A heterotic Kodaira-Spencer theory at one-loop

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I: Introduction and Motivation

Physics / Pheno :

- The world is Quantum !

Mathematics :

- TQFT $\rightsquigarrow$ Geometric / enumerative / topological invariant theory.

Type II String : Topological A model and B model

Heterotic : All sectors are coupled !

Stringy Moduli Problems

String theory/SUSY $\rightsquigarrow$ Geometries with special structure (Calabi-Yau, Instantons, ...)

Spacetime: $M_{10} = M_4 \times X_6$

Supersymmetry: X_6 has special structure.

$$\left\{ \begin{array}{l} \text{deformations of } X_6 \\ \text{Moduli problems} \end{array} \right\} \leftrightsquigarrow \left\{ \begin{array}{l} \text{Interesting} \\ \text{physics in 4d.} \end{array} \right\}$$

3 levels of understanding moduli:

I) Infinitesimal massless spectrum:

- Geometry is described by BPS equations:

$$\mathrm{BPS} = 0 \leftarrow \text{SUSY equations}$$

- Infinitesimal deformation:

$$\delta(\mathrm{BPS}) = 0 \rightsquigarrow \mathcal{D}\alpha = 0$$

- Identify differential $\mathcal{D}$, $\mathcal{D}^2 = 0$.

- Massless fields / infinitesimal symmetries:

$$TM = H^1_{\mathcal{D}} = \frac{\{\text{deformations } \alpha \mid \mathcal{D}\alpha = 0\}}{\{\text{symmetry transformations} \mid \alpha = \mathcal{D}\gamma\}}$$

Infinitesimal symmetries

- $\mathcal{D}$ is usually the differential of an elliptic complex:

$$\cdots \xrightarrow{\mathcal{D}} \Omega^{-1} \xrightarrow{\mathcal{D}} \Omega^0 \xrightarrow{\mathcal{D}} \Omega^1 \rightarrow \Omega^2 \rightarrow \cdots \rightsquigarrow$$ Finite dimensional spectrum.

Exs: deformations of integrable complex structure:

μ: Beltrami differential $\rightsquigarrow [\mu] \in H^{(0,1)}_{\bar\partial}(T^{(1,0)}X) \cong H^{(2,1)}_{\bar\partial}(X)$.

(X Calabi-Yau)

II) Understand Geometry of moduli space $\mathcal{M}$:

- Geometric Structures on $\mathcal{M}$: Complex? Kähler?, ...
- Higher order def.'s; obstructions (Yukawa couplings), smooth directions?, superpotential, ...
- Finite deformations: Solve Maurer-Cartan [Saemann] equation in associated L_∞-algebra:

$$D\alpha + \frac{1}{2}[\alpha,\alpha] \; (+\dots) = 0$$

Ex: Finite def's of complex structure $\mu \in \Omega^{(0,1)}(T^{(1,0)}X)$ solve

$$\bar{\partial}\mu + \frac{1}{2}[\mu,\mu] = 0$$

$\rightsquigarrow$ Diff. graded Lie Algebra.

Tian - Todorov : X Calabi - Yau (or $\partial\bar\partial$ - lemma)

$\Rightarrow$ infinitesimal complex structure moduli are unobstructed.

III) Understand Quantum moduli space:

- Quantize theory (BV-BRST, AKSZ, ...)

- Non-perturbative effects; Instantons, dualities, ...

associated to structure ...

- Compute Invariants: Knot invariants (CS-theory), Donaldson - Thomas, Gromov - Witten, ...

- Find topological theory governing geometric structure?

Exs:

Structure	"Target space"	"World-sheet"
Complex structure	Kodaira-Spencer theory	Wittens B-model
Kähler structure	Kähler-Gravity	Wittens A-model
Various Gauge theories	Chern-Simons Donaldson-Thomas (Hol. CS-theory)	Versions of topological open string

Mirror Symmetry

Open-Closed duality (Conifold transition)

II: The heterotic Superpotential

Given an $SU(3)$-manifold (X, ω, Ω) with a gauge bundle $V \to X$, the heterotic superpotential is [Cardoso '03, Gurrieri '04]

$$W = \int_X (H + i d\omega) \wedge \Omega \qquad (\Omega = e^{-2\phi} \Psi)$$

[Saemann] connection on TX

where $$H = dB + \frac{\alpha'}{4}\left(\omega_{CS}(A) - \omega_{CS}(\nabla)\right) \qquad (0 < \alpha' \ll 1)$$

gauge connection

Note: $H \in \Omega^3(X)$ is a global 3-form.

