The odd integrable system

Nigel Hitchin

ICMAT workshop April 26th 2023

UNIVERSITY OF OXFORD
Mathematical Institute

Oxford Mathematics

HIGGS BUNDLES

- curve C, genus $g > 1$

- vector bundle E, fixed determinant

  Higgs field $\Phi \in H^0(C, \mathrm{End}_0\, E \otimes K)$

- (E, Φ) stable $\Rightarrow$ moduli space $\mathcal{M}$

- $(E, 0)$ stable bundle, moduli space $\mathcal{N}$

  $H^0(C, \mathrm{End}_0\, E \otimes K) \cong$ cotangent space at $[E] \in \mathcal{N}$

THE INTEGRABLE SYSTEM

- $\operatorname{tr}\Phi^m \in H^0(C, K^m)$

- $p : \mathcal{M} \to \mathcal{B} = \bigoplus_{m=2}^{n} H^0(C, K^m)$

  proper map

- $\mathcal{M}$ symplectic

  $\mathcal{B}^* =$ functions on $\mathcal{M}$, Poisson-commuting

- $\dim \mathcal{B} = \dim \mathcal{M}/2$: completely integrable system

NJH, *Stable bundles and integrable systems,* Duke Math.J. **54** (1987), 91–114.

- $\operatorname{tr}\Phi^m \in H^0(C, K^m)$

- $p : \mathcal{M} \to \mathcal{B} = \bigoplus_{m=2}^{n} H^0(C, K^m)$

  proper map

- $\mathcal{M}$ symplectic

  $\mathcal{B}^* =$ functions on $\mathcal{M}$, Poisson-commuting

- $\dim \mathcal{B} = \dim \mathcal{M}/2$: completely integrable system

NJH, *Stable bundles and integrable systems,* Duke Math.J. **54** (1987), 91–114.

M.F.Atiyah & R.Bott, *The Yang-Mills equations over Riemann surfaces, Phil. Trans. R. Soc. Lond. A* **308** (1983) 525–615

- fix C^∞ bundle E

- $\mathcal{A}$ = affine space of $\bar\partial$-operators on E

  $\bar\partial_A : \Omega^0(C,E) \to \Omega^{0,1}(C,E)$

  $\bar\partial_A(fs) = (\bar\partial f)s + f(\bar\partial_A s)$

- $\mathcal{G}^c$ = complex automorphisms

- $\mathcal{A} \times \Omega^0(C, \mathsf{End}_0\, E \otimes K) \sim T^*\mathcal{A}$

- symplectic form $\displaystyle\int_C \mathrm{tr}(\dot{A}_1 \Phi_2) - \mathrm{tr}(\dot{A}_2 \Phi_1)$

translation invariant $\sim$ closed

- $\mathcal{A} \times \Omega^0(C, \mathsf{End}_0\, E \otimes K) \sim T^*\mathcal{A}$

- symplectic form $\displaystyle\int_C \mathrm{tr}(\dot{A}_1 \Phi_2) - \mathrm{tr}(\dot{A}_2 \Phi_1)$

  translation invariant $\sim$ closed

- $\alpha \in \Omega^{0,1}(C, K^{-m+1}), \Phi \in \Omega^0(C, \mathsf{End}_0\, E \otimes K)$

  functions $\displaystyle\int_C \alpha\, \mathrm{tr}\, \Phi^m$

  independent of $A \sim$ Poisson commute

- moment map for $\mathcal{G}^c$: $\mu(A, \Phi) = \bar{\partial}_A \Phi$

  $\mu^{-1}(0)/\mathcal{G}^c = \mathcal{M}$ Higgs bundle moduli space

- function $f = \int_C \alpha \, \mathrm{tr}\, \Phi^m$

  f gauge-invariant, descends to $\mathcal{M}$

- $\bar{\partial}_A \Phi = 0 \Rightarrow f$ depends only on $[\alpha] \in H^1(C, K^{-m+1})$

SYMMETRIC TENSORS

- $T^*\mathcal{N} \subset \mathcal{M}$

  $f = \int_C \alpha \,\mathrm{tr}\, \Phi^m$ degree m polynomial on $\Phi \in T^*_{[E]}\mathcal{N}$

- $=$ section of symmetric power $S^m T$ over $\mathcal{N}$

SYMMETRIC TENSORS

- $T^*\mathcal{N} \subset \mathcal{M}$

  $f = \int_C \alpha \, \mathrm{tr}\, \Phi^m$ degree m polynomial on $\Phi \in T^*_{[E]}\mathcal{N}$

