Mirror symmetry and big algebras
1. Mirror symmetry for Hitchin systems

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Mirror Symmetry

- phenomenon first arose in various forms in string theory
- mathematical predictions (Candelas, de la Ossa, Green, Parkes 1991)
- mathematically it relates the symplectic geometry of a Calabi–Yau manifold X^d to the complex geometry of its mirror Calabi–Yau Y^d
- first aspect is the *topological mirror test* $h^{p,q}(X) = h^{d-p,q}(Y)$
- (Kontsevich 1994) suggests *homological mirror symmetry* $\mathcal{D}^b(Fuk(X,\omega)) \cong \mathcal{D}^b(Coh(Y,I))$
- (Strominger, Yau, Zaslow 1996) suggests a geometrical construction how to obtain Y from X
- many predictions of mirror symmetry have been confirmed - no general understanding yet

SYZ mirror symmetry for Hitchin systems

- C smooth, projective, complex curve
- G complex reductive group; $\mathrm{G}^\vee$ its Langlands dual
  e.g. $\mathrm{PGL}_n^\vee \cong \mathrm{SL}_n$
- (Hitchin 1987) (Simpson 1992):
  $\mathbb{M}_\mathrm{G}$ moduli space of semi-stable G-Higgs bundles (E, Φ)
  - E principal G-bundle
  - $\Phi \in H^0(C; ad(E) \otimes K_C)$ *Higgs field*
- $h_\mathrm{G} : \mathbb{M}_\mathrm{G} \to \mathbb{A}_\mathrm{G} := \mathrm{Spec}(\mathbb{C}[\mathbb{M}_\mathrm{G}])$ *Hitchin map*
  proper, completely integrable Hamiltonian system
  i.e. fibers Lagrangians with respect to symplectic $\omega \in \Omega^2(\mathbb{M}_\mathrm{G})$
- $\exists$ hyperkähler metric $(\mathbb{M}_\mathrm{G}, J) \cong \mathbb{M}_{\mathrm{DR}}(\mathrm{G})$ moduli flat G-bundles
- (Hausel, Thaddeus 2003) $\mathrm{G} = \mathrm{PGL}_n$ (Donagi, Pantev 2012) $\forall\, \mathrm{G}$

$$\begin{tikzcd}
\mathbb{M}_{\mathrm{DR}}(\mathrm{G}) \arrow[dr, "\mathrm{h}_\mathrm{G}"'] & & \mathbb{M}_{\mathrm{DR}}(\mathrm{G}^\vee) \arrow[dl, "\mathrm{h}_{\mathrm{G}^\vee}"] \\
& \mathbb{A}_G \cong \mathbb{A}_{G^\vee} &
\end{tikzcd}$$

dual special Lagrangian fibrations $\Leftrightarrow$ SYZ mirror symmetry

Topological mirror symmetry

- (Hausel, Thaddeus 2003) topological mirror symmetry: $E_{st}(\mathbb{M}_{\mathrm{PGL}_n}) = E_{st}(\mathbb{M}_{\mathrm{SL}_n})$ stringy Hodge numbers agree proved for $n = 2, 3$
  ”topological shadow of equivalence of derived categories of coherent sheaves”
- (Hausel 2013) $\rightsquigarrow$ proposes attack on topological mirror symmetry using (Ngô 2010)’s techniques on the cohomology of the Hitchin fibration in his proof of Langlands–Shelstead fundamental lemma
- (Gröchenig, Wyss, Ziegler 2020) prove topological mirror symmetry for all n using p-adic integration
- (Gröchenig, Wyss, Ziegler 2020) reprove (Ngô 2010)’s geometric stabilisation with p-adic integration
- (Maulik, Shen 2021) prove topological mirror symmetry using (Ngô 2010)’s techniques