$\Rightarrow$ Local $B \in \Omega^2(X)$ transforms under gauge transformations (and diffeo) [Green-Schwarz '84]

Extrema of W (F-term constraints):

$$\delta W = 0 \quad (W=0) \quad \Rightarrow \quad \begin{cases} d\Omega = 0 & (X \text{ is complex}) \\ F^{(0,2)} = \bar\partial_A^2 = 0 & (\text{holomorphic bundle}) \\ H = i(\partial - \bar\partial)\omega & (\text{"Anomaly"}) \end{cases}$$

There are also D-term constraints:

$$d(e^{-2\phi}\omega\wedge\omega) = 0, \qquad \omega\wedge\omega\wedge F = 0$$

dilaton

Moment map interpretation: [Garcia-Fernandez et al '18, '20, Ashmore et al '19]

Together, they form the Hull-Strominger system [Hull '86, Strominger '86].

III: Heterotic Moduli

Vary action away from BPS-locus:

[Ashmore et al '18]

$$\Delta W = S = \int_X \left(\langle g, \bar{D} g \rangle \wedge \Omega + \kappa \wedge \bar{\partial} \kappa \right) + \text{cubic}$$

The (quasi-)topological theory S has features similar to Kodaira-Spencer gravity [BCOV '93], and Donaldson-Thomas theory [DT '98].

Similar theories: [Rosa et al '12, Costello-Li '15, '16, '19, Costello-Williams '21, ...]

$$y \in \Omega^{(0,1)}(\overbrace{T^{*(1,0)} \oplus End(V) \oplus T^{(1,0)}}^{Q}), \quad y = (x, \alpha, \mu)$$

"hermitian", bundle, complex structure deformations

$b \in \Omega^{(0,2)}(X)$, $\kappa \in \Omega^{(3,0)}(X)$.

[Bismut '86, Gualtieri '10, de la Ossa-Svanes '14, McOrist-Svanes '21]:

$$\bar{D} : \Omega^{(0,p)}(Q) \to \Omega^{(0,p+1)}(Q)$$

$$\bar{D}^2 = 0 \iff \begin{cases} \bar{\partial}^2 = 0 \\ \bar{\partial}_A^2 = 0 \\ \underbrace{i\partial\bar{\partial}\omega = \frac{\alpha'}{8}(\mathrm{tr}\, F \wedge F - \mathrm{tr}\, R \wedge R)}_{\text{Bianci Identity}} \end{cases}$$

Note: Quadratic order: $\delta S = 0 \Rightarrow \bar{D}y = 0$

modulo infinitesimal symmetries $[g] \in H^{(0,1)}_{\bar{D}}(Q)$

$\rightsquigarrow$ infinitesimal moduli of "Hull-Strominger" system [Anderson et al '14, delaOssa-Svanes '14, Garcia-Fernandez '15, Ashmore et al '19, McOrist-Svanes '21].

Actually: $\delta S = 0$ & $S = 0$

are the F-term constraints for BPS solutions.

$\Rightarrow$ Maurer-Cartan [Saemann] equation for an L_3-algebra [Ashmore et al '18].

IV : 1-loop partition function

[Saemann]

Standard methods (BV-quantisation, ...) gives

$$|Z_{1\text{-loop}}| = \frac{I(Q)^{\frac{1}{2}}}{I(\Omega^0)}$$

Note : There are anomalies associated with the phase of Z. Cancelled by adding appropriate counter-terms to action.

Also : Defining $I(Q)$, $I(\Omega^0)$ requires a choice of metrics H and g on Q and X.

$\rightsquigarrow$ metric anomalies

Similar to Witten's frame anomaly for CS-theory [Witten '89]

Note also:

$$Z = \int \mathcal{D}[\text{fields}]\, e^{-S}$$

This integal is over all of parameter and moduli space.

$\Rightarrow$ Z is a topological invariant of the associated geometry

Heterotic: (Holomorphic) courant/string algebroid
[Garcia-Fernandez '13, Coimbra et al '14
Garcia-Fernandez et al '18, ...]

