Higher principal bundles and higher gauge theory

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Initial remarks

- Sorry, these are slides ...
  ... but download at https://christiansaemann.de/talks/
- This is an overview:
  - Not always full definitions
  - Don't trust the signs or prefactors
- If you have any questions, please ask!
- Too fast/complex/unclear or too basic, please let me know.

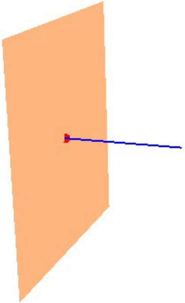
"In these days the angel of topology and the devil of abstract algebra fight for the soul of every individual discipline of mathematics." *Hermann Weyl*

Presented material based on joint work with:
Leron Borsten, Getachew Demessie, Mehran Jalali Farahani, Brano Jurčo, Hyungrok Kim, Sam Palmer, Dominik Rist, Lennart Schmidt, Martin Wolf

Outline

I: The local picture

Why higher gauge theory?

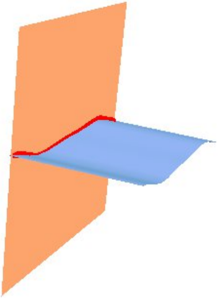


D-branes

- D-branes interact via strings.
- Effective description: theory of endpoints
- Parallel transport of these: Gauge theory
- Study string theory via gauge theory

M5-branes

- M5-branes interact via M2-branes.
- Eff. description: theory of self-dual strings
- Parallel transport: Higher gauge theory
- Long sought $(2,0)$-theory a HGT?



Why higher gauge theory?

More reasons from Physics:

- Kalb–Ramond B-field (connection on gerbe) in string theory
- Higher gauge potentials in supergravity in general
- Tensor hierarchies in gauged supergravity in particular
- 6d superconformal field theories and M5-branes

More reasons from Mathematics:

- Rich and interesting examples of principal bundles:
  Instantons, Monopoles, ADHM construction, twistors, ...
- Should generalize to higher bundles (?!)
- Current definition of higher connections has open questions
- Algebraic structure in generalized geometry
- T-duality as a correspondence of principal 2-bundles

Connections describe parallel transport

Encode gauge theory in parallel transport functor

Mackaay, Picken, 2000

- Every manifold comes with path groupoid $\mathcal{P}M = (PM \rightrightarrows M)$

$$x \xrightarrow{\gamma} y$$

- Gauge group gives rise to delooping groupoid $\mathsf{BG} = (\mathsf{G} \rightrightarrows *)$
- parallel transport functor $\mathrm{hol} : \mathcal{P}M \to \mathsf{BG}$:
  - assigns to each path a group element
  - composition of paths: multiplication of group elements

Formulas do not generalize straightforwardly

Ordinary parallel transport along path:

- holonomy functor hol : path $\gamma \mapsto \mathsf{hol}(\gamma) \in \mathsf{G}$
- $\mathsf{hol}(\gamma) = P\exp(\int_\gamma A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Abelian parallel transport along surface:

- map hol : surface $\sigma \mapsto \mathsf{hol}(\sigma) \in \mathsf{U}(1)$
- $\mathsf{hol}(\sigma) = \exp(\int_\sigma B)$, B: connective structure on gerbe.

Nonabelian case?

Higher parallel transport requires categorification

Non-abelian parallel transport of strings problematic:

$$\begin{tikzcd}[column sep=huge]
\bullet & {} \arrow[l, bend right=50, ""{name=A1}] \arrow[l, ""{name=B1}] \arrow[l, bend left=50, ""{name=C1}] & \bullet \arrow[l, bend right=50, ""{name=A2}] \arrow[l, ""{name=B2}] \arrow[l, bend left=50, ""{name=C2}]
\arrow[Rightarrow, from=A1, to=B1, "g_1"]
\arrow[Rightarrow, from=B1, to=C1, "g_1'"]
\arrow[Rightarrow, from=A2, to=B2, "g_2"]
\arrow[Rightarrow, from=B2, to=C2, "g_2'"]
\end{tikzcd}$$

Consistency of parallel transport requires:

$$(g_1' g_2')(g_1 g_2) = (g_1' g_1)(g_2' g_2)$$

This renders group G abelian.

Eckmann and Hilton, 1962
Physicists 80'ies and 90'ies

Way out: 2-categories, Higher Gauge Theory.

Two operations $\circ$ and $\otimes$ satisfying Interchange Law:

$$(g_1' \otimes g_2') \circ (g_1 \otimes g_2) = (g_1' \circ g_1) \otimes (g_2' \circ g_2) \ .$$

I: The local picture

I.1. Motivation

I.2. Categorification

I.3. Higher gauge algebras: L_∞-algebras

I.4. Homotopy Maurer–Cartan theory

I.5. L_∞-algebras and the BV-formalism

"We'll only use as much category theory as is necessary. Famous last words..."

Dan Abramovich

Constructing higher structures

A mathematical structure ("Bourbaki-style") consists of

- Sets
- Structure Functions
- Structure Equations

Categorification

Sets → Categories

Structure Functions → Structure Functors

Structure Equations → Structure Isomorphisms

\+ Coherence Relations

Note: Process is not unique

- Choice of weakness/strictness:
  - weak: most general
  - semi-, hemi-strict, ...: in between
  - strict: structure isomorphisms trivial
- Choice of coherence relations

Example: Categorified groups

Group:

- Sets: Underlying set G
- Structure: unit $\mathbb{1}$, multiplication, inverse
- Structure equations: associativity, $g^{-1}g = \mathbb{1}$, $\mathbb{1}g = g\mathbb{1} = g$

2-Group:

- Categories: A category $\mathscr{C}$
- Structure functors: unit object $\mathbb{1}$, multiplication $\otimes$, inverse inv
- Structure isomorphisms:
  - associator: $\mathsf{a}_{x,y,z} : (x \otimes y) \otimes z \Rightarrow x \otimes (y \otimes z)$
  - unitors: $\mathsf{l}_x : x \otimes \mathbb{1} \Rightarrow x$ and $\mathsf{r}_x : \mathbb{1} \otimes x \Rightarrow x$
  - $\mathsf{inv}(x) \otimes x \Rightarrow \mathbb{1} \Leftarrow x \otimes \mathsf{inv}(x)$
- Coherence relations ...

more on 2-groups later...

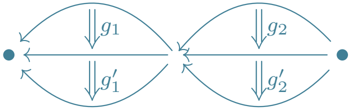
Categorification: Higher dimensional algebra

Higher groups: we are doing higher dimensional algebra.

- Group: can multiply ordered elements in one dimension:

$$a \cdot b \cdot \ldots \cdot d$$

- 2-group: can multiply "vertically" $\circ$ and "horizontally" $\otimes$, i.e. in two dimensions:



$\vdots$

- n-group: can multiply in n dimensions

Example: Lie 2-algebras

Lie algebra:

- Sets: Vector space $\mathfrak{g}$
- Structure: bilinear product $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
- Structure equation:
  - antisymmetry $[u,v] = -[v,u]$
  - Jacobi identity: $[u,[v,w]] - [[u,v],w] - [v,[u,w]] = 0$

Weak Lie 2-algebra: Roytenberg, 2007

- Categories: linear category $\mathfrak{L}$
  (i.e. objects/morphisms are vector spaces)
- Structure functors: functor $[-,-] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$
- Structure isomorphisms:
  - Alternator $\mathsf{Alt} : [v,w] \Rightarrow -[w,v]$
  - Jacobiator $\mathsf{Jac} : [u,[v,w]] - [[u,v],w] - [v,[u,w]] \Rightarrow 0$

more on weak Lie 2-algebras later...

Examples of categorified objects

- Categorified space or 2-space: category $\mathscr{M} = (\mathscr{M}_1 \rightrightarrows \mathscr{M}_0)$
- Lie groupoid: categorified space with invertible morphisms
- 2-group: monoidal category + invertible obj./morph.
- strict 2-group: strict monoidal category + ...
- Strict 2-groups $\cong$ crossed modules of Lie groups
- Categorified principal circle bundles: abelian gerbes
- Categorified principal bundles: principal 2-bundles
- 2-vector spaces: linear category
- Lie 2-algebras: as above
- strict Lie 2-algebras $\cong$ crossed modules of Lie algebras
- semistrict Lie 2-algebras $\cong$ L_∞-algebras in degs. $\{-1,0\}$.
- semistrict Lie n-algebras $\cong$ L_∞-algs. in degs. $\{-n+1,\ldots,0\}$
- weak Lie n-algebras $\cong$ EL_∞-algs. in degs. $\{-n+1,\ldots,0\}$

I: The local picture

From Lie algebras to L_∞-algebras

- First: local and infinitesimal description of higher gauge theory
- Need an analogue of gauge Lie algebra
- An excellent choice: semi-strict higher Lie algebras
- These are the homotopy/∞-algebras of the Lie operad.

From Lie algebras to L_∞-algebras

Lie algebra $\mathfrak{g}$:

$$[-,-]:\mathfrak{g}_\bullet\wedge\mathfrak{g}_\bullet\to\mathfrak{g}_\bullet\ ,$$
$$[a,[b,c]]=[[a,b],c]+[b,[a,c]]$$

Generalize naturally to differential graded Lie alg. $\mathfrak{g}=\bigoplus_{k\in\mathbb{Z}}\mathfrak{g}_k$:

$$\mathrm{d}:\mathfrak{g}_\bullet\to\mathfrak{g}_{\bullet+1}\ ,\quad [-,-]:\mathfrak{g}_\bullet\wedge\mathfrak{g}_\bullet\to\mathfrak{g}_\bullet\ ,$$
$$\mathrm{d}[a,b]=[\mathrm{d}a,b]\pm[a,\mathrm{d}b]\ ,\quad [a,[b,c]]=[[a,b],c]\pm[b,[a,c]]$$

Generalize further to L_∞-algebra:

- Preserve antisymmetry, but lift Jacobi up to homotopy
- Explicitly, introduce $\mu_3:\mathfrak{g}_\bullet\wedge\mathfrak{g}_\bullet\wedge\mathfrak{g}_\bullet\to\mathfrak{g}_{\bullet-1}$ with

$$[a,[b,c]]-[[a,b],c]\mp[b,[a,c]]$$
$$=\mathrm{d}\mu_3(a,b,c)+\mu_3(\mathrm{d}a,b,c)\pm\mu_3(a,\mathrm{d}b,c)\pm\mu_3(a,b,\mathrm{d}c)$$

Systematic construction

Chevalley–Eilenberg algebra of a Lie algebra $\mathfrak{g}$:
Free differential graded commutative algebra $\odot^\bullet \mathfrak{g}[1]^*$

$$\mathsf{CE}(\mathfrak{g}) = \odot^\bullet \mathfrak{g}[1]^* \;,\;\; Q\xi^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} \xi^\beta \xi^\gamma \;\;,\;\; |Q| = 1$$

$Q^2 = 0 \Leftrightarrow$ Jacobi identity

Chevalley–Eilenberg algebra of an L_∞-algebra $\mathfrak{L} = \oplus_{k\in\mathbb{Z}} \mathfrak{L}_k$:
Free differential graded commutative algebra $\odot^\bullet \mathfrak{L}[1]^*$

$$\mathsf{CE}(\mathfrak{L}) = \odot^\bullet \mathfrak{L}[1]^* \;,\;\; Q\xi^\alpha = \sum_n \pm \tfrac{1}{n!} f^\alpha_{\beta_1\dots\beta_n} \xi^{\beta_1} \dots \xi^{\beta_n} \;\;,\;\; |Q| = 1$$

higher brackets: $\mu_n(\tau_{\beta_1}, \dots, \tau_{\beta_n}) = \pm f^\alpha_{\beta_1\dots\beta_n} \tau_\alpha$