(Classical limit of) Homological mirror symmetry

- (Kapustin–Witten 2007) derive (Kontsevich 1994) homological mirror symmetry from S-duality in 4D N=4 SUSY Yang–Mills

$$\begin{array}{ccccc} \mathcal{S}: D^b_{coh}(\mathbb{M}_{\mathrm{DR}}(\mathrm{G})) & \overset{HMS}{\cong} & Fuk(\mathbb{M}_{\mathrm{DR}}(\mathrm{G}^\vee)) & \overset{KW}{\cong} & D^b_{D-mod}(Bun_{\mathrm{G}^\vee}) \\ \rightsquigarrow \lambda\to 0 & & & \hbar\to 0 & \\ \mathcal{S}: D^b_{coh}(\mathbb{M}_{\mathrm{G}}) & \overset{CLHMS}{\cong} & D^b_{coh}(\mathbb{M}_{\mathrm{G}^\vee}) & & \end{array}$$

- $\rightsquigarrow D^b_{coh}(\mathbb{M}_{\mathrm{DR}}(\mathrm{G})) \overset{GLC}{\cong} D^b_{D-mod}(Bun_{\mathrm{G}^\vee})$

  Geometric Langlands Corr. of (Beilinson–Drinfeld, 1995)

- (Donagi, Pantev 2012) explain classical limit of HMS:
  $\mathcal{S}: D^b_{coh}(\mathbb{M}_{\mathrm{G}}) \overset{CLHMS}{\cong} D^b_{coh}(\mathbb{M}_{\mathrm{G}^\vee})$
  generically as Fourier–Mukai transform
  $D^b_{coh}(h_{\mathrm{G}}^{-1}(a)) \overset{FM}{\cong} D^b_{coh}(h_{\mathrm{G}}^{-1}(a)^\vee) \cong D^b_{coh}(h_{\mathrm{G}^\vee}^{-1}(a))$

Enhanced mirror symmetry in the classical limit

- (Kapustin, Witten 2007) introduce enhancements
- propose branes of type (B,A,A) to be mirror to (B,B,B)
- (B,A,A) branes: complex Lagrangians on $(\mathbb{M}_{\mathrm{G}}, \omega)$
  (B,B,B) branes: hyperholomorphic vector bundles in $\mathbb{M}_{\mathrm{G}^\vee}$
- (Hitchin 2016) $\rightsquigarrow$ $\mathbb{M}_{U(n,n)} \subset \mathbb{M}_{\mathrm{GL}_n}$ – (B,A,A) brane in $\mathbb{M}_{\mathrm{GL}_{2n}}$
  mirror: Dirac bundle on $\mathbb{M}_{\mathrm{Sp}_n} \subset \mathbb{M}_{\mathrm{GL}_{2n}}$ – (B,B,B) brane in $\mathbb{M}_{\mathrm{GL}_{2n}}$
  (Hausel–Mellit–Pei 2018) $\rightsquigarrow$ checks mirror of equivariant Euler form of $\mathbb{M}_{U(1,1)}$ matches Dirac bundle's on $\mathbb{M}_{\mathrm{Sp}(1)}$
- (Baraglia–Schaposhnik 2016) $\mathrm{G}_{\mathbb{R}}$ real form of G
  $\mathbb{M}_{\mathrm{G}_{\mathbb{R}}} \subset \mathbb{M}_{\mathrm{G}}$ (B,A,A) mirror should have support in $\mathbb{M}_{\mathrm{G}^\vee_{\mathbb{R}}} \subset \mathbb{M}_{\mathrm{G}^\vee}$
  where $\mathrm{G}^\vee_{\mathbb{R}} \subset \mathrm{G}^\vee$ is the complex reductive *Nadler group* of $\mathrm{G}_{\mathbb{R}}$
- conjectural constructions of (B,A,A) - (B,B,B) mirror pairs:
  (Biswas–García-Prada–Hurtubise 2019)
  (Franco–Gothen–Oliveira–Peón-Nieto 2021)
  (Franco–Jardim 2022)