In particular: Dependence of $Z_{1\text{-loop}}$ on geometric structure says something about higher loop orders

$\rightsquigarrow$ Heterotic "holomorphic anomaly"?

V : Metric Anomalies

From now: Assume $(X, \underbrace{\omega_0}_{\exists \text{ CY metric, and } \omega = \omega_0 + O(\alpha')}, \Omega_0)$ Calabi-Yau.

[Bismut et al '88]: holomorphic bundle $E \to X$:

$$\frac{1}{2\pi}\frac{\partial}{\partial t}\log \mathcal{I}(E) = \frac{1}{2}\int_X \frac{\partial}{\partial t}\left[td\left(\frac{1}{2\pi}(iR_0 + t g_0^{-1}\delta g_0\right) Ch\left(\frac{1}{2\pi}\left(iF + t h^{-1}\delta h\right)\right)\right]_4$$

where F is the curvature of the Chern-connection given by the metric h on E.

Under a variation of the metrics on X and V we get

$$\delta \log |Z| = \frac{\pi}{2}\int_X \delta P_4$$

$$P_4 = \frac{\mathrm{rk}(Q) - 2}{1440}\left(60\, Ch_3(X)\, Ch_1(X) + 20\, Ch_2(X)^2 - 168\, Ch_4(X)\right)$$

$$- \frac{1}{12} Ch_2(X)\, Ch_2(\mathrm{End}\, V) + Ch_4(\mathrm{End}\, V) \quad + \mathcal{O}(\alpha' \cdot \hbar)$$

where:

Adjoint trace

$$Ch_n(X) = \frac{1}{n!} \mathrm{tr}\left(\frac{iR_0}{2\pi}\right)^n, \quad Ch_n(\mathrm{End}\, V) = \frac{1}{n!} \mathrm{Tr}\left(\frac{iF}{2\pi}\right)^n$$

This polynomial can be **factorised** if we restrict to bundles where

$$Ch_4(\mathrm{End}(V)) \propto Ch_2(\mathrm{End}(V))^2$$

True for: $SO(8)$, $SU(3)$, exceptional groups, ...

Note: To avoid requiring $ch_2(X) = ch_2(V) = 0$, we also need

$$G = G_1 \times G_2 \quad \Rightarrow \quad \mathrm{End}(V) \mapsto \mathrm{End}(V_1) \oplus \mathrm{End}(V_2)$$

$$\Rightarrow P_4 = \overset{iii)}{\#\int_X ch_3(X)\, c_1(X)}$$

similar to Euler # anomaly of Kodaira-Spencer [BCOV '93, Pestun-Witten '05]

$$+ \#_1 \overset{i)}{\int_X \left(ch_2(X) - A_1\, ch_2(V_1) \right)^2} + \#_2 \overset{ii)}{\int_X \left(ch_2(X) - A_2\, ch_2(V_2) \right)^2}$$

iii): cancelled by multiplying $|Z|$ by appropriate volume factor [Pestun-Witten '05].

i-ii): May be cancelled if

$$[ch_2(X)] = A_1 [ch_2(V_1)] \, , \quad [ch_2(X)] = A_2 [ch_2(V_2)]$$

by adding strictly background-dependent counter terms.

Similar to Wittens gravitational Chern-Simons term to cancel frame anomaly in CS-theory [Witten '89].

Note: Only for $G = SO(8) \times SO(8)$ and $G = SU(3) \times E_6$ can we also satisfy the 10d constraint:

$$[ch_2(X)] = [ch_2(V)] \qquad \text{(Bianci identity)}$$

VI: Conclusion / Outlook

- We have computed the 1-loop partition of heterotic superpotential and studied its topological properties.

Outlook:

- Truly non-Kähler (How does $\mathcal{I}(E)$ deform?)
- Other dimensions: Anomalies for a similar theory in 10D give constraints [Morgas Ibarra]:

$$\mathrm{rk}(V) = 496 \qquad \text{and} \qquad [ch_2(X)] = [ch_2(V)]$$

$(G = SO(32), E_8 \times E_8)$

- Beyond one-loop:
  - heterotic holomorphic anomaly?

- Higher loop "Invariants" ?

- Non-perturbative invariants : "Knot invariants", world-sheet instantons, NS5 branes, ...