- $=$ section of symmetric power $S^m T$ over $\mathcal{N}$

- Poisson bracket $\{f, g\}$ on $T^*\mathcal{N}$ corresponds to

  symmetric Schouten-Nijenhuis bracket:

  a, b sections of $S^m T, S^n T$, $[a, b]$ section of $S^{m+n-1} T$

THE ODD INTEGRABLE SYSTEM

- invariant alternating form ρ of degree p on $\mathfrak{sl}(n)$

  Higgs fields Φ_i

  $\rho(\Phi_1, \Phi_2, \ldots, \Phi_p) \in H^0(C, K^p)$

- e.g. $\mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]) \in H^0(C, K^3) \cong \mathbf{C}^{5g-5}$

- **Prop**: These define commuting polyvector fields on $\mathcal{N}$

  NJH, *Stable bundles and polyvector fields,* in "Complex and Differential Geometry", W. Ebeling et al (eds.) Springer Proceedings in Mathematics **8**, 135–156, Springer Verlag, Heidelberg (2011).

- sections of $\Lambda^p T$ polyvector fields

- alternating Schouten-Nijenhuis bracket

- a, b sections of $\Lambda^p T, \Lambda^q T \Rightarrow [a, b]$ section of $\Lambda^{p+q-1} T$

- $p = q = 1$: Lie bracket of vector fields

- $[a, fb] = f[a, b] + i(df)a \wedge b$ (interior product)

$SL(2,\mathbf{C})$ BUNDLES

- invariant 3-form $\mathrm{tr}(A_1[A_2,A_3])$

- $\mathfrak{sl}(2,\mathbf{C}) \cong \mathbf{C}^3$

  $[A,B] \sim \mathbf{a}\times\mathbf{b}$ vector cross product

  $\mathrm{tr}(AB) \sim \mathbf{a}\cdot\mathbf{b}$

$SL(2, \mathbf{C})$ BUNDLES

- invariant 3-form $\mathrm{tr}(A_1[A_2, A_3])$

- $\mathfrak{sl}(2, \mathbf{C}) \cong \mathbf{C}^3$

  $[A, B] \sim \mathbf{a} \times \mathbf{b}$ vector cross product

  $\mathrm{tr}(AB) \sim \mathbf{a} \cdot \mathbf{b}$

- vector identity $(\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3))^2 = \det(\mathbf{a}_i \cdot \mathbf{a}_j)$

- $-2\det(\mathrm{tr}(\Phi_i \Phi_j)) = \mathrm{tr}(\Phi_1[\Phi_2, \Phi_3])^2 \in H^0(C, K^6)$

SUPERGEOMETRY

- Supermanifold:

  $\mathcal{N}$ + sheaf $\mathcal{O}(\Lambda^* T)$ of exterior algebras = "functions"

- Schouten bracket: $[a, \ ] : \mathcal{O}(\Lambda^* T) \to \mathcal{O}(\Lambda^* T)$ derivation

- derivation of functions = vector field

- "odd symplectic manifold"

EXAMPLES

C GENUS 2

- C: $y^2 = f(x) = (x - x_1)\ldots(x - x_6)$

  involution $\sigma(x, y) = (x, -y)$

- $H^0(C, K^3) : (c_0 y + c_1 + c_2 x + c_3 x^2 + c_4 x^3)\dfrac{dx^3}{y^3}$

- E rank 2, $\dim \mathcal{N} = 3g - 3 = 3$

  $\Lambda^3 T =$ (anticanonical) line bundle $\Rightarrow$ all sections commute

- rank 2 $\Lambda^2 E$ trivial

- $\mathcal{N} = \mathrm{P}^3$, $\Lambda^3 T \cong \mathcal{O}(4)$

- involution acts trivially $\Rightarrow \mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]) = \lambda \dfrac{y dx^3}{y^3}$

- $E = L \oplus L^* \Rightarrow$ Kummer surface $\mathrm{Jac}(C)/\mathbf{Z}_2 \subset \mathrm{P}^3$

  .... quartic surface $\sim$ section of $\Lambda^3 T$

REMARK...