$Q^2 = 0 \Leftrightarrow$ homotopy Jacobi identities

Example: semi-strict Lie 2-algebras

2-term L_∞-algebra or semi-strict Lie 2-algebra:

- Underlying vector space: $\mathfrak{L} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0$
- Thus: free dgca $\odot^\bullet \mathfrak{L}[1]$ generated by w^a, v^i of degs. 1 and 2
- Most general differential:

$$Qw^a = -m_i^a v^i - \tfrac{1}{2} m_{bc}^a w^b w^c$$
$$Qv^i = -m_{aj}^i w^a v^j - \tfrac{1}{3!} m_{abc}^i w^a w^b w^c$$

- Structure constants induce higher products μ_i on $\mathfrak{L}$:

$$\mu_1(\tau_i) = m_i^a \tau_a \ , \qquad \mu_2(\tau_a, \tau_b) = m_{ab}^c \tau_c \ ,$$
$$\mu_2(\tau_a, \tau_i) = m_{ai}^j \tau_j \ , \qquad \mu_3(\tau_a, \tau_b, \tau_c) = m_{abc}^i \tau_i$$

- $Q^2 = 0$: Higher or homotopy Jacobi identity, e.g.

$$\mu_2(w_1, \mu_2(w_2, w_3)) + \text{cycl.} = \mu_1(\mu_3(w_1, w_2, w_3))$$

Direct description

L_∞-algebra in "bracket picture":

- Graded vector space
  $\mathfrak{L} = \cdots \oplus \mathfrak{L}_{-2} \oplus \mathfrak{L}_{-1} \oplus \mathfrak{L}_0 \oplus \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots$
- μ_1 is a differential, hence (cochain) complex:
  $$\ldots \xrightarrow{\mu_1} \mathfrak{L}_{-2} \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \mathfrak{L}_2 \xrightarrow{\mu_1} \ldots$$
- Graded totally antisymmetric multilinear products
  $$\mu_i : \wedge^i \mathfrak{L} \to \mathfrak{L} \ , \quad |\mu_i| = 2 - i$$
- Satisfying higher/homotopy Jacobi identity:
  $$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(n)}) = 0$$

Remark on operadic background:

- Actually constructed $\mathcal{L}ie_\infty$-algebras
- Used Koszul duality $\mathcal{L}ie \leftrightarrow \mathcal{C}om$

L_∞-algebras: Homotopy Jacobi Identities

Homotopy Jacobi identity:

$$\sum_{i+j=n} \sum_{\sigma \in \mathrm{Sh}(i,n-i)} \pm \mu_{i+1}(\mu_j(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \ldots, \ell_{\sigma(n)}) = 0$$

First few homotopy Jacobi identities:

- $\mu_1(\mu_1(\ell)) = 0$:
  μ_1 is a differential turning $\mathfrak{L}_\bullet$ into a complex
- $\mu_1(\mu_2(\ell_1, \ell_2)) = \mu_2(\mu_1(\ell_1), \ell_2) \pm \mu_2(\ell_1, \mu_1(\ell_2))$:
  μ_1 is a derivation with respect to μ_2
- $\mu_2(\mu_2(\ell_1, \ell_2), \ell_3) + \text{cycl.} = \pm \mu_1(\mu_3(\ell_1, \ell_2, \ell_3))$:
  Jacobi identity violated in a controlled way, by cocycle
  (last: typical for higher structures.)

L_∞-algebras are generalizations of dg Lie algebras.

L_∞-algebras: Examples

Simple examples:

- Trivial L_∞-algebra: $\ldots \xrightarrow{\mu_1} * \xrightarrow{\mu_1} * \xrightarrow{\mu_1} * \xrightarrow{\mu_1} \ldots$
- Lie algebra $\mathfrak{g}$: $\mathfrak{L}_0 = \mathfrak{g}$, $\ldots \xrightarrow{\mu_1} * \xrightarrow{\mu_1} \mathfrak{g} \xrightarrow{\mu_1} * \xrightarrow{\mu_1} \ldots$
- Lie algebra $\mathfrak{g}$, representation $\rho : \mathfrak{g} \mapsto \mathsf{End}(V)$

$$\mathfrak{L}_0 = \mathfrak{g}\ , \quad \mathfrak{L}_{-1} = V \qquad \mu_1 = 0\ , \quad \begin{array}{c} \mu_2(\ell_1, \ell_2) = [\ell_1, \ell_2] \\ \mu_2(\ell_1, v) = \rho(\ell_1)v \end{array}$$

- de Rham complex on some M: $\mathfrak{L}_\bullet = \Omega^\bullet(M)$, $\mu_1 = \mathrm{d}$, $\mu_2 = 0$

Above: strict L_∞-algebras, that is $\mu_i = 0$ for $i > 2$.

More interesting: string Lie 2-algebra (higher spin Lie algebra)

$$\mathfrak{L}_0 = \mathfrak{spin}(n)\ , \quad \mathfrak{L}_{-1} = \mathbb{R} \qquad \mu_1 = 0\ , \quad \begin{array}{c} \mu_2(\ell_1, \ell_2) = [\ell_1, \ell_2] \\ \mu_3(\ell_1, \ell_2, \ell_3) = (\ell_1, [\ell_2, \ell_3]) \end{array}$$

L_∞-algebras: Strict Morphisms

What are morphisms of L_∞-algebras?

Recall: L_∞-algebra has underlying complex $(\mathfrak{L}_\bullet, \mu_1)$

Naively: Chain maps $\phi_1 : \mathfrak{L} \to \mathfrak{L}'$

$$\begin{tikzcd}
\dots \arrow[r, "\mu_1"] & \mathfrak{L}_{-1} \arrow[r, "\mu_1"] \arrow[d, "\phi_1"] & \mathfrak{L}_0 \arrow[r, "\mu_1"] \arrow[d, "\phi_1"] & \mathfrak{L}_1 \arrow[r, "\mu_1"] \arrow[d, "\phi_1"] & \dots \\
\dots \arrow[r, "\mu_1"] & \mathfrak{L}'_{-1} \arrow[r, "\mu'_1"] & \mathfrak{L}'_0 \arrow[r, "\mu'_1"] & \mathfrak{L}'_1 \arrow[r, "\mu'_1"] & \dots
\end{tikzcd}$$

such that

$$\phi_1(\mu_i(\ell_1, \dots, \ell_i)) = \mu_i(\phi_1(\ell_1), \dots, \phi_1(\ell_1)) \; .$$

These give only strict morphisms, more later.

L_∞-algebras: General morphisms

L_∞-algebras have much more general morphisms:

- Morphisms in Chevalley–Eilenberg-picture clear:

$$\mathsf{CE}(\mathfrak{L}') \xrightarrow{\Phi} \mathsf{CE}(\mathfrak{L}) \;, \quad Q \circ \Phi = \Phi \circ Q'$$

- Morphisms of L_∞-algebras $\phi : \mathfrak{L} \to \mathfrak{L}'$ induced:

$$\phi_i : \mathfrak{L}^{\wedge i} \to \mathfrak{L}' \;, \quad |\phi_i| = 1 - i \;, \quad \phi_{1*} : H^\bullet_{\mu_1}(\mathfrak{L}) \to H^\bullet_{\mu'_1}(\mathfrak{L}')$$

  (One can write out the detailed relations...)

The notion of quasi-isomorphism of chain complexes generalizes:

- L_∞-algebras $\mathfrak{L}$ and $\mathfrak{L}'$ quasi-isomorphic:
  There is a $\phi : \mathfrak{L} \to \mathfrak{L}'$ with $\phi_1 : H^\bullet_{\mu_1}(\mathfrak{L}) \cong H^\bullet_{\mu_1}(\mathfrak{L}')$

Quasi-isomorphisms are the right notion of equivalence.

Useful structural theorems

Strictification/rectification theorem
Any L_∞-algebra is quasi-isomorphic to a strict one
i.e. a differential graded Lie algebra.

But: strictified L_∞-algebra usually much larger than original one.

Homotopy transfer
Given a contracting homotopy from a chain complex $\mathfrak{L}$ to another one $\mathfrak{L}'$, we can transport an L_∞-algebra structure on $\mathfrak{L}$ to $\mathfrak{L}'$.

The latter implies:

Minimal model theorem
Any L_∞-algebra is quasi-isomorphicm to an L_∞-algebra structure on its cohomology, the minimal model.

L_∞-algebras: Inner Products

Inner product on Lie algebra $\mathfrak{g}$: $\langle -,- \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

- positive definite/non-degenerate
- symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1, [\ell_2, \ell_3] \rangle = \langle \ell_2, [\ell_3, \ell_1] \rangle$$

generalized naturally (more later) to

Cyclic structure on L_∞-algebra $\mathfrak{L}$: $\langle -,- \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$

- non-degenerate
- graded symmetric
- bilinear
- satisfying cyclic relation:

$$\langle \ell_1, \mu_i(\ell_2, \ldots, \ell_{1+i}) \rangle = \pm \langle \ell_2, \mu_i(\ell_3, \ldots, \ell_{1+i}, \ell_1) \rangle$$

I: The local picture

I.1. Motivation

I.2. Categorification

I.3. Higher gauge algebras: L_∞-algebras

I.4. Homotopy Maurer–Cartan theory

I.5. L_∞-algebras and the BV-formalism

"One ring to rule them all ..."

L_∞-algebras come with their own gauge theory

Maurer–Cartan equation for differential graded Lie algebra, $(\mathfrak{g}, \mathrm{d})$:

$$\mathrm{d}a + \tfrac{1}{2}[a,a] = 0~, \quad a \in \mathfrak{g}~.$$

Recall: L_∞-algebras are generalizations of dg Lie algebras.

Homotopy Maurer–Cartan eqn: (a: gauge potential $\quad f$: curvature)

$$f := \mu_1(a) + \tfrac{1}{2}\mu_2(a,a) + \tfrac{1}{3!}\mu_3(a,a,a) + \cdots = 0~, \quad a \in \mathfrak{L}_1$$

(Higher) gauge transformations: homotopies.

Bianchi identity:

$$\mu_1(f) - \mu_2(f,a) + \tfrac{1}{2}\mu_3(f,a,a) - \tfrac{1}{3!}\mu_4(f,a,a,a) + \cdots = 0~.$$

Homotopy Maurer–Cartan Action:

$$S_{\mathrm{MC}}[a] := \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \ldots, a) \rangle_{\mathfrak{L}}~.$$

Later: Any (...) field theory is a hMC theory for some L_∞-algebra.

