Moduli spaces of holomorphic bundles framed on a real hypersurface

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Recall: Let X be a projective complex algebraic manifold. A class $\mathcal{C}$ of coherent sheaves on X with fixed Hilbert polynomial χ is called *bounded* if there exists a coherent sheaf $\mathcal{V}$ on X such that for any $\mathcal{E} \in \mathcal{C}$ there exists a sheaf epimorphism $\mathcal{V} \twoheadrightarrow \mathcal{E}$.

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such that, for any $\mathcal{E} \in \mathcal{C}$, there exists $t \in K$ with $\mathcal{E} \simeq \mathcal{F}_t$.

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- **Remark:** If $\mathcal{C}$ is bounded, then any sequence $(\mathcal{E}_n)_n$ of $\mathcal{C}$ has a subsequence $(\mathcal{E}_{n_k})_k$ which is *convergent in complex geometric sense*, i.e. we have $\mathcal{E}_{n_k} \simeq \mathcal{F}_{t_k}$ for a *flat family* $(\mathcal{F}_t)_{t\in T}$ and a convergent sequence $(t_k)_k$ of T.

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  Examples:

Figure: Two compact moduli spaces on class VII surfaces. In red: the non-stable semistable locus. Not complex spaces!

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- Let now g be a Gauduchon metric on X.
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- **To prove:** Any sequence $(\mathcal{E}_n)_n$ of g-stable bundles (with fixed topological type and fixed determinant line bundle) admits a subsequence which is convergent in the complex geometric sense explained above.

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A_∞ defines a holomorphic structure $\mathcal{E}_\infty$ on E_∞.

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The first notion uses analytic methods (estimates for solutions of non-linear PDE), the second uses homological algebraic tools in complex geometry, e.g. *flatness*.

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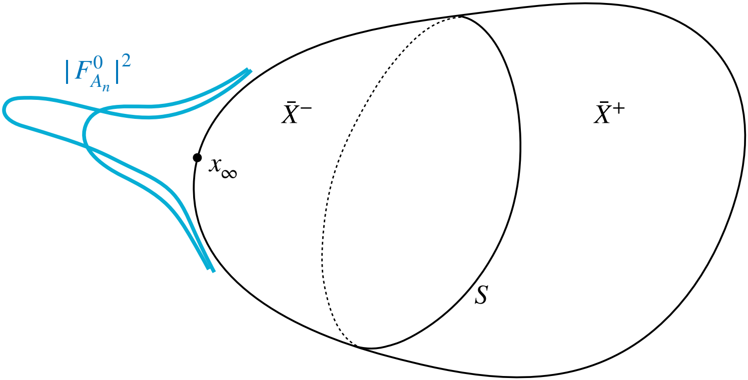

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1. The moduli spaces of boundary framed holomorphic bundles (of fixed topological type) are infinite dimensional.
2. We can no longer describe the local structure of such a moduli space using complex geometric deformation theory (because we don't have a deformation theory on cplx manifolds with boundary).

New idea: Recall that, in our situation, the boundary $\partial\bar{X}^-$ is a closed real hypersurface in a *closed* complex manifold X.

Definition:

Let $S \subset X$ be a closed, separating, real hypersurface. An S-framed holomorphic bundle of class $\mathcal{C}^\kappa$ is a pair $(\mathcal{E}, \theta)$ where

1. $\mathcal{E}$ is a holomorphic bundle on X,
2. θ is a trivialization of class $\mathcal{C}^\kappa$ of $\mathcal{E}|_S$.

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Theorem ([Te]):

The obtained map $\mathcal{M}_S(E) \to \mathcal{M}_{\partial\bar{X}^-}(E^-) \times_{\mathcal{C}} \mathcal{M}_{\partial\bar{X}^+}(E^+)$ is a homeomorphism.

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2. Stein's generalization for "Lipshitz spaces" of Whitney extension theorem.

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Let E be the trivial $\mathrm{SL}(2,\mathbb{C})$ differentiable bundle on $\mathbb{P}^1_{\mathbb{C}}$.

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The isomorphism theorem and the remark above gives a homeomorphism

$$\mathcal{M}_{S^1}(E) \xrightarrow{\simeq} \mathcal{M}_{\partial\bar{X}^-}(E^-) \times \mathcal{M}_{\partial\bar{X}^+}(E^+),$$

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By Grothendieck theorem, any holomorphic $\mathrm{SL}(2,\mathbb{C})$ bundle on $\mathbb{P}^1_\mathbb{C}$ is isomorphic to one of the bundles $\mathcal{E}_m = \mathcal{O}(m) \oplus \mathcal{O}(-m)$ with $m \in \mathbb{N}$.

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Theorem:

The moduli space $\mathcal{M}_{S^1}(E)$ is a smooth infinite dimensional complex analytic space, which comes with a natural stratification $\mathcal{M}_{S^1}(E) = \coprod_{m\in\mathbb{N}} \mathcal{M}_m$.

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$$\mathcal{G}_m:=\left\{\begin{pmatrix} a & P|_{S^1}\\ 0 & a^{-1}\end{pmatrix}\middle|\ a\in\mathbb{C},\ P\in\mathbb{C}[z],\ \deg(P)\leq 2m\right\}.$$

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$$\mathcal{M}_m \simeq \mathcal{C}^\kappa(S^1, \mathrm{SL}(2, \mathbb{C})) \big/ \mathcal{G}_m .$$

In order to understand how the strata $\mathcal{M}_m$ fit together in the moduli space $\mathcal{M}_{S^1}(E)$ one needs an explicit description of the ...

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Note that
$$\overline{\mathcal{M}_m} \subset \bigcup_{l \geq m} \mathcal{M}_l$$

In particular $\mathcal{M}_0$ is open and dense.
We obtain:

Corollary:

The product
$$\mathcal{C}^\kappa(S^1, \mathrm{SU}(2))/_{\mathrm{SU}(2)} \times \mathcal{C}^\kappa(S^1, \mathrm{SU}(2))/_{\mathrm{SU}(2)}$$
has a natural stratification $(\mathcal{M}_m)_{m \in \mathbb{N}}$, with
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In order to understand how the strata $\mathcal{M}_m$ fit together in the moduli space $\mathcal{M}_{S^1}(E)$ one needs an explicit description of the … *moduli stack of* $\mathrm{SL}(2,\mathbb{C})$*-bundles on* $\mathbb{P}^1_{\mathbb{C}}$.

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THANK YOU!