- Relations to world-sheet models ? Chiral de Rham complex [Linshaw] and vertex operator algebras applied to heterotic with applications to (0,2) mirror symmetry : [Witten '92, Witten '05, Kapustin '05, Melnikov-Sharpe '11, Melnikov et al '12, Gorbounov et al '16, de la Ossa-Fiset '18, Álvarez-Cónsul et al '20, '23, ...]

Hints of world-sheet - Target space duality :

[Pestun-Witten '05] :

$$Z_{1\text{-loop}}(\mathrm{IIB}) = Z_{1\text{-loop}}(\text{Generalised Hitchin functional})$$

[Costello-Williams '21]:

$$\left\{\begin{array}{l}\text{Algebra of symmetries of}\\ \text{heterotic } \beta\text{-}\gamma\text{-system /}\\ \text{chiral deRham complex}\end{array}\right\} = \left\{\begin{array}{l}\text{Algebra of a}\\ \text{certain type I}\\ \text{Kodaira-Spencer}\\ \text{gravitational theory}\end{array}\right\}$$

- Other potentials? $\mathcal{N}=1$ heterotic comes with a Kähler potential $\mathcal{K}$ [Candelas et al '16, '18, McOrist-Sisca '19, Garcia-Fernandez et al '20]

Type II large volume: $Z_{1\text{-loop}}(\mathrm{IIA}) = Z_{1\text{-loop}}(\mathrm{IIB})$

It is interesting to speculate about relations between $Z(W)$ and $Z(\mathcal{K})$. Connect to (0,2) mirror symmetry? In particular, W is holomorphic, and so perturbatively protected.

Thank You !

Gauge anomalies

This talk: Ignore gravitational anomalies

From now: $\bar{D} = \bar{\partial}$.

$$\Rightarrow \quad S = \int_X (\mu^a \bar{\partial} x_a + \mathrm{tr}\, \alpha\, \bar{\partial}_A \alpha) \wedge \Omega + \int_X \kappa\, \bar{\partial} \sigma \quad (*)$$

The phase of Z has a gauge-transformation:

$$\delta \log Z \propto i \int_X \mathrm{Tr}(F \wedge F \wedge F \gamma), \quad \gamma \in \Omega^0(\mathrm{End}(V))$$

Adjoint trace

Focus on bundles where $\mathrm{Tr}(T^4) \propto \mathrm{tr}(T^2)^2$

$$\Rightarrow \quad \delta \log Z \propto i\hbar \int_X (\mathrm{tr}\, F^2)\, \mathrm{tr}\, F_\gamma$$

Fix by adding counter-term to classical action:

$$S_c = i \int (\mathrm{tr}\, F^2) \wedge X$$

and demand: $\quad \delta_\gamma X \propto \hbar\, \mathrm{tr}\, F_\gamma \quad (**)$

However, the classical theory $(*)$ is not invariant under $(**)$.

BUT: Classical action is invariant modulo EOM's

$\Rightarrow$ Theory is gauge invariant modulo $O(\hbar^2)$.

"Toy-Model": Chern-Simons

$$S_{CS}(A) = \int_{M_3} tr\left(A\,dA + \frac{2}{3}A^3\right)$$

$$A \in \Omega^1(\mathfrak{g}) \quad , \quad \dim(M_3) = 3.$$

EOM : $F(A_0) = dA_0 + A_0 \wedge A_0 = 0.$

1-loop Action : $A = A_0 + \alpha \quad , \alpha \in \Omega^1(\mathrm{End}(V))$

$$\Rightarrow \quad S(\alpha) = \int_{M_3} \alpha\, d_0\, \alpha \quad , \quad d_0 = d + A_0 .$$

Partition function:

$$Z(M_3) = \frac{1}{\mathrm{Vol}(G)} \int \mathcal{D}\alpha \, e^{-S(\alpha)}$$

No metric $\rightsquigarrow$ Topological invariant of M_3.

$S(\alpha)$ has gauge symmetry: $\alpha \to \alpha + d_0 \epsilon$.

$\Rightarrow$ Quantise using BV-BRST formalism.

OR: Physical Approach [e.g. Pestun-Witten '05]

Often sufficient for quadratic actions...