- U line bundle U^2 trivial

- $\Lambda^2 E$ fixed $E \mapsto E \otimes U$

  action of $\Gamma = H^1(C, \mathbf{Z}_2) \cong \mathbf{Z}_2^{2g}$

- $\mathrm{End}_0\, E$ invariant

  $\Rightarrow$ trivector fields $\alpha(\mathrm{tr}(\Phi_1[\Phi_2, \Phi_3]))$ are Γ-invariant

GENUS 3

- rank 2 $\Lambda^2 E$ trivial

  $\mathcal{N} \cong$ Coble quartic hypersurface in P^7

  M.S.Narasimhan & S.Ramanan, 2Θ *systems on abelian varieties*, Vector bundles and algebraic varieties, Bombay 1984, OUP (1987) 415 – 427.

- $\dim H^0(\mathcal{N}, \Lambda^3 T) \geq 73$

- $10 = 5g - 5$ Γ-invariant sections

- $\dim \mathcal{N} = 6$, $[a, b] \in H^0(\mathcal{N}, \Lambda^5 T)$

SHEAF COHOMOLOGY

- odd integrable system $\Rightarrow$ commutative action on **sheaf** $\mathcal{O}(\Lambda^* T)$

- $\Rightarrow$ action on cohomology $H^*(\mathcal{N}, \Lambda^* T)$

- odd integrable system $\Rightarrow$ commutative action on **sheaf** $\mathcal{O}(\Lambda^* T)$

- $\Rightarrow$ action on cohomology $H^*(\mathcal{N}, \Lambda^* T)$

- $H^1(\mathcal{N}, T) \cong H^1(C, K^*)$

  M.S.Narasimhan & S.Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve,* Ann. of Math. **101** (1975), 391–417.

- bracket action $H^0(\mathcal{N}, \Lambda^p T) : H^1(\mathcal{N}, T) \to H^1(\mathcal{N}, \Lambda^p T)$

- odd integrable system $\Rightarrow$ commutative action on **sheaf** $\mathcal{O}(\Lambda^* T)$

- $\Rightarrow$ action on cohomology $H^*(\mathcal{N}, \Lambda^* T)$

- $H^1(\mathcal{N}, T) \cong H^1(C, K^*)$

  M.S.Narasimhan & S.Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve,* Ann. of Math. **101** (1975), 391–417.

- bracket action $H^0(\mathcal{N}, \Lambda^p T) : H^1(\mathcal{N}, T) \to H^1(\mathcal{N}, \Lambda^p T)$

$$H^0(\mathcal{N}, \Lambda^p T) \otimes H^1(\mathcal{N}, T) \to H^1(\mathcal{N}, \Lambda^p T)$$

- bracket action $H^1(\mathcal{N},T) : H^0(\mathcal{N},\Lambda^p T) \to H^1(\mathcal{N},\Lambda^p T)$

- $\alpha \in H^1(\mathcal{N},T)$ Kodaira-Spencer class for deformation

  $[\alpha,\rho] \in H^1(\mathcal{N},\Lambda^p T) =$ obstruction to extending ρ

- $= 0$ for ρ defined by an invariant form ...

THE SYMMETRIC VERSION

- $H^*(\mathcal{N}, S^*T) = \bigoplus_{p,q} H^q(\mathcal{N}, S^pT)$

- $H^q(\mathcal{N}, S^pT) \Rightarrow H^q(T^*\mathcal{N}, \mathcal{O})$ degree p under $\mathbf{C}^*$ action

- Hartogs: $\Rightarrow H^q(\mathcal{M}, \mathcal{O})$ Higgs bundle moduli space

  $\ldots$ for $q + 1 <$ codimension of semistable bundles

- $H^1(\mathcal{N}, T) \Rightarrow H^1(\mathcal{M}, \mathcal{O})$

- integrable system $p : \mathcal{M} \to \mathcal{B}$

  generic fibre abelian variety A

  Hamiltonian vector fields tangential to A

- Leray spectral sequence $H^1(\mathcal{M}, \mathcal{O}) \sim H^0(\mathcal{B}, H^1(A, \mathcal{O}))$

- translations on A act trivially on $H^1(A, \mathcal{O})$

  $\Rightarrow$ trivial action on $H^1(\mathcal{N}, T)$

- $H^p(A, \mathcal{O}) \cong \Lambda^p(H^1(A, \mathcal{O}))$

- trivial action on $H^p(\mathcal{M}, \mathcal{O})$

- A Lagrangian $\Rightarrow H^1(A, \mathcal{O}) \cong T^*_{p(A)}\mathcal{B}$

- stack: $H^*(\mathrm{Bun}_G, S^*T) \cong$ algebraic exterior forms on $\mathcal{B}$

  E.Frenkel & C.Teleman, *Geometric Langlands correspondence near opers,* arXiv:1306.0876

- symmetric case: trivial action on $H^*(\mathcal{N}, S^*T)$

- alternating case: trivial action on part generated multiplicatively by $H^1(\mathcal{N}, T)$ and the polyvector fields

Question: What is $H^*(\mathcal{N}, \Lambda^*T)$?

