fiber of $\Sigma_\theta \to X$ has n-points.

Suppose (E,θ) is equipped with a non-degenerate symmetric pairing C.

Proposition (Li-M) If (E,θ) is regular semisimple, there exists a unique decoupled harmonic metric h^C of (E,θ) compatible with C.

- Let $\pi : \Sigma_\theta \to X$ denote the projection. There exists a line bundle L on Σ_θ such that $\pi_*(L) = (E,\theta)$.
- There exists a non-degenerate symmetric pairing C_L of L such that $C = \pi_*(C_L)$.
- We obtain a flat metric h_L of L as $h_L(v,v) = |C_L(v,v)|$.
- $\pi_*(h_L)$ is the desired decoupled harmonic metric.

Remark The uniqueness is claimed for decoupled harmonic metrics compatible with C. In general, there are many other harmonic metrics compatible with C.

Some estimates

Let $\mathrm{Harm}(E,\theta,C)$ be the set of harmonic metrics of (E,θ) compatible with C. For any $h \in \mathrm{Harm}(E,\theta,C)$, we obtain the automorphism $s(h^C,h)$ of E determined by

$$h(u,v) = h^C\big(s(h^C,h)u,v\big) \quad u,v \in C^\infty(X,E).$$

Let $X' \subset X$ be a relatively compact open subset.

Estimate 1 (Li-M)

There exists $B_1 > 0$, depending only on X, X' and Σ_θ, such that

$$\sup_{X'} \big|s(h^C,h)\big|_{h^C} \leq B_1 \quad (\forall h \in \mathrm{Harm}(E,\theta,C)).$$

Estimate 2 (M-Szabo)

There exists $B_2, B_3 > 0$ depending only on X, X' and Σ_θ, such that

$$\sup_{X'} \big|s(h^C,h) - \mathrm{id}_E\big|_{h^C} \leq B_2 \exp(-B_3 t) \quad (\forall t \geq 1,\ \forall h \in \mathrm{Harm}(E,t\theta,C)).$$

Remark Higher derivatives are also dominated similarly.

These estimates follow from a variant of Simpson's main estimates and elementary calculations for matrices.

Asymptotic behaviour of large-scale solutions

Let X be a compact Riemann surface. Let (E,θ) be a stable Higgs bundle of degree 0 on X. Suppose the spectral curve is smooth. We fix a flat metric $h_{\det(E)}$ of $\det(E)$. Let h_t be the unique harmonic metric of $(E,t\theta)$ such that $\det(h_t)=h_{\det(E)}$.

We would like to study the behaviour of h_t as $t\to\infty$.

Symmetric case (easy)

If (E,θ) is equipped with a non-degenerate symmetric pairing, it is easy to study. Let D denote the set of the critical values of $\Sigma_\theta \to X$. There exists the decoupled harmonic metric h^C of $(E,\theta)_{|X\setminus D}$ compatible with C. The following theorem is a direct consequence of Estimate 2.

Theorem (M-Szabo) For any compact subset $K\subset X\setminus D$, there exists $\beta(K)>0$ such that

$$\sup_K \big|s(h^C,h_t)-\mathrm{id}_E\big|_{h^C}=O\big(\exp(-\beta(K)t)\big)\quad (t\to\infty).$$

Estimate 2 is also useful even in the case (E,θ) is not equipped with C.

Limiting configuration (Mazzeo-Swoboda-Weiss-Witt, Fredrickson)

There exists a line bundle L on Σ_θ with an isomorphism $\pi_*(L) \simeq (E,\theta)$.
Let $Q \in \Sigma_\theta$ be the critical point of $\pi : \Sigma_\theta \to X$. Let $r(Q)$ denote the ramification index at Q, i.e., for an appropriate local coordinate ζ_Q around Q, $\pi(\zeta_Q) = \zeta^{r(Q)}$.
Because

$$\deg(L) - \sum_Q \frac{1}{2}(r(Q)-1) = 0,$$

there exists a flat metric h_L of $L_{|\Sigma_\theta \setminus \pi^{-1}(D)}$ such that

$$h_L(v_Q, v_Q) = |\zeta_Q|^{-r(Q)+1} \quad \exists \text{ frame } v_Q \text{ of } L \text{ around } Q$$