Construction: L_∞-algebras from tensor products

"dg commutative algebra $\otimes$ L_∞-algebra yields L_∞-algebra"

Example: $\Omega^\bullet(M,\mathfrak{L}) := \Omega^\bullet(M)\otimes\mathfrak{L} = \bigoplus_{k\in\mathbb{Z}}\Omega^\bullet_k(M,\mathfrak{L})$:

- $\Omega^\bullet_k(M,\mathfrak{L}) := \Omega^0(M)\otimes\mathfrak{L}_k \oplus \Omega^1(M)\otimes\mathfrak{L}_{k-1}\oplus\cdots\oplus\Omega^d(M)\otimes\mathfrak{L}_{k-d}$
- Higher products:

$$\begin{aligned}\hat{\mu}_1(\alpha_1\otimes\ell_1) &:= \mathrm{d}\alpha_1\otimes\ell_1 \pm \alpha_1\otimes\mu_1(\ell_1)\\ \hat{\mu}_i(\alpha_1\otimes\ell_1,\ldots,\alpha_i\otimes\ell_i) &:= \pm(\alpha_1\wedge\ldots\wedge\alpha_i)\otimes\mu_i(\ell_1,\ldots,\ell_i)\end{aligned}$$

- Cyclic structure for compact manifolds and cyclic $\mathfrak{L}$:

$$\langle\alpha_1\otimes\ell_1,\alpha_2\otimes\ell_2\rangle_{\Omega^\bullet(M,\mathfrak{L})} := \pm\int_M \alpha_1\wedge\alpha_2\ \langle\ell_1,\ell_2\rangle_{\mathfrak{L}}$$

Example: Chern–Simons theory

Tensor product L_∞-algebra $\hat{\mathfrak{L}} = \Omega^\bullet(M) \otimes \mathfrak{g}$ with $\mathfrak{g}$ Lie algebra:

- gauge potential

$$A \in \hat{\mathfrak{L}}_1 = \Omega^1(M) \otimes \mathfrak{g}$$

- higher products:

$$\hat{\mu}_1 = \mathrm{d} \quad \text{and} \quad \mu_2 = \wedge \otimes [-,-]$$

- Homotopy Maurer–Cartan equation:

$$F := \mathrm{d}A + \tfrac{1}{2}[A, A] = 0$$

- Homotopy Maurer–Cartan action:

$$S_{\mathrm{MC}}[A] := \int_M \left\langle \tfrac{1}{2}(A, \mathrm{d}A) + \tfrac{1}{3!}(A, [A, A]) \right\rangle .$$

Example: 4d Higher Chern–Simons theory

For $d = 4$, need cyclic 2-term L_∞-algebra: $\mathfrak{L} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0$.

Tensor product L_∞-algebra $\hat{\mathfrak{L}} = \Omega^\bullet(M) \otimes \mathfrak{L}$:

- gauge potential
$$A + B \in \hat{\mathfrak{L}}_1 = \Omega^1(M) \otimes \mathfrak{L}_0 \quad \oplus \quad \Omega^2(M) \otimes \mathfrak{L}_{-1}$$
- higher products are $\hat{\mu}_1 = \mathrm{d} + \mu_1$, μ_2, μ_3
- Homotopy Maurer–Cartan equation:
$$F = \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \mu_1(B)$$
$$H = \mathrm{d}B + \mu_2(A, B) + \tfrac{1}{3!}\mu_3(A, A, A)$$
- Homotopy Maurer–Cartan action:
$$S_{\mathrm{MC}} = \int_M \Big\{ \langle B, \mathrm{d}A + \tfrac{1}{2}\mu_2(A, A) + \tfrac{1}{2}\mu_1(B) \rangle_{\mathfrak{L}} + \\ + \tfrac{1}{4!}\langle \mu_3(A, A, A), A \rangle_{\mathfrak{L}} \Big\}$$

Generalizes to arbitrary dimensions $d \geq 3$!

Alternative picture

Connection: splitting of Atiyah algebroid sequence

$$0 \longrightarrow P \times_{\mathsf{G}} \mathsf{Lie}(\mathsf{G}) \longrightarrow TP/\mathsf{G} \longrightarrow TM \longrightarrow 0$$

Atiyah, 1957

Related approach: Cartan, Kotov, Strobl, Schreiber, ...

- Gauge potential dually as morphism of graded com algebras:

$$a^* : \mathsf{CE}(\mathfrak{g}) \to \Omega^\bullet(M) \quad , \quad \xi^\alpha \mapsto A^a_\mu \mathrm{d}x^\mu := a^*(\xi^a)$$

- Curvature: failure of a to be morphism of dgcas:

$$F^a := (\mathrm{d} \circ a^* - a^* \circ Q)(\xi^a) = \mathrm{d}A^a + \tfrac{1}{2} f^a_{bc} A^b \wedge A^c$$

- Infinitesimal gauge transformations: flat homotopies
- gca morphisms vs. dgca morphisms a bit strange?

Non-flat connections

- Double CE algebra to Weil algebra $\mathsf{W}(\mathfrak{g}) := \mathsf{CE}(\mathfrak{inn}(\mathfrak{g}))$

$$\mathsf{W}(\mathfrak{g}) := C^\infty(\overset{\sigma\xi^\alpha}{T}[1]\overset{\xi^\alpha}{\mathfrak{g}}[1])\ ,\quad Q = Q_{\mathrm{CE}} + \sigma\ ,\quad \sigma Q_{\mathrm{CE}} = -Q_{\mathrm{CE}}\sigma$$

- Potentials/curvatures/Bianchi identities from dgca-morphisms

$$(A,F) : \mathsf{W}(\mathfrak{g}) \to \Omega^\bullet(M) = W(M)$$
$$\xi^\alpha \mapsto A^\alpha$$
$$(\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \mapsto F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A,A])^\alpha$$
$$Q(\sigma\xi^\alpha) = -f^\alpha_{\beta\gamma}(\sigma\xi^\alpha)\xi^\beta \mapsto (\nabla F)^\alpha = 0$$

- Gauge transformations: homotopies between dgca-morphisms
- Topological invariants: invariant polynomials in $\mathsf{W}(\mathfrak{g})$

Example: higher gauge theory with $\mathfrak{string}(n)$

- Recall:

$$\mathfrak{string}(n) = (\mathbb{R} \xrightarrow{0} \mathfrak{spin}(n)) \ , \ \ \mu_2 = [-,-] \ , \ \ \mu_3 = (-,[-,-])$$

- Weil algebra:

$$\mathsf{W}(\mathfrak{g}) := \mathcal{C}^\infty(T[1]\mathfrak{g}[1]) = \mathcal{C}^\infty(\mathfrak{g}[1] \oplus \mathfrak{g}[2]) \ , \quad \sigma : \mathfrak{g}^*[1] \xrightarrow{\cong} \mathfrak{g}^*[2]$$
$$Q|_{\mathcal{C}^\infty(\mathfrak{g}[1])} = Q_{\rm CE} + \sigma \ , \quad Q_{\rm CE}\sigma = -\sigma Q_{\rm CE}$$

- Potentials/curvatures/Bianchi identities from dgca-morphisms

$$(A,B,F,H) : \mathsf{W}(\mathfrak{g}) \quad \longrightarrow \quad \Omega^\bullet(M) = W(M)$$
$$\xi^\alpha \ \longmapsto \ A^\alpha \in \Omega^1(M) \qquad \text{and} \qquad b \ \longmapsto \ B \in \Omega^2(M)$$
$$(\sigma\xi^\alpha) = Q\xi^\alpha + \tfrac{1}{2} f^\alpha_{\beta\gamma}\xi^\beta\xi^\gamma \ \longmapsto \ F^\alpha = (\mathrm{d}A + \tfrac{1}{2}[A,A])^\alpha$$
$$(\sigma b) = Qb - \tfrac{1}{3!} f_{\alpha\beta\gamma}\xi^\alpha\xi^\beta\xi^\gamma \ \longmapsto \ H = \mathrm{d}B - \tfrac{1}{3!}(A,[A,A])$$

- Bianchi identities: $\nabla F = 0$ and $\mathrm{d}H = -\frac{1}{2}(\mathrm{d}A,[A,A])$
- Gauge trafos and top. invariants derived as above

I: The local picture

Classical BRST-BV formalism

Classical space of observables:

$$\frac{\text{Functionals on fields } \mathfrak{F}}{\text{ideal } \mathfrak{I} := \langle \text{LHS of eom} \rangle \qquad \text{gauge symmetry } \mathfrak{G}}$$

Observation:

- Orbit spaces are often not nice
- Better: derived quotient
  - Consider action groupoid
  - quotient space in cohomology

BRST formalism: Modding out gauge symmetry

Action Lie groupoid ("derived quotient")

$$(\text{symmetry group} \ltimes \text{field space}) \rightrightarrows \text{field space}$$
$$\Phi \xrightarrow{(g,\Phi)} g \rhd \Phi$$

This differentiates to the action Lie algebroid

$$\mathfrak{F}_{\mathrm{BRST}} := \big(\text{Lie(symmetry group)} \ltimes \text{field space} \to \text{field space}\big)$$

BRST complex is the dgca-description of this Lie algebroid.

Chevalley–Eilenberg resolution:

$$0 \longrightarrow \mathscr{C}^\infty(\mathfrak{F}/\mathfrak{G}) \cong H^0(\mathfrak{F}/\mathfrak{G}) \hookrightarrow \mathscr{C}^\infty_0(\mathfrak{F}_{\mathrm{BRST}}) \xrightarrow{Q} \mathscr{C}^\infty_1(\mathfrak{F}_{\mathrm{BRST}}) \xrightarrow{Q} \cdots$$

Antifields: Equations of motion

Classical observables:

field configurations modulo symmetries satisfying eom

- Field space $\mathfrak{F}$
- Enlarged: $\mathfrak{F}_{\rm BV} := T^*[-1]\mathfrak{F}$ coords. fields Φ^A, "antifields" Φ^+_A
- Natural symplectic form, Poisson braacket: "anti-bracket"
- $S_{\rm BV}$ defines $Q_{\rm BV} = \{S_{\rm BV}, -\}$ with $Q^2_{\rm BV} = 0$
- Note: $Q_{\rm BV}\Phi^+_A = \{S_{\rm BV}, \Phi^+_A\} = \delta_{\Phi^A} S$, classical eoms.
- Note: $Q_{\rm BV}(\mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F})) = \mathfrak{I}$, ideal vanishing on solutions

Koszul–Tate resolution:

$$\cdots \xrightarrow{Q} \mathscr{C}^\infty_{-1}(T^*[-1]\mathfrak{F}) \xrightarrow{Q} \mathscr{C}^\infty_0(T^*[-1]\mathfrak{F}) \longrightarrow H^0(T^*[-1]\mathfrak{F}) = \mathscr{C}^\infty(\mathfrak{F})/\mathfrak{I} \longrightarrow 0$$

Classical BRST-BV formalism

Essentially:

Classical BRST-BV formalism = Chevalley–Eilenberg resolution + Koszul–Tate resolution

We have

$$S_{\mathrm{BV}}\ ,\quad Q_{\mathrm{BV}} := \{S_{\mathrm{BV}}, -\}\ ,\quad Q_{\mathrm{BV}}^2 = 0$$

Note:

- BV-complex is a free differential graded commutative algebra
- Dually, it defines an L_∞-algebra
- Its homotopy Maurer–Cartan action is the usual action.
- There is an extension of this picture producing full BV action.

Perturb. quantum field theory and homotopy algebras

There is the following dictionary:

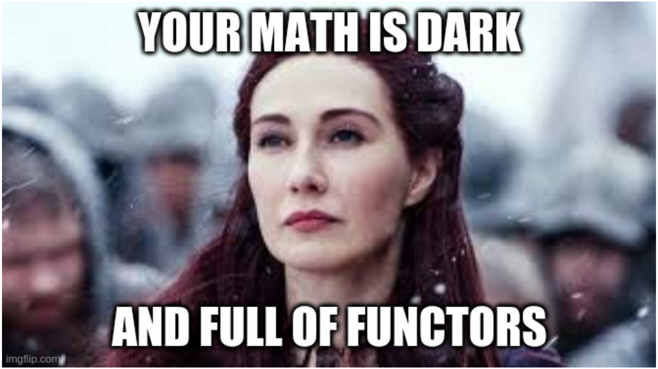
Perturb. quantum field theory	Homotopy algebra
classical action	cyclic L_∞-algebra
tree-level scattering amplitude	minimal model
integrating out fields	homotopy transfer
semi-classical equivalence	quasi-isomorphism
loop level considerations	ext. to loop homotopy algebra
Behrends–Giele recursion	geometric series in homolog. pert. lemma
colour-kinematic duality	homotopy $BV^{\blacksquare}$-algebra
⋮	⋮

Summary

- String/M-theory require higher principal bundles
  Effective description expected to be higher gauge theories
- We can construct new relevant structures by categorification.
- Higher Lie algebras are conveniently modelled by L_∞-algebras.
- L_∞-algebras come with their own gauge theories.
- Any field theory is the gauge theory of an L_∞-algebra.