- $H^1(\mathcal{N}, T^*) \cong \mathbf{C}$, $c_1 > 0$

  Akizuki-Nakano $H^q(\mathcal{N}, \Lambda^{n-p}T^* \otimes K^*) = 0$ for $p < q$

  $\Rightarrow$ if $p < q$, $H^q(\mathcal{N}, \Lambda^p T) = 0$.

- $H^1(\mathcal{N}, T^*) \cong \mathbf{C}$, $c_1 > 0$

  Akizuki-Nakano $H^q(\mathcal{N}, \Lambda^{n-p} T^* \otimes K^*) = 0$ for $p < q$

  $\Rightarrow$ if $p < q$, $H^q(\mathcal{N}, \Lambda^p T) = 0$.

- Kodaira if $q > 0$, $H^q(\mathcal{N}, K^*) = H^{3g-3-q}(\mathcal{N}, K^2)^* = 0$

  Verlinde formula:

  $\dim H^0(\mathcal{N}, \Lambda^{3g-3} T) = 3^{g-1} 2^{2g-1} \pm 2^{2g-1} + 3^{g-1}$

- $H^1(\mathcal{N}, T^*) \cong \mathbf{C}$, $c_1 > 0$

  Akizuki-Nakano $H^q(\mathcal{N}, \Lambda^{n-p} T^* \otimes K^*) = 0$ for $p < q$

  $\Rightarrow$ if $p < q$, $H^q(\mathcal{N}, \Lambda^p T) = 0$.

- Kodaira if $q > 0$, $H^q(\mathcal{N}, K^*) = H^{3g-3-q}(\mathcal{N}, K^2)^* = 0$

  Verlinde formula:

  $\dim H^0(\mathcal{N}, \Lambda^{3g-3} T) = 3^{g-1} 2^{2g-1} \pm 2^{2g-1} + 3^{g-1}$

- $\dim(H^0(\mathcal{N}, \Lambda^{3g-3} T))^\Gamma = \dfrac{3^g \pm 1}{2}$

- $g > 2$ $H^0(\mathcal{N}, T) = 0$, $\dim H^1(\mathcal{N}, T) = 3g - 3$

  M.S.Narasimhan & S.Ramanan (1975)

- $g > 2$ $\dim H^0(\mathcal{N}, \Lambda^3 T) \geq 5g - 5$

  $g > 4$ $H^0(\mathcal{N}, \Lambda^2 T) = 0$

  NJH (2011)

genus 2 odd degree: $\mathcal{N}$ = intersection of two quadrics in P^5

$\dim H^q(\mathcal{N}, \Lambda^p T)$:

1			
0	0		
0	3	15	
0	0	0	19
	0	0	0
		0	0
			0

genus 2 odd degree: $\mathcal{N}$ = intersection of two quadrics in P^5

dim $H^q(\mathcal{N}, \Lambda^p T)$:

1

0 0

0 3 15

0 0 0 19 $H^0(\mathcal{N}, \Lambda^3 T)$

0 0 0

0 0

0

HOCHSCHILD COHOMOLOGY

- automorphisms of $\mathcal{O}(\Lambda^* T)$ and $H^*(\mathcal{N}, \Lambda^* T)$

- $$\bigoplus_{p+q=n} H^q(\mathcal{N}, \Lambda^p T) \stackrel{HKR}{\cong} HH^n(\mathcal{N})$$

  as vector spaces...

- automorphisms of $\mathcal{O}(\Lambda^* T)$ and $H^*(\mathcal{N}, \Lambda^* T)$

- $$\bigoplus_{p+q=n} H^q(\mathcal{N}, \Lambda^p T) \stackrel{HKR}{\cong} HH^n(\mathcal{N})$$

  as vector spaces...