We obtain the decoupled harmonic metric $h_\infty = \pi_*(h_L)$ of $(E,\theta)_{|X\setminus D}$ (limiting configuration). We can normalize $\det(h_\infty) = h_{\det(E)}$ (C^∞ on X). We would like to show

$$\lim_{t\to\infty} h_t = h_\infty.$$

Locally defined symmetric pairing

Lemma Let $P \in D$. On a neighbourhood X_P of P, there exists a non-degenerate symmetric pairing C_P of $(E,\theta)_{|X_P}$ such that $C_{P|X_P\setminus\{P\}}$ is compatible with $h_{\infty|X_P\setminus\{P\}}$.

- $T^*X_P \cap \Sigma_\theta = \bigsqcup_{Q\in T_P^*X\cap\Sigma\theta}(\Sigma_\theta)_Q$ (the connected components)
- Let ζ_Q be a holomorphic coordinate of $(\Sigma_\theta)_Q$ such that $\zeta_Q(Q)=0$.
- Let L_Q denote the restriction of L to $(\Sigma_\theta)_Q$. There exists a frame v_Q of L_Q such that
$$h_L(v_Q,v_Q) = |\zeta_Q|^{-r(Q)+1}.$$
- We define
$$C_Q : L_Q \otimes L_Q \longrightarrow \mathscr{O}_{(\Sigma_\theta)_Q}\big((r(Q)-1)Q\big), \quad C_Q(v_Q,v_Q) = \zeta^{-r(Q)+1}.$$
- $C_P = \bigoplus_Q \pi_*(C_Q)$ induces a non-degenerate symmetric pairing of $(E,\theta)_{|X_P\setminus\{P\}}$ compatible with h_∞.
- Moreover, C_P induces a non-degenerate symmetric pairing of $(E,\theta)_{|X_P}$.

This pairing C_P is useful in the study of the asymptotic behaviour of large solutions.

Local models and approximate solutions

- For each $t > 0$, we obtain a harmonic metric $h_{P,t}$ of $(E, t\theta)_{|X_P}$ such that $h_{P,t|\partial X_P} = h_{\infty|\partial X_P}$ as a solution of the Dirichlet problem.
- Because h_∞ is compatible with C_P, $h_{P,t}$ is compatible with C_P.
- By Estimate 2, on any compact subset $K \subset X_P \setminus \{P\}$,

$$\sup_K \big| s(h_\infty, h_{P,t}) - \mathrm{id} \big|_{h_\infty} \leq B_2(K) \exp\big(-B_3(K)t\big) \quad (t \to \infty).$$

- By patching h_∞ and $h_{P,t}$ $(P \in D)$, we construct Hermitian metrics $\widetilde{h}_t$ of E such that $\det(\widetilde{h}_t) = h_{\det(E)}$ and

$$\sup_X \big| R(\widetilde{h}_t) + t^2 [\theta, \theta^\dagger_{\widetilde{h}_t}] \big|_{\widetilde{h}_t, g_X} = O(e^{-\varepsilon t}) \quad (\exists \varepsilon > 0).$$

$$\sup_K \big| s(h_\infty, \widetilde{h}_t) - \mathrm{id} \big|_{h_\infty} = O(e^{-\beta(K)t}) \quad (\forall K \subset X \setminus D \text{ compact } \exists \beta(K) > 0)$$

Theorem (M-Szabo) $\sup_X \big| s(\widetilde{h}_t, h) - \mathrm{id} \big|_{\widetilde{h}_t} = O(e^{-\delta t}) \quad (\delta > 0).$
The higher derivatives are also dominated similarly.