II: The global picture

II: The global picture



Categorification of groups

Categorification

$$\text{Sets} \to \text{Categories}$$
$$\text{Structure Functions} \to \text{Structure Functors}$$
$$\text{Structure Equations} \to \text{Structure Isomorphisms}$$
$$+ \text{ Coherence Relations}$$

Group $\to$ 2-Group

- Set G $\to$ Category $\mathscr{G}$
- product, identity ($\mathbb{1} : * \to \mathsf{G}$), inverse $\to$ Functors
- $a(bc) = (ab)c \to$ Associator $\mathsf{a} : a \otimes (b \otimes c) \Rightarrow (a \otimes b) \otimes c$
- $\mathbb{1}a = a\mathbb{1} = a \to$ Unitors $\mathsf{l}_a : a \otimes \mathbb{1} \Rightarrow a$, $\mathsf{r}_a : \mathbb{1} \otimes a \Rightarrow a$
- $aa^{-1} = a^{-1}a = \mathbb{1} \to$ weak inv. $\mathsf{inv}(x) \otimes x \Rightarrow \mathbb{1} \Leftarrow x \otimes \mathsf{inv}(x)$

Strict Lie 2-groups

Simplification:

$$\text{strict Lie 2-groups} \xleftrightarrow{1:1} \text{Crossed modules of Lie groups}$$

Lie crossed module

Pair of Lie groups (G,H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd : \mathsf{G} \times \mathsf{H} \to \mathsf{H}$
- group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$ such that

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$$

Equivalence: every strict 2-group is of the form Baez,Lauda 2003

$$\begin{tikzcd} \mathsf{G} \ltimes \mathsf{H} \arrow[r, shift left] \arrow[r, shift right] & \mathsf{G} \end{tikzcd}\ , \qquad \begin{tikzcd} g & & \mathsf{t}(h^{-1})g \arrow[ll, bend right, "{(g,h)}"'] \end{tikzcd}$$

$$(g_1,h_1) \otimes (g_2,h_2) := (g_1 g_2, (g_1 \rhd h_2) h_1)\ ,$$

$$\mathsf{inv}(g_1,h_1) := (g_1^{-1}, g_1^{-1} \rhd h_1^{-1})$$

Examples of Lie Crossed Modules

Lie crossed module

Pair of Lie groups (G, H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd : \mathsf{G} \times \mathsf{H} \to \mathsf{H}$
- group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$ such that

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$.
- Abelian Lie group G, Lie crossed module: $\mathsf{BG} = (\mathsf{G} \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(\mathsf{G} \xrightarrow{\mathsf{t}} \mathsf{Aut}(\mathsf{G}))$

Note:

- CMs of Lie groups differentiate to CMs of Lie algebras.
- CMs of Lie algebras are the same as 2-term dg-Lie algebras.

Example: The Lie 2-group TD_n

TD_n:

$$\mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times \mathsf{U}(1) \rightrightarrows \mathbb{R}^{2n}$$

$$\begin{tikzcd}
\xi & \xi - m_1 \arrow[l, bend right, "{(\xi,m_1,\phi_1)}"'] & \xi - m_1 - m_2 \arrow[l, bend right, "{(\xi-m_1,m_2,\phi_2)}"'] \arrow[ll, bend left, "{(\xi,m_1+m_2,\phi_1+\phi_2)}"]
\end{tikzcd}$$

$$\mathsf{id}_\xi := (\xi, 0, 0)\ , \quad (\xi, m, \phi)^{-1} := (\xi - m, -m, -\phi)$$

$$(\xi_1, m_1, \phi_1) \otimes (\xi_2, m_2, \phi_2) := (\xi_1 + \xi_2, m_1 + m_2, \phi_1 + \phi_2 - \langle \xi_1, m_2 \rangle)$$

$$\mathsf{inv}(\xi, m, \phi) := (-\xi, -m, -\phi - \langle \xi, m \rangle)$$

$$\left\langle \begin{pmatrix} \hat{\xi}_1 \\ \check{\xi}_2 \end{pmatrix}, \begin{pmatrix} \hat{\xi}_1 \\ \check{\xi}_2 \end{pmatrix} \right\rangle = \check{\xi}_1 \hat{\xi}_2$$

As a crossed module of Lie groups:

$$\mathsf{TD}_n \ := \ \left(\mathbb{Z}^{2n} \times \mathsf{U}(1) \xrightarrow{\ \mathsf{t}\ } \mathbb{R}^{2n}\right)\ ,$$

$$\mathsf{t}(m, \phi) = m\ , \quad \xi \rhd (m, \phi) = \phi - \langle \xi, m \rangle\ ,$$

2-group model of the string group

- Higher analogue of the spin group: String group

  Stolz, Teichner, Witten, ...
- Defined (up to homotopy) as the 3-connected cover of $\mathsf{Spin}(n)$
- Whitehead tower, iteratively delete homotopy groups

  $$\ldots \to \mathsf{String}(n) \to \mathsf{Spin}(n) \to \mathsf{Spin}(n) \to \mathsf{SO}(n) \to \mathsf{O}(n)$$
- Definition only up to homotopy, as a group: ∞-dimensional

2-group model of $\mathsf{String}(n)$

2-group $\mathscr{G}$ with 2-group morphism $\Phi : \mathscr{G} \to \mathsf{Spin}(n)$
such that geometric realization of Φ is 3-connected cover.

These 2-groups are perhaps the most important ones.

more later

2-group model of the string group

2-group model of String(n)
2-group $\mathscr{G}$ with 2-group morphism $\Phi : \mathscr{G} \to \mathsf{Spin}(n)$
such that geometric realization of Φ is 3-connected cover.

Strict 2-group model as crossed module Baez et al. (2005)

$$\mathscr{G} = \Big(L_0\widehat{\mathsf{Spin}(n)} \longrightarrow P_0\mathsf{Spin}(n) \Big)$$

- $L_0\mathsf{Spin}(n)$ and $P_0\mathsf{Spin}(n)$ are based loop and path spaces
- $L_0\widehat{\mathsf{Spin}(n)}$ is the Kac–Moody central extension
- Products are pointwise
- Action $\triangleright$ is complicated

There is also a finite-dimensional, weak 2-group model.
It's very complicated and not explicit. Schommer-Pries (2009)

Homomorphisms of 2-groups

Just as L_∞-algebras: many more morphisms!

- As categories: monoidal functors
- As crossed modules: butterflies

Examples:

$$\text{Lie group } \mathsf{U}(1)^{\times n} \quad \cong \quad \mathscr{G} = (\mathbb{Z}^n \longrightarrow \mathbb{R}^n)$$
$$\text{Lie group } \mathsf{G} \quad \cong \quad \mathscr{G} = (L_0\mathsf{G} \longrightarrow P_0\mathsf{G})$$

II: The global picture

Descriptions of principal bundles

Principal bundles can be described in many different ways:

- Total space perspective:
  Fibration $P \to M$, locally trivial, fiberwise G-action, ...
- Čech cocycles:
  Cover $\sqcup_i U_i \to M$, $g_{ij} : U_i \cap U_j \to \mathsf{G}$
- Maps into classifying space:
  $f : M \to \mathsf{BG}$, $P = f^*\mathsf{EG}$
- $\vdots$

All of them are more or less readily categorified.

This yields higher principal bundles or (non-abelian) gerbes.

Principal fiber bundles, topologically

Essentially, all definitions of principal bundles have higher versions.

Here: Čech cocycle description subordinate to a cover

- Surjective submersion $\sigma : Y \twoheadrightarrow X$, e.g. $Y = \sqcup_a U_a$
- Čech groupoid:

$$\check{\mathscr{C}}(\sigma) \;:\; Y \times_X Y \rightrightarrows Y \;, \quad (y_1, y_2) \circ (y_2, y_3) = (y_1, y_3)$$

- Principal G-bundle: functor $g : \check{\mathscr{C}}(\sigma) \to \mathsf{BG} = (\mathsf{G} \rightrightarrows *)$

$$\begin{tikzcd} Y \times_X Y \arrow[r, "g"] \arrow[d, shift left] \arrow[d, shift right] & \mathsf{G} \arrow[d, shift left] \arrow[d, shift right] \\ Y \arrow[r, "*"] & * \end{tikzcd} \qquad g(y_1, y_2) g(y_2, y_3) = g(y_1, y_3)$$

- Equivalences/bundle isomorphisms: natural isomorphisms

Principal 2-bundles, topologically

- Lie 2-group, e.g. $\mathcal{G} = (\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G})$
- Principal G-bundle: weak 2-functors $g : \check{\mathscr{C}}(\sigma) \to \mathsf{B}\mathcal{G}$
- After unpacking this for $\mathcal{G} = (\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G})$:

$$g \in \Omega^0(Y^{[2]}, \mathsf{G}) \quad \text{and} \quad h \in \Omega^0(Y^{[3]}, \mathsf{H})$$

$$\mathsf{t}(h_{abc}) g_{ab} g_{bc} = g_{ac} \quad \text{and} \quad h_{acd} h_{abc} = h_{abd}(g_{ab} \triangleright h_{bcd})$$

- Equivalences/bundle isomorphisms: natural 2-isomorphisms
- Higher bundle isomorphisms: 2-modifications
- Special cases:
  - $\mathsf{H} = *$: principal G-bundle
  - $\mathsf{H} = \mathsf{U}(1)$, $\mathsf{G} = *$: abelian gerbe
- Similarly: groupoid bundles, 2- and n-groupoid bundles

Application: Differentiation of Lie n-groups

With our above definition, we can differentiate Lie 2-groups.