Finite-dimensional case:

- S_{ij} symmetric pos. definite matrix

- $\int dx_1 dx_2 \dots dx_n \, \mathrm{Exp}\left(-\frac{1}{2}\sum_{ij} x_i S_{ij} x_j\right) \propto \frac{1}{\det(S)^{\frac{1}{2}}}$

Generalisation to field theory:

Choose metric g on M_3:

$\Rightarrow$ Hodge decomposition: $\Omega^1 = d_0\Omega^0 \oplus d_0^+\Omega^2$

(we ignore finite-dimensional space of harmonic forms)

$$\Rightarrow Z = \frac{1}{\mathrm{Vol}(G)} \int_{\alpha \in d_0\Omega^0} \int_{\alpha \in d_0^+\Omega^2} \mathcal{D}\alpha \, \exp(-S(\alpha))$$

$$= \frac{1}{\mathrm{Vol}(G)} \frac{\mathrm{Vol}(d_0\Omega^0)}{\det(d_0 : d_0^\dagger\Omega^2 \to d_0\Omega^1)^{\frac{1}{2}}}$$

Goal: Write answer in terms of determinants of elliptic operators (laplacians), whose determinants can be regularised.

Denominator: $\det(d_0 : d_0^\dagger\Omega^2 \to d_0\Omega^1)^{\frac{1}{2}}$

We define:

$$\det(d_0|_{d_0^\dagger\Omega^2}) := \det(d_0^\dagger d_0|_{d_0^\dagger\Omega^2})^{\frac{1}{2}}$$

Note:

$$\det(d_0^\dagger d_0|_{d_0^\dagger\Omega^2}) = \frac{\det(d_0^\dagger d_0|_{d_0^\dagger\Omega^2})\det(d_0 d_0^\dagger/_{d_0\Omega^0})}{\det(d_0 d_0^\dagger/_{d_0\Omega^0})}$$

$$= \frac{\det(\Delta')}{\det\left(d_0 d_0^{+} \big/_{d_0 \Omega^0}\right)}$$

Consider eigenvector $d_0 \alpha$ of $d_0 d_0^{+}$ on $d_0 \Omega^0$:

$$d_0 d_0^{+} d_0 \alpha = \lambda d_0 \alpha$$

$$\Updownarrow \qquad (d_0 \text{ is invertible on } \mathrm{Im}(d_0^{+}))$$

$$\underbrace{d_0^{+} d_0}_{\Delta^0} \alpha = \lambda \alpha$$

$\Rightarrow \quad \det\left(d_0 d_0^{+} \big|_{d_0 \Omega^0}\right) = \det(\Delta^0)$

$\Rightarrow \quad \det\left(d_0^{+} d_0 \big|_{d_0^{+} \Omega^2}\right) = \dfrac{\det(\Delta^1)}{\det(\Delta^0)}$

Numerator: $\quad \mathrm{Vol}(d_0 \Omega^0)$.

Recall: Given Linear operator $A : V \to W$

$$\Rightarrow \quad \mathrm{Vol}(AV) = \det(A) \frac{\mathrm{Vol}(V)}{\mathrm{Vol}(\ker(A))}$$

$$\Rightarrow \quad \mathrm{Vol}(d_0\Omega^0) = \underbrace{\det\left(d_0|_{\Omega^0}\right)}_{\det(\Delta^0)^{\frac{1}{2}}} \frac{\mathrm{Vol}(\Omega^0)}{\mathrm{Vol}\Big[\underbrace{\ker\left(d_0|_{\Omega^0}\right)}_{H^0(M_3)}\Big]} = 1$$

$$\Rightarrow \quad \mathrm{Vol}(d_0\Omega^0) = \det(\Delta^0)^{\frac{1}{2}}\, \mathrm{Vol}(\Omega^0)$$

Volume of gauge-transformations

Collect:

$$Z(M_3) = \frac{1}{\cancel{\mathrm{Vol}(\mathcal{G})}} \frac{\det(\Delta^0)^{\frac{3}{4}}}{\det(\Delta^1)^{\frac{1}{4}}}\, \cancel{\mathrm{Vol}(\Omega^0)}$$

$$\Rightarrow \quad Z(M_3) = \frac{\overbrace{|\Delta^0|}^{\text{Regularise}}{}^{\frac{3}{4}}}{\overbrace{|\Delta^1|}{}^{\frac{1}{4}}}$$

This is the Ray-Singer torsion of M_3.