Outline of the proof

Step 1 We set $s_t = s(\widetilde{h}_t, h_t)$. We have $\det(s_t) = 1$ and

$$\Delta_{g_X} \mathrm{Tr}(s_t) = \sqrt{-1}\Lambda \mathrm{Tr}\Big(\big(R(\widetilde{h}_t) + t^2[\theta, \theta^{\dagger}_{\widetilde{h}_t}]\big) \cdot s_t\Big) - \big|\overline{\partial}_E(s_t)s_t^{-1/2}\big|^2_{\widetilde{h}_t} - \big|[t\theta, s_t]s_t^{-1/2}\big|^2_{\widetilde{h}_t}.$$

We obtain

$$\big\|\overline{\partial}_E(s_t)s_t^{-1/2}\big\|^2_{L^2,\widetilde{h}_t,g_X} + \big\|[t\theta, s_t]s_t^{-1/2}\big\|^2_{L^2,\widetilde{h}_t,g_X} = O(e^{-\varepsilon t}) \quad (\exists \varepsilon > 0)$$

We set $b_t = \sup_X \mathrm{Tr}(s_t)$. Note that $b_t \geq \mathrm{rank}(E)$, and $b_t = \mathrm{rank}(E)$ if and only if $s_t = \mathrm{id}$. We set $u_t = b_t^{-1} s_t$.

- $\exists B > 0$ such that $|u_t|_{\widetilde{h}_t} \leq B$, and $\int_X |\overline{\partial} u_t|^2_{\widetilde{h}_t} + \big|[\theta, u_t]\big|^2_{\widetilde{h}_t} \to 0$ as $t \to \infty$.
- $\exists$ subsequence $u_{t(i)}$ convergent to u_∞ in L_1^2 locally on $X \setminus D$.

> *Lemma* u_∞ is non-zero, bounded with respect to h_∞, and satisfies $\overline{\partial} u_\infty = [\theta, u_\infty] = 0$. Moreover, u_∞ is self-adjoint with respect to h_∞.

- By looking at $(E,\theta)_{|X\setminus D}$ with h_∞, it is easy to see that such u_∞ is the multiplication of a positive constant.
- Hence, b_t and b_t^{-1} are bounded. Together with $\det(s_t) = 1$, we obtain $s_t \to \mathrm{id}_E$.

Step 2 By the convergence $s_t \to \mathrm{id}_E$, we obtain

$$\left\|\overline{\partial}(s_t)\right\|^2_{L^2,\widetilde{h}_t,g_X} + \left\|t[\theta, s_t]\right\|^2_{L^2,\widetilde{h}_t,g_X} = O(e^{-\varepsilon t}) \quad (\exists \varepsilon > 0).$$

Because s_t are self-adjoint, we also obtain

$$\left\|\partial_{\widetilde{h}_t}(s_t)\right\|^2_{L^2,\widetilde{h}_t,g_X} + \left\|t[\theta^\dagger_{\widetilde{h}_t}, s_t]\right\|^2_{L^2,\widetilde{h}_t,g_X} = O(e^{-\varepsilon t}).$$

By a variant of Simpson's main estimate,

$$\sup_K \left|\overline{\partial}(s_t^{-1}\partial_{\widetilde{h}_t} s_t)\right|_{\widetilde{h}_t,g_X} = O\left(e^{-\beta(K)t}\right) \quad (\forall K \subset X \setminus D \text{ compact}, \exists \beta(K) > 0)$$

We obtain

$$\sup_K \left|\partial_{\widetilde{h}_t}(s_t)\right|_{\widetilde{h}_t,g_X} = O\left(e^{-\beta'(K)t}\right), \quad \sup_K \left|\overline{\partial}(s_t)\right|_{\widetilde{h}_t,g_X} = O\left(e^{-\beta'(K)t}\right).$$

It means

$$\sup_K \left|\nabla_{\widetilde{h}_t}(s_t)\right|_{\widetilde{h}_t,g_X} = O\left(e^{-\beta'(K)t}\right).$$

Step 3 Let $P \in X \setminus D$. On a neighbourhood X_P, there exists a decomposition

$$(E,\theta)_{|X_P} = \bigoplus_{i=1}^{n} (E_{P,i}, \theta_{P,i}) \quad (\operatorname{rank} E_{P,i} = 1).$$

We have the decomposition $s_t = \sum (s_t)_{P,i,j} \quad (s_t)_{P,i,j} \in \mathrm{Hom}(E_{P,j}, E_{P,i})$.

By a variant of Simpson's main estimate, we obtain $|(s_t)_{P,i,j}|_{\widetilde{h}_t} = O(e^{-\beta(X_P)t}) \quad (i \neq j)$.