Hecke (t' Hooft) and Wilson operators

- (Kapustin–Witten 2007) proposes t'Hooft and Wilson operators; in the classical limit:
  $\mathcal{H}^{\mu}_{c} : D^{b}_{coh}(\mathbb{M}_{\mathrm{G}}) \to D^{b}_{coh}(\mathbb{M}_{\mathrm{G}})$ *Hecke operator* (B,A,A)
  $$\begin{array}{rcl} \mathcal{W}^{\mu}_{c} : D^{b}_{coh}(\mathbb{M}_{\mathrm{G}^{\vee}}) & \to & D^{b}_{coh}(\mathbb{M}_{\mathrm{G}^{\vee}}) \\ \mathcal{F} & \mapsto & \mathcal{F} \otimes \rho_{\mu}(\mathbb{E}^{\vee})_{c} \end{array}$$ *Wilson operator* (B,B,B)
  $c \in C$; $\mu \in X_{*}^{+}(\mathrm{G}) = X^{*}_{+}(\mathrm{G}^{\vee})$ dominant cocharacter;
  $\rho_{\mu} : \mathrm{G}^{\vee} \to \mathrm{GL}(V^{\mu})$ μ-highest weight representation;
  $\mathbb{E}^{\vee}$ universal $\mathrm{G}^{\vee}$-bundle on $\mathbb{M}_{\mathrm{G}^{\vee}} \times C$
- intertwine $\mathcal{S}$: $\mathcal{H}^{\mu}_{c} \circ \mathcal{S} = \mathcal{S} \circ \mathcal{W}^{\mu}_{c}$
- test for $\mathcal{O}_{\mathbb{M}_{\mathrm{G}^{\vee}}} \in D^{b}_{coh}(\mathbb{M}_{\mathrm{G}^{\vee}})$:
  $\mathcal{H}^{\mu}_{c}(\mathcal{S}(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^{\vee}}})) = \mathcal{H}^{\mu}_{c}(\mathcal{O}_{W_0^+}) = \mathcal{S}(\mathcal{W}^{\mu}_{c}(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^{\vee}}})) = \mathcal{S}(\rho^{\mu}(\mathbb{E}^{\vee})_{c})$
- the Hecke transform of the Hitchin section $\mathcal{H}^{\mu}_{c}(\mathcal{O}_{W_0^+})$ is supported at a union of Lagrangian upward flows

Lagrangian upward flows in $\mathbb{M}$

- $\mathbb{M} := \mathbb{M}_{\mathrm{PGL}_n} \ni (E, \Phi); \Phi \in \mathrm{End}_0(E) \otimes K_C$
- $\begin{array}{cccc} h: & \mathbb{M} & \to & \mathbb{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E,\Phi) & \mapsto & \det(x - \Phi) \end{array}$ Hitchin map
- $\mathbb{C}^\times \circlearrowright \mathbb{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; *semiprojective*:
  1. $\mathbb{M}^{\mathbb{C}^\times}$ projective
  2. $\lim_{\lambda \to 0} \lambda\mathcal{E}$ exists for every $\mathcal{E} \in \mathbb{M}$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times} \rightsquigarrow W^+_{\mathcal{E}} := \{\mathcal{F} \in \mathbb{M} | \lim_{\lambda \to 0} \lambda\mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W^+_{\mathcal{E}} \subset \mathbb{M}$ locally closed $\cong T^+_{\mathcal{E}}\mathbb{M}$
- $\lambda^*(\omega) = \lambda\omega \rightsquigarrow W^+_{\mathcal{E}} \subset (\mathbb{M}, \omega)$ is Lagrangian
- $\mathbb{M} = \coprod_{\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}} W^+_{\mathcal{E}}$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow W^+_{\mathcal{E}}$ closed $\Leftrightarrow h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathbb{A}$ proper
- Motivating Problem: find coordinates s.t.

$$h_{\mathcal{E}} := h|_{W^+_{\mathcal{E}}} : W^+_{\mathcal{E}} \to \mathbb{A}$$

becomes explicit!