To differentiate Lie n-group $\mathscr{G}$: Ševera (2006)

- Consider moduli of functor defining principal $\mathscr{G}$-bundles:

  $$X \mapsto \text{descent data subordinate to } Y = X \times \mathbb{R}^{0|1} \twoheadrightarrow X$$

  (1-jet of $\mathscr{G}$, "representable presheaf")
- Moduli generate free dg com algebra
- This is Chevalley–Eilenberg algebra of L_∞-algebra $\mathsf{Lie}(\mathscr{G})$

Application: Differentiation of Lie n-groups

$$X \mapsto \text{descent data subordinate to } Y = X \times \mathbb{R}^{0|1} \twoheadrightarrow X$$

Example: Lie group G:

$$g : X \times \mathbb{R}^{0|2} \to \mathsf{G}~, \quad g(\theta_0,\theta_1)g(\theta_1,\theta_2) = g(\theta_0,\theta_2)~.$$

This implies

$$g(0,\theta) = g(\theta,0)^{-1}$$
$$g(\theta_0,\theta_1) = g(\theta_0,0)(g(\theta_1,0))^{-1}$$
$$g(\theta_0,0) = \mathbb{1} + \alpha\theta_0~, \quad \alpha \in \mathsf{Lie}(\mathsf{G})[1]~.$$

$$g(\theta_0,\theta_1) = \mathbb{1} + \alpha(\theta_0 - \theta_1) + \tfrac{1}{2}[\alpha,\alpha]\theta_0\theta_1~.$$

Differential:

$$Qg(\theta_0,\theta_1) := \frac{\partial}{\partial\varepsilon} g(\theta_0+\varepsilon,\theta_1+\varepsilon)$$
$$Q\alpha ~= -\tfrac{1}{2}[\alpha,\alpha]$$

Observations

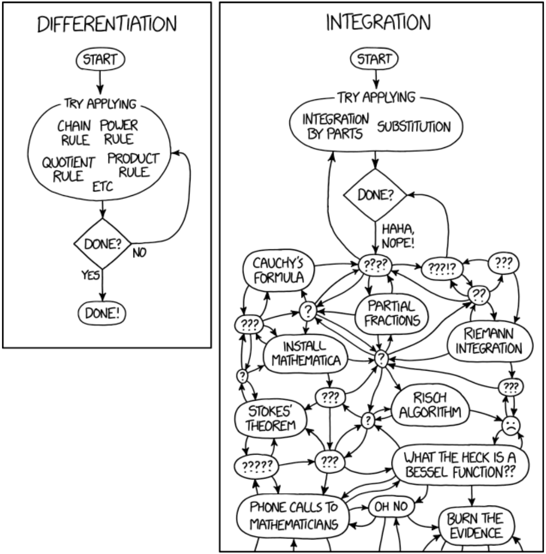
- Differentiating a CM of Lie groups yields CM of Lie algebras.
- Differentiating a weak Lie 2-group yields 2-term L_∞-algebra.
- There is a formal integration procedure; results very large

Classification results: Baez, Lauda and Baez, Crans (2003)

- Any Lie 2-group equivalent to "minimal" 2-group given by Lie group G, Representation V, cocycle $H^3(\mathsf{G}, V)$
- Any semi-strict Lie 2-algebra equivalent to minimal model Lie algebra $\mathfrak{g}$, Representation V, cocycle $H^3(\mathfrak{g}, V)$

This does not integrate!

In particular:
$\mathsf{String}(3)$ differentiating to minimal $\mathfrak{string}(3)$ very complicated!
Schommer-Pries (2009)


DIFFERENTIATION
START
TRY APPLYING
CHAIN RULE
POWER RULE
QUOTIENT RULE
PRODUCT RULE
ETC
DONE?
NO
YES
DONE!
INTEGRATION
START
TRY APPLYING
INTEGRATION BY PARTS
SUBSTITUTION
DONE?
HAHA, NOPE!
CAUCHY'S FORMULA
????
???!?
???
??
?
PARTIAL FRACTIONS
???
INSTALL MATHEMATICA
RIEMANN INTEGRATION
?
?
???
???
RISCH ALGORITHM
STOKES' THEOREM
?
WHAT THE HECK IS A BESSEL FUNCTION??
?????
???
PHONE CALLS TO MATHEMATICIANS
OH NO
BURN THE EVIDENCE

II: The global picture

A bit harder: Connections

Many ways of defining connections on principal 2-bundles:

- Algebraic geometry — Breen, Messing (2005)
- Higher bundle gerbes — Aschieri, Cantini, Jurčo (2005)
- Formally integrating infinitesimal description — Fiorenza, Sati, Schreiber (2012)
- Isomorphisms of Maurer–Cartan forms (enhancing Ševera's differentiation) — Jurčo, CS, Wolf (2014)
- Forms on total 2-space — Waldorf (2016)
- Atiyah algebroid sequence — M Farahani, CS (soon)

They all yield essentially the same objects.

A bit harder: Connections

Data obtained for 2-group $\mathsf{G} \ltimes \mathsf{H} \rightrightarrows \mathsf{G}$ and Lie 2-algebra $\mathfrak{g} \ltimes \mathfrak{h} \rightrightarrows \mathfrak{g}$:

$h \in \Omega^0(Y^{[3]}, \mathsf{H})$ $\quad \Lambda \in \Omega^1(Y^{[2]}, \mathfrak{h})$ $\quad B \in \Omega^2(Y, \mathfrak{h})$ $\quad \delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$

$g \in \Omega^0(Y^{[2]}, \mathsf{G})$ $\quad A \in \Omega^1(Y, \mathfrak{g})$

Local curvature forms as in the infinitesimal case:

$$F_a = \mathrm{d}A_a + \tfrac{1}{2}[A_a, A_a] - \mathsf{t}(B_a)$$
$$H_a = \mathrm{d}B_a + A_a \rhd B_a + T_\delta$$

- Note: δ sticks out unnaturally
- Dropped in most later work (Schreiber, Waldorf (2009), ...)
- Price to pay: $F_a = 0$
- Gauge transformations differentiate to infinitesimal description

Principal 2-Bundles

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \triangleright h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \triangleright h'_{abc})h_{ab}(g_{ab} \triangleright h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a]$	$F_a = \mathrm{d}A_a + \frac{1}{2}[A_a, A_a] - \mathsf{t}(B_a) \overset{!}{=} 0$ $H_a = \mathrm{d}B_a + A_a \triangleright B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \triangleright B_a + \tilde{A}_a \triangleright \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a (flat) principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = \mathsf{BU}(1)$-bundle is an abelian gerbe.

Side remark: Avoiding higher geometry with loop spaces

An abelian gerbe over M "is" a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

3-form $H = dd(\mathscr{G})$ on $M \xrightarrow{\mathcal{T}}$ 2-form $F = c_1(P)$ on $\mathcal{L}M$

Consider the following double fibration:

$$\begin{tikzcd} & \mathcal{L}M \times S^1 \arrow[dl, "ev"'] \arrow[dr, "pr"] & \\ M & & \mathcal{L}M \end{tikzcd}$$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: reparameterization invariant, chain map, ...

Can't live with or without fake curvature?

$$\mathcal{F} := \mathrm{d}A + \tfrac{1}{2}[A,A] + \mathsf{t}(B) \stackrel{!}{=} 0$$

Without this condition:

- Higher parallel transport is not reparameterization invariant
- Closure of gauge transformations and composition of cocycles:

$$(g_{23}^{-1}g_{12}^{-1}) \rhd (h_{123}^{-1}(\mathcal{F}_1 \rhd h_{123})) \stackrel{!}{=} 0$$

- 6d Self-duality equation $H = \star H$ is not gauge-covariant:

$$H \to \tilde{H} = g \rhd H - \mathcal{F} \rhd \Lambda$$

With this condition:

- Principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is flat principal G-bundle.
- Higher connections are locally abelian!

Gastel (2019), CS, Schmidt (2020)

- Reason for bias against non-abelian gerbes in string theory(?)

Argument

- Lie 2-group (crossed module) $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G}, \rhd)$, $\quad (\mathfrak{h} \xrightarrow{\mathsf{t}} \mathfrak{g}, \rhd)$
- Potential forms: $A \in \Omega^1(\mathbb{R}^d, \mathfrak{g})$, $B \in \Omega^2(\mathbb{R}^d, \mathfrak{h})$
- Fake flatness: $\mathcal{F} := \mathrm{d}A + \frac{1}{2}[A, A] + \mathsf{t}(B) = 0$
- Gauge transformations: $g \in \Omega^0(\mathbb{R}^d, \mathsf{G})$, $\Lambda \in \Omega^1(\mathbb{R}^d, \mathfrak{h})$

$$A \mapsto \tilde{A} = g^{-1}Ag + g^{-1}\mathrm{d}g + \mathsf{t}(\Lambda_1)$$

$$B \mapsto \tilde{B} = g^{-1} \rhd B + \mathrm{d}\Lambda_1 + \tilde{A} \rhd \Lambda_1 + \tfrac{1}{2}[\Lambda_1, \Lambda_1]$$

- A and gauge transformations restrict to $\mathsf{G}^\circ = \mathsf{G}/\mathrm{im}(\mathsf{t})$
- $F^\circ = 0$ and non-abelian Poincaré lemma: gauge with $\tilde{A}^\circ = 0$
- $\tilde{A} \in \mathrm{im}(\mathsf{t})$, gauge away with Λ-transformation: $\tilde{\tilde{A}} = 0$
- connection is abelian with $\tilde{\tilde{B}} \in \ker(\mathsf{t})$!

CS, Schmidt (2020), see also Gastel (2018)

Comparing to physics: heterotic supergravity

How to construct "good" curvatures for non-abelian gauge potentials in presence of Kalb–Ramond B-field?

Answers in the literature:

- Use Chern-Simons terms:

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A]\ , \quad H = \mathrm{d}B + (A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])$$

Bergshoeff et al. (1982), Chapline et al. (1983)

- This is at odds with the "conventional" non-abelian gerbes:

$$F = \mathrm{d}A + \tfrac{1}{2}[A, A]\ , \quad H = \mathrm{d}B - \tfrac{1}{3}(A, [A, A])$$

Breen/Messing (2001), Aschieri, Cantini, Jurco (2003)

Local adjustment for skeletal string algebra

Example: Skeletal string Lie 2-algebra: $\mathfrak{string}(n) = (\mathbb{R} \to \mathfrak{spin}(n))$

- Adjusted Weil algebra: Fiorenza, Sati, Schreiber (2011)

$$Q_{\mathrm{W}} t^\alpha = -\tfrac{1}{2} f^\alpha_{\beta\gamma} t^\beta t^\gamma + \hat{t}^\alpha \qquad Q_{\mathrm{W}} r = \tfrac{1}{3!} f_{\alpha\beta\gamma} t^\alpha t^\beta t^\gamma - \kappa_{\alpha\beta} t^\alpha \hat{t}^\beta + \hat{r}$$

$$Q_{\mathrm{W}} \hat{t}^\alpha = -f^\alpha_{\beta\gamma} t^\beta \hat{t}^\gamma \qquad Q_{\mathrm{W}} \hat{r} = \kappa_{\alpha\beta} \hat{t}^\alpha \hat{t}^\beta$$

- Adjustment governed by Killing form $\kappa_{\alpha\beta}$.
- Gauge potentials:

$$(A, B) \in \Omega^1(U) \otimes \mathfrak{g} \oplus \Omega^2(U)$$

- Curvatures:

$$\begin{aligned} F &:= \mathrm{d}A + \tfrac{1}{2}[A, A] \\ H &:= \mathrm{d}B - \tfrac{1}{3!}\mu_3(A, A, A) + \chi_{\mathrm{sk}}(A, F) \\ &= \mathrm{d}B + \underbrace{(A, \mathrm{d}A) + \tfrac{1}{3}(A, [A, A])}_{\mathrm{cs}(A)} \end{aligned}$$

- Induces modified gauge transformations and Bianchi identities.

General local adjustment

Adjustment (local form) CS, Schmidt (2020)

An adjustment is a redefinition of the Weil algebra preserving the projection onto the Chevalley–Eilenberg algebra such that the resulting gauge transformations close generically.