- ... as algebras if take product with $\sqrt{\mathrm{td}}$

$$\sqrt{\mathrm{td}} = (1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \cdots)^{1/2}$$

$$= 1 + \frac{c_1}{4} + \frac{c_1^2 + 4c_2}{96} + \frac{4c_1 c_2 - c_1^3}{3.2^7} \cdots$$

- Hochschild cohomology $HH(M)$:

  invariant of derived category $\mathcal{D}(M)$

- $HH^2(M) =$ first order deformations

- $g > 4, H^0(\mathcal{N}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(\mathcal{N}) \cong H^1(\mathcal{N}, T) \cong H^1(C, K^*) \sim$ deformations of C

Narasimhan conjecture:

$\mathcal{N}$ moduli space of rank 2 bundles E, odd degree, there is a semi-orthogonal decomposition:

$$\mathcal{D}^b(\mathcal{N}) = \langle \mathcal{D}^b(pt), \mathcal{D}^b(pt), \mathcal{D}^b(C), \mathcal{D}^b(C), \dots,$$
$$\dots \mathcal{D}^b(C^{(g-2)}), \mathcal{D}^b(C^{(g-2)}), \mathcal{D}^b(C^{(g-1)}) \rangle$$

where $C^{(n)}$ = symmetric product of C

J.Tevelev & S.Torres, *The BGMN conjecture via stable pairs,* arXiv:2108.11951v4.

K.Xu & S.-T.Yau, *Semiorthogonal decomposition of* $D^b(\mathrm{B}un_2^L)$, arXiv:2108.13353.

- $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ semi-orthogonal $\Rightarrow$ exact sequence

  $$\to HH^p(\mathcal{A}) \to HH^p(\mathcal{A}_1) \oplus HH^p(\mathcal{A}_2) \to HH^{p+1}(P_1, P_2) \to$$

- $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ semi-orthogonal $\Rightarrow$ exact sequence

  $\to HH^p(\mathcal{A}) \to HH^p(\mathcal{A}_1) \oplus HH^p(\mathcal{A}_2) \to HH^{p+1}(P_1, P_2) \to$

- $HH^*(\mathcal{N}) \to HH^*(C^{(g-1)})$

  $H^*(\mathcal{N}, \Lambda^* T) \to H^*(C^{(g-1)}, \Lambda^* T)$

- What is $H^*(C^{(g-1)}, \Lambda^* T)$?

- $H^0(C^{(n)}, \Lambda^p T^*) = H^0(C^n, \Lambda^p T^*)^{S_n} \Rightarrow H^0(C^{(n)}, \Lambda^p T) = 0$

  I.G.Macdonald, (1962)

- $H^1(C^{(n)}, T) \cong H^1(C, K^*)$

  G.Kempf (1980), B.Fantechi (1994)

$HH^2(\mathcal{N})$ = DEFORMATIONS

- $H^0(C^{(n)}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(C^{(n)}) \cong H^1(C^{(n)}, T) \oplus H^2(C^{(n)}, \mathcal{O}) \cong H^1(C, K^*) \oplus H^2(C^{(n)}, \mathcal{O})$

  $(\dim H^2(C^{(n)}, \mathcal{O}) = g(g-1)/2)$

$HH^2(\mathcal{N})$ = DEFORMATIONS

- $H^0(C^{(n)}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(C^{(n)}) \cong H^1(C^{(n)}, T) \bigoplus H^2(C^{(n)}, \mathcal{O}) \cong H^1(C, K^*) \bigoplus H^2(C^{(n)}, \mathcal{O})$

  $(\dim H^2(C^{(n)}, \mathcal{O}) = g(g-1)/2)$

- $g > 4, H^0(\mathcal{N}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(\mathcal{N}) \cong H^1(\mathcal{N}, T) \cong H^1(C, K^*)$

$HH^2(\mathcal{N})$ = DEFORMATIONS

- $H^0(C^{(n)}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(C^{(n)}) \cong H^1(C^{(n)}, T) \oplus H^2(C^{(n)}, \mathcal{O}) \cong H^1(C, K^*) \oplus H^2(C^{(n)}, \mathcal{O})$

  $(\dim H^2(C^{(n)}, \mathcal{O}) = g(g-1)/2)$

- $g > 4, H^0(\mathcal{N}, \Lambda^2 T) = 0 \Rightarrow$

  $HH^2(\mathcal{N}) \cong H^1(\mathcal{N}, T) \cong H^1(C, K^*)$

- $HH^2(\mathcal{N}) \to HH^2(C^{(n)})$

  $H^1(C, K^*) \to H^1(C, K^*) \oplus H^2(C^{(n)}, \mathcal{O})$

Prop: $H^p(\mathcal{N}, \Lambda^p T) \neq 0$ for $p \leq g-1$.