Examples of very stable Higgs bundles

- (Laumon 1988) $\rightsquigarrow$ $(E, 0)$ very stable for generic E
- (Peón-Nieto 2023) $\rightsquigarrow$ class. generic very stable $(V_1 \oplus V_2, \Phi_{12})$
- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \cdots \oplus K_C^{1-n}$
- $a = (a_2, \ldots, a_n) \in \mathbb{A} = H^0(K_C^2) \times \cdots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & 0 \ldots & 0 & a_n \\ 1 & 0 \ldots & 0 & a_{n-1} \\ \vdots & & \vdots & \\ 0 & \ldots 1 & 0 & a_2 \\ 0 & \ldots & 1 & 0 \end{pmatrix} : E_0 \to E_0 K_C$ companion matrix
- $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathbb{M}^{\mathbb{C}^\times}$ *canonical uniformising Higgs bundle*
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ *Hitchin section* $\Rightarrow$ very stable
- $c \in C$, $E_k := \mathcal{O}_C \oplus K_C^{-1} \ldots \oplus K_C^{-k}(c) \oplus \ldots K_C^{1-n}(c)$,

$$s_c \in H^0(\mathcal{O}_C(c)), \ \Phi_k := \begin{array}{c} \quad\quad k \\ \begin{pmatrix} 0 & \ldots 0 & \ldots 0 \\ 1 & \ldots 0 & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots s_c & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots & 1\ 0 \end{pmatrix} \end{array} : E_k \to E_k K_C$$

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_k := (E_k, \Phi_k)$ is very stable.

- proof by noticing $W_k^+ := W_{\mathcal{E}_k}^+ = \mathcal{H}_c^{\omega_k}(W_0^+)$
  ω_k kth fundamental character of SL_n, minuscule

Multiplicity algebra of $h_{\mathcal{E}}$

- *multiplicity algebra* of $h_{\mathcal{E}} = (h_1, .., h_N) : \mathbb{C}^N \cong W^+_{\mathcal{E}} \to \mathbb{A} \cong \mathbb{C}^N$:

  $Q_{h_{\mathcal{E}}} := \mathbb{C}[W^+_{\mathcal{E}} \cap h^{-1}(0)] = \mathbb{C}[W^+_{\mathcal{E}}]/(h^{-1}(\mathfrak{m}_0)) = \mathbb{C}[x_1, \ldots, x_N]/(h_1, \ldots, h_N)$
- notion due to (Arnold et al. 1982)
- $\dim(Q_{h_{\mathcal{E}}}) < \infty \Leftrightarrow h_{\mathcal{E}}$ is proper $\Leftrightarrow W^+_{\mathcal{E}} \subset \mathbb{M}_n$ closed $\Leftrightarrow W^+_{\mathcal{E}} \cap h^{-1}(0) = \{\mathcal{E}\} :\Leftrightarrow \mathcal{E}$ *very stable*
- in this case $m_{\mathcal{E}} := \dim(Q_{h_{\mathcal{E}}})$ is the *multiplicity* of $h^{-1}_{\mathcal{E}}(0)$
- as $h_{\mathcal{E}}$ is $\mathbb{C}^\times$-equivariant $\rightsquigarrow \mathbb{C}^\times \circlearrowright Q_{h_{\mathcal{E}}} \rightsquigarrow Q_{h_{\mathcal{E}}} = \bigoplus_{k=0}^{m} Q^k_{h_{\mathcal{E}}}$ s.t.
  - $Q^m_{h_{\mathcal{E}}} \cong \mathbb{C} Jac(h_{\mathcal{E}})$
  - $Q^i_{h_{\mathcal{E}}} \times Q^{m-i}_{h_{\mathcal{E}}} \to Q^m_{h_{\mathcal{E}}}$ nondegenerate $\rightsquigarrow$ Poincaré duality ring
  - $\sum_i \dim(Q^i_{h_{\mathcal{E}}}) t^i = \frac{\chi_{\mathbb{C}^\times}(Sym(T^{+*}_{\mathcal{E}}))}{\chi_{\mathbb{C}^\times}(Sym(\mathbb{A}^*))} = m_{\mathcal{E}}(t) \in \mathbb{N}[t]$ monic, palindromic

    equivariant multiplicity of (Hausel–Hitchin, 2022)
- Problem: can we determine $Q_{h_{\mathcal{E}}}$ explicitly?
- (Hitchin 2022) $\rightsquigarrow Q_{h_{\mathcal{E}}}$ for $\mathcal{E} = (E, 0)$ rank 2, genus $2, 3$