Because $\det(s_t) = 1$, we obtain $\prod_{i=1}^{n} (s_t)_{P,i,i} - 1 = O\big(e^{-\beta'(X_P)t}\big)$.

Step 4

Let γ be a loop on $X \setminus D$ such that $\gamma(0) = \gamma(1) = P$. We obtain a permutation of the components $E_{P,i}$ as the monodromy of $\Sigma_\theta \to X$ along γ. Because $|\nabla_{\widetilde{h}_t}(s_t)|_{\widetilde{h}_t, g_X} = O(e^{-\beta t})$ along γ (Step 2), we obtain

$$(s_t)_{P,i,i} - (s_t)_{P,j,j} = O(e^{-\beta''(X_P)t}) \quad (\forall i, j).$$

Because $\prod_{i=1}^{n} (s_t)_{P,i,i} - 1 = O(e^{-\beta'(X_P)t})$ by Step 3, we obtain

$$(s_t)_{P,i,i} - 1 = O(e^{-\beta'''(X_P)t}) \quad (\forall i).$$

Namely, $\sup_K \big| s_t - \mathrm{id} \big|_{\widetilde{h}_t} = O(e^{-\beta(K)t}) \quad (\forall K \subset X \setminus D \text{ compact } \exists \beta(K) > 0)$.

Step 5 Let $N(D)$ be a neighbourhood of D.

$$\sup_{X\setminus N(D)} \left|s_t - \mathrm{id}\right|_{\widetilde{h}_t} = O(e^{-\beta t}).$$

$$\Delta_{g_X} \mathrm{Tr}(s_t - \mathrm{id}) \leq O(e^{-\beta t}).$$

By using the maximum principle for subharmonic functions, we obtain

$$\mathrm{Tr}(s_t - \mathrm{id}) = O(e^{-\beta t}).$$

It implies $|s_t - \mathrm{id}|_{\widetilde{h}_t} = O(e^{-\beta t})$.

Another result: An existence result of harmonic metrics

A Higgs bundle is called generically regular semisimple if there exists a discrete subset $D \subset X$ such that $(E, \theta)_{|X \setminus D}$ is regular semisimple.

Theorem (Li-M)

Let (E, θ) be a generically regular semisimple equipped with a non-degenerate symmetric pairing C on X (not necessarily compact). Then, (E, θ) has a harmonic metric compatible with C.

Remark We can apply this theorem to Higgs bundles in the Hitchin section.

Remark If $X = \overline{X} \setminus D$ for a compact Riemann surface $\overline{X}$ and a finite point D, and if Σ_θ is meromorphic along D, we can classify harmonic metrics the associated filtered Higgs bundles. We do not have to care about polystability condition in the symmetric case.

Outline of the proof

- We take an increasing family of open subsets $X_1 \subset X_2 \subset \cdots$ such that (i) X_i are relatively compact in X_{i+1}, (ii) $\bigcup X_i = X$, (iii) ∂X_i are smooth, (iv) $\partial X_i \cap D = \emptyset$.
- Let h_0 be a Hermitian metric of E compatible with C. As a solution of the Dirichlet problem, we obtain harmonic metrics h_i of $E_{|X_i}$ such that $h_{i|\partial X_i} = h_{0|\partial X_i}$. h_i are also compatible with C.
- Let $N(D)$ denote a neighbourhood of D such that $\partial X_i \cap N(D) = \emptyset$. By Estimate 1, for each i, there exists $B_i > 0$ such that

$$\sup_{X_i \setminus N(D)} \left(|s(h^C, h_j)|_{h^C} + |s(h^C, h_j)^{-1}|_{h^C} \right) \leq B_i \quad (\forall j > i).$$

- By using the subharmonicity of $\mathrm{Tr}(s(h_{i+1}, h_j))$, for each i, there exists $B_i' > 0$ such that

$$\sup_{X_i} \left(|s(h_{i+1}, h_j)|_{h^C} + |s(h_{i+1}, h_j)^{-1}|_{h^C} \right) \leq B_i' \quad (\forall j > i).$$

- Then, we can show the existence of a convergent subsequence by a standard argument.