Physicists would say:

- Unadjusted Weil algebra $\rightarrow$ BRST complex is open
- Adjusted Weil algebra $\rightarrow$ BRST complex closes off-shell

Remarks:

- Appears also for Lie algebroid gauge theories
- Extends to higher L_∞-algebras
- Examples later

Global adjustment

Many (not all!) higher gauge groups come with

Adjustment of 2-group $\mathcal{G}$: CS, Schmidt (2020), Rist, CS, Wolf (2022)

Map $\kappa : \mathcal{G} \times \mathsf{Lie}(\mathcal{G}) \to \mathsf{Lie}(\mathcal{G})$ of degree -1 such that

$$
\begin{aligned}
&(g_2^{-1} g_1^{-1}) \rhd (h^{-1}(X \rhd h)) + g_2^{-1} \rhd \kappa(g_1, X) \\
&\qquad + \kappa(g_2, g_1^{-1} X g_1 - \mathsf{t}(\kappa(g_1, X))) - \kappa(\mathsf{t}(h) g_1 g_2, X) \;=\; 0
\end{aligned}
$$

for all $g_{1,2} \in \mathcal{G}_0$ and $X \in \mathsf{Lie}(\mathcal{G})_0$.

Remarks:

- Adjustment is additional algebraic datum
- Necessary for consistent definition of invariant polynomials.
- specifies $\delta \in \Omega^2(Y^{[2]}, \mathfrak{h})$ in terms of g and F
- Adjustment of curvature/cocycle/coboundary relations
- Can drop fake flatness condition, all problems go away

Fully adjusted principal 2-bundles

One can construct a global picture

Dominik Rist, CS, Martin Wolf, (2022)

Cocycle description

$$
\begin{aligned}
h_{ikl}h_{ijk} &= h_{ijl}(g_{ij} \rhd h_{jkl}) \\
g_{ik} &= \mathsf{t}(h_{ijk})g_{ij}g_{jk} \\
\Lambda_{ik} &= \Lambda_{jk} + g_{jk}^{-1} \rhd \Lambda_{ij} - g_{ik}^{-1} \rhd (h_{ijk}\nabla_i h_{ijk}^{-1}) \ , \\
A_j &= g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}\mathrm{d}g_{ij} - \mathsf{t}(\Lambda_{ij}) \ , \\
B_j &= g_{ij}^{-1} \rhd B_i + \mathrm{d}\Lambda_{ij} + A_j \rhd \Lambda_{ij} + \tfrac{1}{2}[\Lambda_{ij}, \Lambda_{ij}] - \kappa(g_{ij}, F_i)
\end{aligned}
$$

Example: Principal bundles with connection as higher bundles

- Gauge group G is equivalent to 2-group $\mathcal{G} = (L_0\mathsf{G} \to P_0\mathsf{G})$
- $\kappa(g, \alpha) : p \mapsto (\mathbb{1} - p \cdot \flat)(g^{-1}\alpha g - \alpha)$
  where G matrix Lie group, $\flat g$ denotes the endpoint of path g

Properties of adjustment

Without Adjustment	With Adjustment
Gauge transformations only close generically if $F = 0$	Gauge transformations close
Parallel transport only consistent if $F = 0$	Parallel transport consistent
$H = \star H$ only transforms covariantly if $F = 0$	$H = \star H$ transforms covariantly
Matches SUGRA expectations only if $F = 0$	Matches SUGRA expectations

Further properties:

- Adjustments also improve properties of invariant polynomials
- $\exists$ higher gauge groups that do not allow for an adjustment
- There is an adjusted version of parallel transport.

Teaser for part III

Evident question:

Where do the structure constants for adjustment come from?

Observation:
There is a family of quasi-isomorphic weak Lie 2-algebras

$$\mathfrak{string}^{\mathrm{wk},\alpha}_{\mathrm{sk}}(\mathfrak{g}) \;:=\; (\mathbb{R} \xrightarrow{0} \mathfrak{g}) \;,$$
$$\varepsilon_1(r) = 0 \;,$$
$$\varepsilon_2(x_1, x_2) = [x_1, x_2] \;, \quad \varepsilon_2(x_1, r) = 0 \;,$$
$$\varepsilon_3(x_1, x_2, x_3) = (1-\alpha)(x_1, [x_2, x_3]) \;,$$
$$\mathsf{alt}(x_1, x_2) = -2\alpha(x_1, x_2)$$

Conjecture:

Adjustment data from alternators in weak Lie n-algebras

II: The global picture

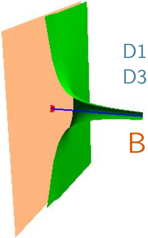
Dirac Monopole

Dirac Monopole

- Dirac postulated in 1931 a new particle: magnetic monopole
- Hopf discovered in 1931 the principal $\mathsf{U}(1)$-bundle $S^3 \to S^2$
- "Fundamental" circle bundle (c_1=volume form on S^2)
- Total space carries group structure: $S^3 \cong \mathsf{SU}(2)$.

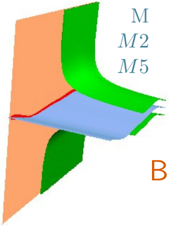
The self-dual string

Question: What is a monopole in M-theory?



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

Bogomolny monopole eqn.
$F = \nabla^2 = *\nabla\Phi$ on $\mathbb{R}^3$

Abelian self-dual string eqn.
$H := \mathrm{d}B = *\mathrm{d}\Phi$ on $\mathbb{R}^4$

Principal fiber bundles from cosets

Useful construction:

- Symmetric space G/H
  i.e. H is subgroup of stabilizer G^σ of some involution σ
- Implies Cartan decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \ , \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m} \ , \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

- Principal bundle $P = \mathsf{G} \to \mathsf{G}/\mathsf{H}$
- Connection on P given by $\mathrm{pr}_{\mathfrak{h}}(\Theta_{\mathsf{G}})$
- Curvature is 2-form on G/H due to Maurer–Cartan eqn.

Examples: Trautman (1977)

- Dirac monopole: $\mathsf{SU}(2) \to \mathsf{SU}(2)/\mathsf{U}(1) \cong S^2$
- Doubled instanton: $\mathsf{Spin}(5) \to \mathsf{Spin}(5)/\mathsf{Spin}(4) \cong S^4$
- Many other solutions to YM and YM-Einstein.

"Categorifying" monopoles and instantons

- Dirac monopole: $\mathsf{Spin}(3) \to \mathsf{Spin}(3)/\mathsf{Spin}(2) \cong S^2$
- Doubled instanton: $\mathsf{Spin}(5) \to \mathsf{Spin}(5)/\mathsf{Spin}(4) \cong S^4$

Idea: Replace $\mathsf{Spin}(n)$ with $\mathsf{String}(n)$ Roberts (2014)

Results:

- Higher Dirac monop.: $\mathsf{String}(3) \to \mathsf{String}(3)/\mathsf{String}(2) \cong S^2$
- Higher Instanton: $\mathsf{String}(5) \to \mathsf{String}(5)/\mathsf{String}(4) \cong S^4$

Beyond topology:

- Full adjusted differential cocycle data Rist, CS, Wolf (2022)
- First ... example non-Abelian principal 2-bundle
- String structure on S^4
- The latter is both a higher instanton and the non-abelian self-dual string, i.e. a higher non-abelian monopole.

Summary

- We categorified Lie groups to Lie 2-groups.
- A useful description were crossed modules of Lie groups.
- Analogue of Čech cocycles for principal 2-bundles clear
- Could use these to differentiate Lie 2-groups.
- Connections are harder; naive definitions lead to problems.
- Adjustment: modify curvatures such that gauge transformations close generically
- Solves all problems, in particular requirement of fake-flatness
- Explicit example of a principal 2-bundle with connection: Higher monopole or fundamental non-abelian self-dual string

III: Applications

III.1. T-duality with higher spaces
III.2. Penrose–Ward transform
III.3. 6d superconformal field theories
III.4. Tensor hierarchies

III: Applications

Outline T-duality

T-duality

Roughly: Equivalence of string theory for strings moving in two different spaces with 1-cycles.

- Simplest example: $\check{M} = M^9 \times T^1$ and $\hat{M} = M^9 \times (T^1)^*$
- Low-energy limit: supergravity with fields (g, B, ϕ)
- Metric g: Kaluza–Klein metric from connection on $\mathsf{U}(1)$-bundle
- 2-form B-field connective structure on a gerbe

Geometric string background:

- A (Riemannian) manifold X
- A principal/affine torus bundle $\pi : P \to X$ (with connection)
- An abelian gerbe (with connection) $\mathscr{G}$ on the total space of P

Topological T-duality

E.g. for a principal circle bundle $\check{P} \to X$ and gerbe $\check{\mathscr{G}} \to \check{P}$:

$$\begin{tikzcd} \overset{\check{H}\in \mathrm{H}^3(P,\mathbb{Z})}{\check{\mathscr{G}}} \arrow[r] & \overset{\check{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\check{P}} \arrow[dr, "\check{\pi}"] & \\ & & X \end{tikzcd}$$

Recall:

- Principal circle bundles over X:
  characterized by 1st Chern class $c_1 = \check{F} \in \mathrm{H}^2(X, \mathbb{Z})$
- Abelian gerbe over $\check{P}$:
  characterized by Dixmier–Douady class $dd = \check{H} \in \mathrm{H}^3(\check{P}, \mathbb{Z})$

Topological T-duality

E.g. for a principal circle bundle $\check{P} \to X$ and gerbe $\check{\mathscr{G}} \to \check{P}$:

$$\begin{tikzcd} \overset{H \in \mathrm{H}^3(\check{P},\mathbb{Z})}{\check{\mathscr{G}}} \arrow[r] & \overset{F \in \mathrm{H}^2(X,\mathbb{Z})}{\check{P}} \arrow[dr, "\check{\pi}"] & \\ & & X \end{tikzcd}$$

Topological T-duality from exactness of the Gysin sequence

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

Bouwknegt, Evslin, Hannabuss, Mathai (2004)

Topological T-duality

E.g. for a principal circle bundle $\check{P} \to X$ and gerbe $\check{\mathscr{G}} \to \check{P}$:

$$
\begin{tikzcd}
\overset{\check{H}\in \mathrm{H}^3(\check{P},\mathbb{Z})}{\check{\mathscr{G}}} \arrow[r] & \overset{\check{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\check{P}} \arrow[dr, "\check{\pi}"] & & \overset{\hat{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\hat{P}} \arrow[dl, "\hat{\pi}"'] \\
& & X &
\end{tikzcd}
$$

Topological T-duality from exactness of the Gysin sequence

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

1) Pushforward $\check{\pi}_*\check{H}$ yields Chern class $\hat{F}$ of new circle bundle

Topological T-duality

E.g. for a principal circle bundle $\check{P} \to X$ and gerbe $\check{\mathscr{G}} \to \check{P}$:

$$\begin{tikzcd}
\overset{\check{H}\in \mathrm{H}^3(\check{P},\mathbb{Z})}{\check{\mathscr{G}}} \arrow[r] & \overset{\check{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\check{P}} \arrow[dr, "\check{\pi}"] & & \overset{\hat{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\hat{P}} \arrow[dl, "\hat{\pi}"'] & \overset{\hat{H}\in \mathrm{H}^3(\hat{P},\mathbb{Z})}{\hat{\mathscr{G}}} \arrow[l] \\
& & X & &
\end{tikzcd}$$

Topological T-duality from exactness of the Gysin sequence

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\,\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

1) Pushforward $\check{\pi}_*\check{H}$ yields Chern class $\hat{F}$ of new circle bundle
2) $\check{F} \cup \hat{F} = \hat{F} \cup \check{F} = 0$, so $\check{F} = \hat{\pi}_*\hat{H}$ for some $\hat{H}$ $\to$ new gerbe

Topological T-duality

E.g. for a principal circle bundle $\check{P} \to X$ and gerbe $\check{\mathscr{G}} \to \check{P}$:

$$\begin{tikzcd}
\overset{\check{H}\in \mathrm{H}^3(\check{P},\mathbb{Z})}{\check{\mathscr{G}}} \arrow[r] & \overset{\check{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\check{P}} \arrow[dr, "\check{\pi}"] & & \overset{\hat{F}\in \mathrm{H}^2(X,\mathbb{Z})}{\hat{P}} \arrow[dl, "\hat{\pi}"'] & \overset{\hat{H}\in \mathrm{H}^3(\hat{P},\mathbb{Z})}{\hat{\mathscr{G}}} \arrow[l] \\
& & X & &
\end{tikzcd}$$

Topological T-duality from exactness of the Gysin sequence

$$\ldots \to \mathrm{H}^3(X,\mathbb{Z}) \xrightarrow{\pi^*} \mathrm{H}^3(P,\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{H}^2(X,\mathbb{Z}) \xrightarrow{F\cup} \mathrm{H}^4(X,\mathbb{Z}) \to \ldots$$

1) Pushforward $\check{\pi}_*\check{H}$ yields Chern class $\hat{F}$ of new circle bundle
2) $\check{F}\cup\hat{F} = \hat{F}\cup\check{F} = 0$, so $\check{F} = \hat{\pi}_*\hat{H}$ for some $\hat{H}$ $\to$ new gerbe

Topological T-duality here:

$$(\check{F},\check{H}) = (\pi_*\hat{H},\check{H}) \longleftrightarrow (\hat{F},\hat{H}) = (\pi_*\check{H},\hat{H})$$

Severe topology change, "$M\times S^1$"-backgrounds not sufficient!