NB: $H^{3g-3}(\mathcal{N}, \Lambda^{3g-3}T) = H^0(\mathcal{N}, K^2)^* = 0$

Proof:

- $\rho : HH^2(\mathcal{N}) \to HH^2(C^{(p)})$

  $H^1(\mathcal{N}, T) \to H^1(C^{(p)}, T) \oplus H^2(C^{(p)}, \mathcal{O})$

- $a_i \in H^1(\mathcal{N}, T)$, $a_1 a_2 \ldots a_p \in H^p(\mathcal{N}, \Lambda^p T)$

- $\sqrt{\mathrm{td}}$ acts trivially on $H^p(\mathcal{N}, \Lambda^p T)$ since $H^{p+k}(\mathcal{N}, \Lambda^{p-k} T) = 0$

- ... if $a_1 a_2 \ldots a_p = 0$ then $\rho(a_1)\rho(a_2)\ldots\rho(a_p) = 0 \in HH^p(C^{(p)})$

- local deformation of C: punctured disc around $p \in C$

$$\text{Čech cocycle } \left[\frac{1}{z}\frac{\partial}{\partial z}\right] \in H^1(C, K^*)$$

- Serre duality $H^0(C, K^2)$ residue of $h(z)dz^2\frac{1}{z}\frac{\partial}{\partial z} = h(0)$

$$= \text{evaluation of } H^0(C, K^2) \text{ at } p$$

- local deformation of C: punctured disc around $p \in C$

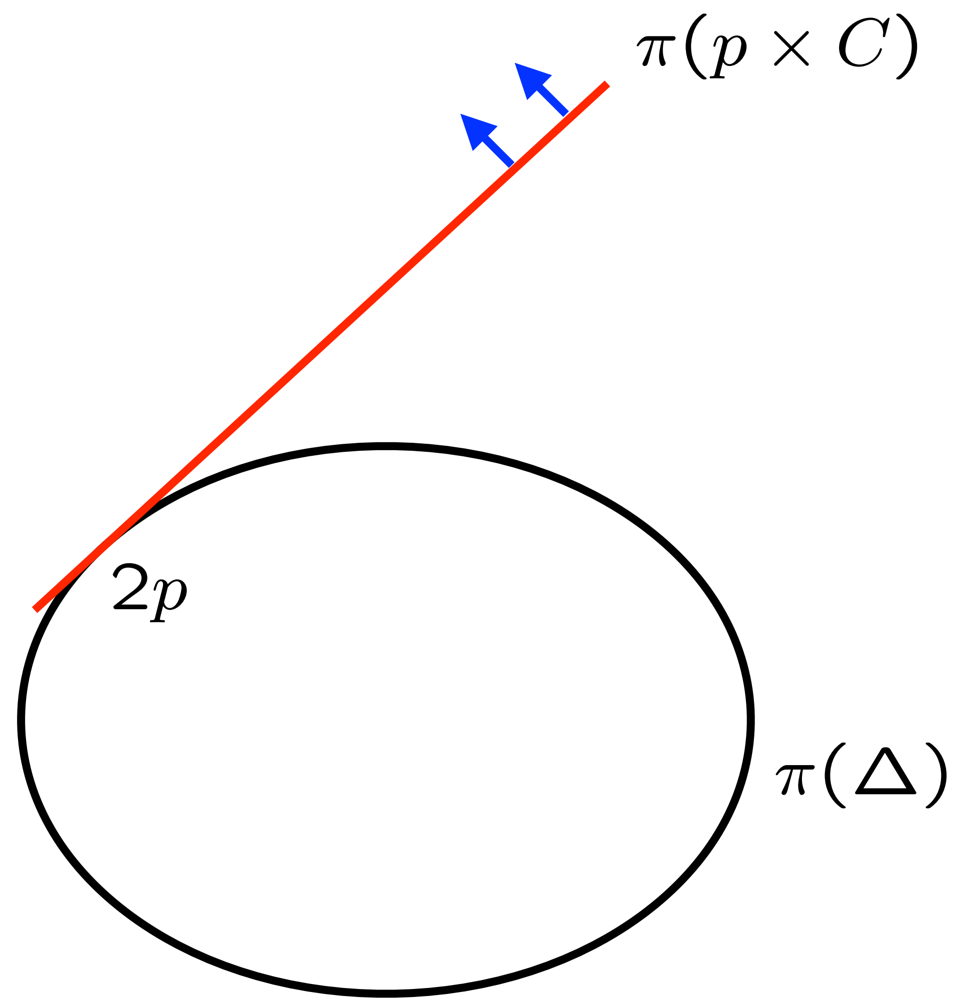
Čech cocycle $\left[\frac{1}{z}\frac{\partial}{\partial z}\right] \in H^1(C, K^*)$