Topological T-duality, geometrically

T-correspondence:

$$
\begin{tikzcd}
 & & \mathscr{G}_{\mathrm{C}} = \check{\mathsf{p}}^*\check{\mathscr{G}} \otimes \hat{\mathsf{p}}^*\hat{\mathscr{G}}^{-1} \cong \mathcal{I} \arrow[d] & & \\
 & & \check{P} \times_X \hat{P} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & & \\
\check{\mathscr{G}} \arrow[r] & \check{P} \arrow[dr, "\check{\pi}"] & & \hat{P} \arrow[dl, "\hat{\pi}"'] & \hat{\mathscr{G}} \arrow[l] \\
 & & X & &
\end{tikzcd}
$$

Bunke, Rumpf, Schick (2005, 2006)

Principal 2-bundles (without connections) over X:

$$
\begin{tikzcd}
 & \mathscr{P}_{\mathrm{C}} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\
\check{\mathscr{P}} & & \hat{\mathscr{P}}
\end{tikzcd}
$$

Nikolaus, Waldorf (2018)

A 2-group fibration

- String backgrounds: principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles for 2-group $\mathsf{TB}_n^{\mathsf{F2}}$
- There is an 2-group TD_n with $\mathsf{BTD}_n \cong \mathsf{BTB}_n^{\mathsf{F2}}$:

$$\mathsf{TD}_n \;:=\; \big(\mathbb{Z}^{2n} \times \mathsf{U}(1) \xrightarrow{\;\mathsf{t}\;} \mathbb{R}^{2n}\big)\ ,$$
$$\mathsf{t}(m,\phi) = m\ , \quad \xi \rhd (m,\phi) = \phi - \langle \xi, m\rangle\ ,$$

- T-duality 2-group $\mathscr{GO}(d,d;\mathbb{Z})$ is autom. 2-group of TD_n
- Double fibration of 2-groups:

$$\begin{tikzcd} & \mathsf{TD}_n \arrow[dl, "\check{\phi}"'] \arrow[dr, "\hat{\phi}"] & \\ \mathsf{TB}_n^{\mathsf{F2}} & & \mathsf{TB}_n^{\mathsf{F2}} \end{tikzcd}$$

  - $\hat{\phi}$: strict morphism
  - $\check{\phi} = \hat{\phi} \circ \phi_{\mathsf{flip}}$ with $\phi_{\mathsf{flip}} : \mathsf{TD}_n \to \mathsf{TD}_n$

Geometric T-duality: Topological picture

2-group double fibration induces double fib. of principal 2-bundles:

$$\begin{tikzcd} & \mathscr{P}_{\mathrm{C}} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\ \check{\mathscr{P}} & & \hat{\mathscr{P}} \end{tikzcd}$$

- $\mathscr{P}_C$ is a principal TD_n-bundle
- $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$ are principal $\mathsf{TB}_n^{\mathsf{F2}}$-bundles
- Gerbe and circle fibration combined into 2-bundles $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- This describes geometric ($F^2 \leftrightarrow F^2$) topological T-duality

Nikolaus, Waldorf (2018)

Geometric T-duality: Full Picture

$$\begin{tikzcd} & \mathscr{P}_{\mathrm{C}} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\ \check{\mathscr{P}} & & \hat{\mathscr{P}} \end{tikzcd}$$

- Differential refinement: i.e. (g, B, ϕ) Kim, CS (2022)
  - TD_n comes with very natural adjustment map: $\langle -, - \rangle$
  - (interestingly, $\mathsf{TB}_n^{\mathsf{F2}}$ does not...)
  - Have topological and full connection data on $\mathscr{P}_C$
  - Can reconstruct gerbe and bundle data on $\check{\mathscr{P}}$ and $\hat{\mathscr{P}}$
- Generalization to affine torus bundles: use $\mathsf{GL}(n, \mathbb{Z}) \ltimes \mathsf{TD}_n$
- Extend further to groupoid bundle to accommodate dilaton ϕ

Explicit example: Geometric T-duality with nilmanifolds

Kim, CS (2022)

$$
\begin{tikzcd}
 & \mathscr{P}_{\mathrm{C}} \arrow[dl, "\check{\mathsf{p}}"'] \arrow[dr, "\hat{\mathsf{p}}"] & \\
\check{\mathscr{P}} & & \hat{\mathscr{P}}
\end{tikzcd}
$$

Lie 2-group:

$$\mathsf{TD}_1 \;:=\; \big(\mathbb{Z}^2 \times \mathsf{U}(1) \xrightarrow{\mathsf{t}} \mathbb{R}^2\big)$$

Topological cocycle data:

$$g = \begin{pmatrix} \hat{\xi} \\ \check{\xi} \end{pmatrix}\,, \quad \begin{aligned} \hat{\xi}(x,y;x',y') &= \ell(x'-x)y\,, \\ \check{\xi}(x,y;x',y') &= k(x'-x)y\,, \end{aligned}$$

$$h = \begin{pmatrix} \hat{m} \\ \check{m} \\ \phi \end{pmatrix}\,, \quad \begin{aligned} \hat{m}(x,y;x',y';x'',y'') &= -\ell(x''-x')(y'-y) \\ \check{m}(x,y;x',y';x'',y'') &= -k(x''-x')(y'-y) \\ \phi &= \tfrac{1}{2}k\ell\big(y'(xx''-xx'-x'x'')-(x''-x')(y'^2-y^2)x\big) \end{aligned}$$

Cocycle data of differential refinement:

$$A = \begin{pmatrix} \check{A} \\ \hat{A} \end{pmatrix} = \begin{pmatrix} kx\,\mathrm{d}y \\ \ell x\,\mathrm{d}y \end{pmatrix}\,, \quad B = 0\,, \quad \Lambda = \tfrac{1}{2}k\ell(xx'\,\mathrm{d}y + (xy + x'y' + y^2(x'-x))\,\mathrm{d}x)$$

Reconstruction procedure for both string backgrounds fully.

Buscher rules

Full verification:

- This formalism reproduces the Buscher rules locally.

  Waldorf (2022).

Altogether:

Full description of geometric T-duality with non-trivial topology

Also:

Extension to non-geometric T-dualities

III: Applications

Twistor description of higher gauge theories

Recall the principle of the Penrose-Ward transform:

- Interested in field equations that are equivalent to flatness of connections along subspaces of spacetime M
- Establish a double fibration

$$
\begin{tikzcd}
 & F \arrow[dl] \arrow[dr] & \\
P & & M
\end{tikzcd}
$$

  P: twistor space, moduli space of subspaces in M
  F: correspondence space
- $H^n(P, \mathfrak{S})$ (e.g. vector bundles) $\xleftrightarrow{1:1}$ sols. to field equations.
- Explicitly appearing: gauge transformations, moduli, symmetries of the equations, etc.

Known Examples of Twistor Descriptions

$$\begin{tikzcd} & \mathbb{C}^4 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ \mathbb{C}P^3_\circ & & \mathbb{C}^4 \end{tikzcd}$$

Instantons
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^3 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ T\mathbb{C}P^1 & & \mathbb{C}^3 \end{tikzcd}$$

Monopoles
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ P^{5|6} & & \mathbb{C}^{4|12} \end{tikzcd}$$

(Super) Yang-Mills
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^6 \times \mathbb{C}P^3 \arrow[dl] \arrow[dr] & \\ P^6 & & \mathbb{C}^6 \end{tikzcd}$$

abelian $H = \star H$
hol. gerbe

Hughston, Murray, Eastwood, CS & M.Wolf, Mason et al.

Note: last twistor space reduces nicely to the above ones.

Idea: Put a non-abelian gerbe on the last twistor space.

Penrose–Ward transform with gerbes

$$\begin{tikzcd} & \mathbb{C}^{6|16} \times \mathbb{C}P^3 \arrow[dl] \arrow[dr] & \\ P^{6|4} & & \mathbb{C}^{6|16} \end{tikzcd}$$

non-abelian self-dual tensor multiplet
hol. principal 2-bundle
hol. principal 3-bundle

CS & M Wolf, 1205.3108, 1305.4870

Note:

- $P^{6|4}$ is a straightforward SUSY generalization of P^6
- EOMs, abelian: $H = \star H$, $F = \mathsf{t}(B)$, $\nabla\!\!\!/ \psi = 0$, $\Box\phi = 0$
- $\mathcal{N} = (2,0)$ SC non-abelian tensor multiplet EOMs!
- EOMs on superspace seem to restrictive...