- Serre duality $H^0(C, K^2)$ residue of $h(z)dz^2\frac{1}{z}\frac{\partial}{\partial z} = h(0)$

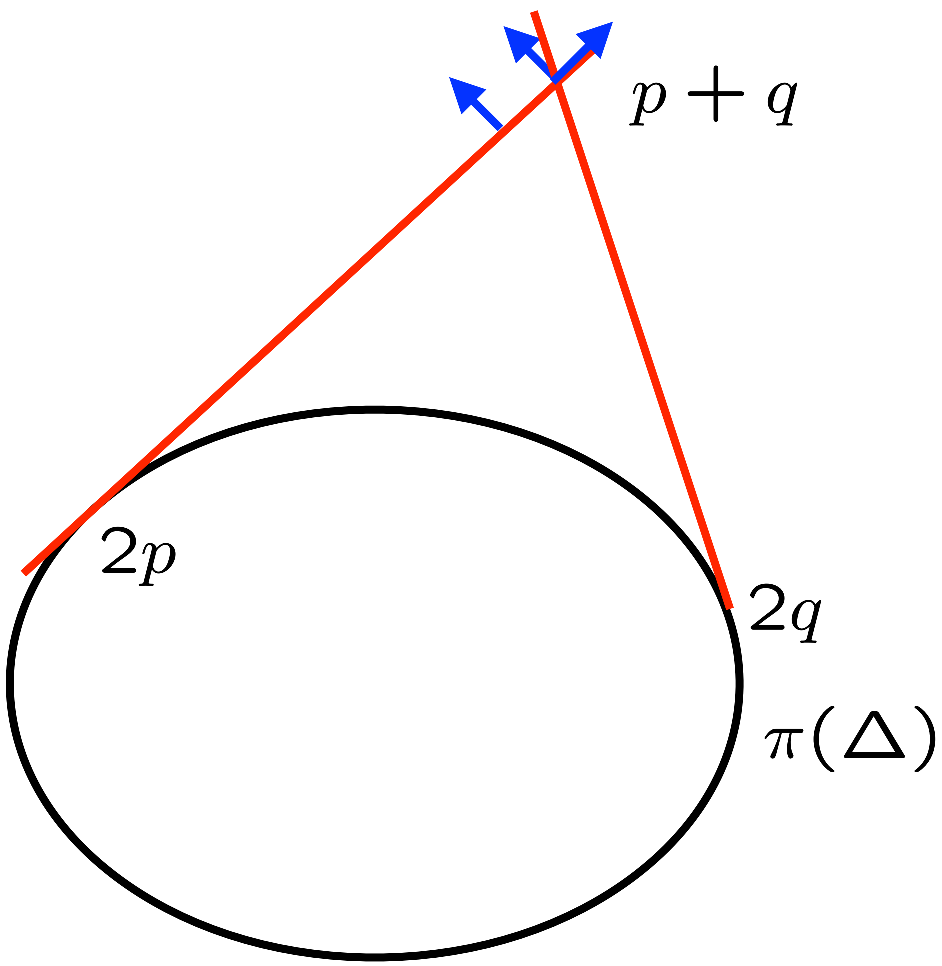
$=$ evaluation of $H^0(C, K^2)$ at p $\quad ev_p \in H^1(C, K^*)$

- $C^{(2)} = C \times C/S_2, \quad \pi : C \times C \to C^{(2)}$

$H^1(C, K^*) \cong H^1(C^{(2)}, T)$



$\pi(p \times C)$
$2p$
$\pi(\Delta)$



$$ev_p \wedge ev_q \in H^2(C^{(2)}, \Lambda^2 T) = H^2(C^{(2)}, K^*_{C^{(2)}})$$

Serre dual $H^0(C^{(2)}, K^2_{C^{(2)}})$: evaluate at $p+q$

$$\not\parallel$$

$$0$$

QUESTIONS

- is $H^p(\mathcal{N}, \Lambda^p T) = 0$ for $p \geq g$?

QUESTIONS

- is $H^p(\mathcal{N}, \Lambda^p T) = 0$ for $p \geq g$?

- is $H^p(\mathcal{N}, \Lambda^p T)$ generated by $H^1(\mathcal{N}, T)$?

QUESTIONS

- is $H^p(\mathcal{N}, \Lambda^p T) = 0$ for $p \geq g$?

- is $H^p(\mathcal{N}, \Lambda^p T)$ generated by $H^1(\mathcal{N}, T)$?

- are the $5g - 5$ holomorphic sections of $\Lambda^3 T$ all of them?

- ... or are they a maximal commuting subspace?

QUESTIONS

- is $H^p(\mathcal{N}, \Lambda^p T) = 0$ for $p \geq g$?

- is $H^p(\mathcal{N}, \Lambda^p T)$ generated by $H^1(\mathcal{N}, T)$?

- are the $5g - 5$ holomorphic sections of $\Lambda^3 T$ all of them?

- ... or are they a maximal commuting subspace?

- is the link with $C^{(n)}$ related to $\mathbf{C}^*$ fixed points on $\mathcal{M}$?

- $\mathcal{D}^b(C^{(n)}) \subset \mathcal{D}^b(\mathcal{N})$

  Poincaré sheaf on $C^{(n)} \times \mathcal{N}$

- universal bundle E over $C \times \mathcal{N}$

- $x_1, \ldots, x_n \in C^n$ universal bundle $E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_n}$

  descends to $C^{(n)} \times \mathcal{N}$

- $x_1, \ldots, x_n \in C^n$ universal bundle $E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_n}$

  defined on Higgs bundle moduli space $\mathcal{M}$

- $=$ mirror of upward flow of $x_1, \ldots, x_n \in C^{(n)}$ $=$ fixed point of $\mathbf{C}^*$-action on $\mathcal{M}$

T.Hausel & NJH, *Very stable Higgs bundles, equivariant multiplicity and mirror symmetry* , Inventiones Mathematicae **228** (2022) 893–989

FINAL QUESTION

How does this relate to the **even** integrable system?

- generic fibre $A \cong \mathrm{Prym}(S, C)$

  $A \cap T^*\mathcal{N} = A \backslash U$ codimension $\geq g$

- generic fibre $A \cong \mathrm{Prym}(S, C)$

  $A \cap T^*\mathcal{N} = A \backslash U$ codimension $\geq g$

- $\Lambda^q T \cong \Lambda^{n-q} T^* \otimes K^*$

  $\rho \in H^0(\mathcal{N}, \Lambda^{n-q} T^* \otimes K^*)$ pull back to

  $p^*\rho \in H^0(A \backslash U, \Lambda^{n-q} T^* \otimes p^* K^*) \cong H^0(A \backslash U, \Lambda^q T \otimes p^* K^*)$

- generic fibre $A \cong \mathsf{Prym}(S, C)$

  $A \cap T^*\mathcal{N} = A \backslash U$ codimension $\geq g$

- $\Lambda^q T \cong \Lambda^{n-q} T^* \otimes K^*$

  $\rho \in H^0(\mathcal{N}, \Lambda^{n-q} T^* \otimes K^*)$ pull back to

  $p^*\rho \in H^0(A \backslash U, \Lambda^{n-q} T^* \otimes p^* K^*) \cong H^0(A \backslash U, \Lambda^q T \otimes p^* K^*)$

- $p^* K^* \cong \mathcal{O}(m\Theta)$

  $p^*\rho$ extends to A as a section of $\Lambda^q T \otimes \mathcal{O}(m\Theta)$

- ramification divisor

  $s \in H^0(A, \mathcal{O}(m\Theta)) \Rightarrow \sigma = s^{-1} p^* \rho$

  commuting meromorphic polyvector fields on A

- ramification divisor

  $s \in H^0(A, \mathcal{O}(m\Theta)) \Rightarrow \sigma = s^{-1} p^* \rho$

  commuting meromorphic polyvector fields on A

- tautological 1-form $\eta \in p^* T^*$ on $T^* \mathcal{N}$

  interior product $i(\eta)\sigma$

- A Lagrangian $\Rightarrow d\eta = 0$

  if $[\sigma_1, \sigma_2] = 0$ then $[i(\eta)\sigma_1, \sigma_2] + [\sigma_1, i(\eta)\sigma_2] = 0$