Detailed Penrose-Ward Transform

- Double fibration: $P^6 \xleftarrow{\pi_1} \mathbb{C}^6 \times \mathbb{C}P^3 \xrightarrow{\pi_1} \mathbb{C}^6$
- M^4-trivial holomorphic gerbe over P^6:

$$g_{ab} = \mathsf{t}(h_{ab}^{-1}) g_a g_b^{-1} \quad \text{and} \quad h_{abc} = h_{ac}^{-1} h_{ab} (g_{ab} \rhd h_{bc})$$

Penrose-Ward transfrom:

- Introduce relative differential forms along π_1

$$a_a := g_a^{-1} \mathrm{d}_{\pi_1} g_a \quad \text{and} \quad b_{ab} := g_a^{-1} \rhd (\mathrm{d}_{\pi_1} h_{ab}\, h_{ab}^{-1})$$

- We have $b_{ab} + b_{bc} + b_{ca} = 0$ and $H^1(\mathbb{C}^6 \times \mathbb{C}P^3, \Omega^1_\pi) = 0$: Split once more: $b_{ab} = b_a - b_b$
- Define global relative differential forms:

$$A_a := a_a - \mathsf{t}(b_a) \quad \text{and} \quad B_a := -(\mathrm{d}_{\pi_1} b_a - b_a \wedge b_a + a_a \rhd b_a)$$

- Field equations:

$$\begin{aligned} F_{\pi_1} &:= \mathrm{d}A_{\pi_1} + \tfrac{1}{2}[A_{\pi_1}, A_{\pi_1}] &&= \mathsf{t}(B_{\pi_1}) \\ H_{\pi_1} &:= \mathrm{d}B_{\pi_1} + A_{\pi_1} \rhd B_{\pi_1} &&= 0 \end{aligned}$$

Resulting Superspace Constraint Equations

Supersymmetric case, coordinates x^{AB}, η_I^A on space-time:

Field expansions

$$F_{\pi_1} = -\tfrac{1}{4} e_A \wedge e_B \lambda_C \, \varepsilon^{ABCD} F_D{}^E \lambda_E + \tfrac{1}{2} e_A \lambda_B \wedge e_I^{EF} \lambda_E \, \varepsilon^{ABCD} \, F_{CD}{}^I_F + \\ + \tfrac{1}{2} e_I^{CA} \lambda_C \wedge e_J^{DB} \lambda_D \, F_{AB}^{IJ} \, ,$$

$$H_{\pi_1} = -\tfrac{1}{3} e_A \wedge e_B \wedge e_C \lambda_D \varepsilon^{ABCD} \, H^{EF} \lambda_E \lambda_F + \\ - \tfrac{1}{4} e_A \wedge e_B \lambda_C \, \varepsilon^{ABCD} \wedge e_I^{EF} \lambda_E \, (H_D{}^{GI}{}_F)_0 \lambda_G + \\ + \tfrac{1}{4} e_A \lambda_B \wedge e_I^{EF} \lambda_E \wedge e_J^{GH} \lambda_G \, \varepsilon^{ABCD} \, (H_{CD}{}^{IJ}_{FH})_0 + \\ + \tfrac{1}{6} e_I^{DA} \lambda_D \wedge e_J^{EB} \lambda_E \wedge e_K^{FC} \lambda_F \, H_{ABC}^{IJK} \, ,$$

Constraint equations

$$F_A{}^B = \mathsf{t}(B_A{}^B) \, , \quad F_{AB}{}^I_C = \mathsf{t}(B_{AB}{}^I_C) \, , \quad F_{AB}^{IJ} = \mathsf{t}(B_{AB}^{IJ}) \, ,$$

$$H^{AB} = 0 \, , \qquad H_A{}^{BI}{}_C = \delta_C^B \psi_A^I - \tfrac{1}{4} \delta_A^B \psi_C^I \, ,$$

$$H_{AB}{}^{IJ}_{CD} = \varepsilon_{ABCD} \phi^{IJ} \, , \qquad H_{ABC}^{IJK} = 0 \, ,$$

Yet another twistor space

New twistor space parameterizing hyperplanes in $\mathbb{C}^4$:

$$
\begin{tikzcd}
& \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\
P^3 & & \mathbb{C}^4
\end{tikzcd}
$$

self-dual strings
hol. principal 2-bundle
hol. principal 3-bundle

CS & M Wolf, 1111.2539, 1205.3108, 1305.4870

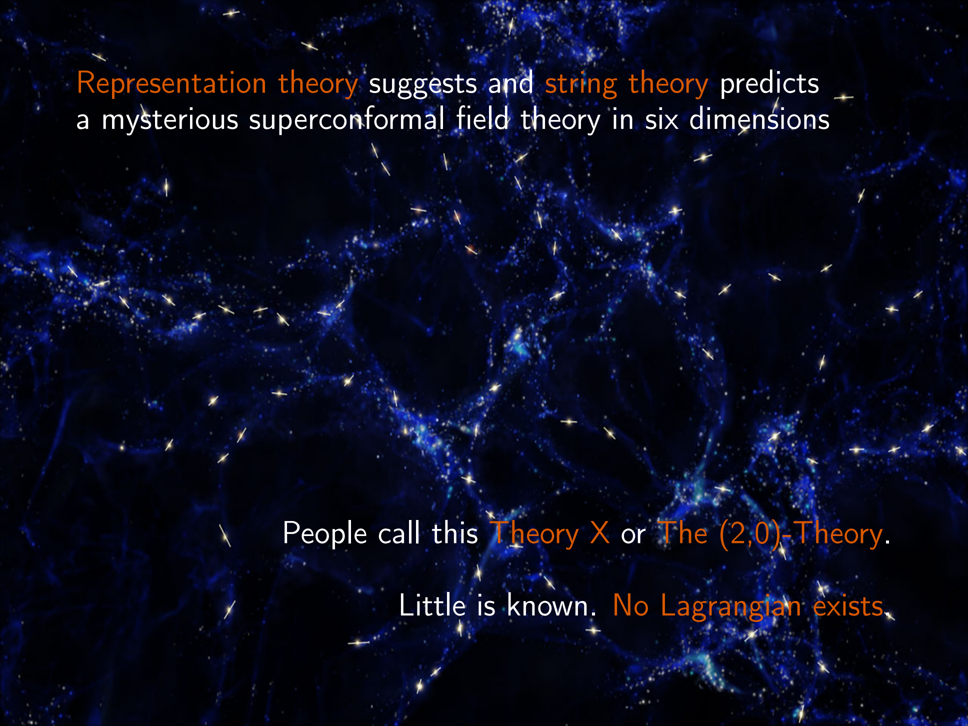
Note:

- Hyperplane twistor space P^3: $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$.
- The spheres $\mathbb{C}P^1 \times \mathbb{C}P^1$ parameterize an α- and a β-plane.
- The span of both is a hyperplane.
- Nonabelian self-dual string equations: $H = \star \mathrm{d}_A \Phi$, $F = \mathsf{t}(B)$.
- Reduces nicely to the monopole twistor space: $\mathcal{O}(2) \to \mathbb{C}P^1$.

III: Applications

III.1. T-duality with higher spaces
III.2. Penrose–Ward transform
III.3. 6d superconformal field theories
III.4. Tensor hierarchies

Warning: Much of the following is supposed to be impossible.



Representation theory suggests and string theory predicts a mysterious superconformal field theory in six dimensions

People call this Theory X or The (2,0)-Theory.

Little is known. No Lagrangian exists.

We know:

- It describes stacks of M5-branes with gravity turned off (just as Yang–Mills theory describes stack of D-branes)
- It has Wilson surfaces as observables (just as Yang–Mills has Wilson lines) It is a theory of ("self-dual") strings

Conjecture

The (2,0)-theory is classically a higher gauge theory.

"But Witten has said there is no Lagrangian!"

"... by hunting for unicorns we may find other creatures that are useful in understanding the theory more generally."

Neil Lambert

Wish:

Reality:

or

What we know about the (2,0)-theory

Pre-history:

- Conformal QFTs: particularly interesting and important
- Conformal algebra on $\mathbb{R}^{p,q}$: $\mathfrak{so}(p+1,q+1)$
- Supersymmetric extensions only for $p+q\leq 6$ Nahm, 1978
- Examples for $p+q\leq 4$ known for long time
- Belief: $p+q=4$ maximum for interacting QFTs

String/M-theory:

- First identified in type IIB superstring theory Witten, 1995
- Also appears in M-theory Witten, Strominger 1995/1996
- Physical field content is clear: $\mathcal{N}=(2,0)$ supermultiplet in 6D
- Gravity decouples
- Contains 2-form B-field with $H=\star H$
- A theory of strings, observables: Wilson surfaces
- Huge interest, would improve our understanding significantly.
- Classical description claimed to be impossible.

Objections to classical description

- Non-abelian higher parallel transport of strings difficult
  $\Rightarrow$ Non-abelian principal 2-bundles are fine.
- Action for self-dual 2-forms difficult
  $\Rightarrow$ BV-formalism: lift any eom to an action, also Sen's work
- No coupling constant, no continuous deformation of free action
  $\Rightarrow$ Same for Chern–Simons theory, M2-brane models
- Dimensional reduction to 5d and 4d unclear
  $\Rightarrow$ We give an example of a reduction
- Other points, e.g. parts of action such as $\int \phi F^2$ unbounded
  $\Rightarrow$ same issues as in F-theory.
- Unclear how to realize certain gauge groups ("Tachikawa test")
  $\Rightarrow$?? Let's cross that bridge when we get there.

An action for self-dual fields

Regard self-duality as homotopy Maurer–Cartan equation:

$$\Big(\underbrace{\overset{\lambda}{\Omega^0(M)}}_{\mathfrak{L}_{-1}} \xrightarrow{\mathrm{d}} \underbrace{\overset{\Lambda}{\Omega^1(M)}}_{\mathfrak{L}_0} \xrightarrow{\mathrm{d}} \underbrace{\overset{B}{\Omega^2(M)}}_{\mathfrak{L}_1} \xrightarrow{-(\mathrm{d}-\star\mathrm{d})} \underbrace{\overset{H_-}{\Omega^3_-(M)}}_{\mathfrak{L}_2}\Big)$$

Take cotangent bundle to have action:

$$\left(\begin{array}{cccccccc} \overset{\lambda}{\Omega^0(M)} \xrightarrow{\mathrm{d}} & \overset{\Lambda}{\Omega^1(M)} \xrightarrow{\mathrm{d}} & \overset{B}{\Omega^2(M)} \xrightarrow{-(\mathrm{d}-\star\mathrm{d})} & \overset{H_-}{\Omega^3_-(M)} & & & \\ & & & & & & \\ & & \overset{H_-^+}{\Omega^3_+(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{B^+}{\Omega^2(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{\Lambda^+}{\Omega^1(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{\lambda^+}{\Omega^0(M)} \end{array}\right)$$

Compensate unwanted fields by kernel injection:

$$\left(\begin{array}{cccccc} \overset{\lambda}{\Omega^0(M)} \xrightarrow{\mathrm{d}} & \overset{\Lambda}{\Omega^1(M)} \xrightarrow{\mathrm{d}} & \overset{B}{\Omega^2(M)} \xrightarrow{-(\mathrm{d}-\star\mathrm{d})} & \overset{H^-}{\Omega^3_-(M)} \xrightarrow{\delta} & \overset{\alpha^\dagger}{\Omega^2_+(M)} & \\ & \oplus & \oplus & \oplus & \oplus & \\ & \overset{\alpha}{\Omega^2_+(M)} \xrightarrow{\mathrm{d}} & \overset{H_-^\dagger}{\Omega^3_+(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{B^\dagger}{\Omega^2(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{\Lambda^\dagger}{\Omega^1(M)} \xrightarrow{\mathrm{d}^\dagger} & \overset{\lambda^\dagger}{\Omega^0(M)} \end{array}\right)$$

Adjusted metric string structures

- We know the BPS states $\Rightarrow$ gauge structure: $\mathfrak{string}(n)$.
- String algebra not cyclic, cf. $\hat{\mathfrak{g}}_{\rm sk} = \big(\ \mathbb{R}_r \longrightarrow \mathfrak{g}_t \ \big)$
- Construct $T^*[-2]\hat{\mathfrak{g}}_{\rm sk}$, imitating BV-formalism

  Lennart Schmidt+CS, 2017

- Result:

$$\hat{\mathfrak{g}}^{\omega}_{\rm sk} = \left(\begin{array}{ccccc} \mathfrak{g}^*_v \xrightarrow{\mu_1 = \mathsf{id}} & \mathfrak{g}^*_u & & \mathbb{R}^*_s & \xrightarrow{\mu_1 = \mathsf{id}} \mathbb{R}^*_p \\ & \oplus & & \oplus & \oplus \\ & \mathbb{R}_q & \xrightarrow{\mu_1 = \mathsf{id}} & \mathbb{R}_r & \mathfrak{g}_t \end{array} \right)$$

- Adjustments for metric string structures

  Lennart Schmidt+CS, 2019

Classically well-defined (1,0)-action

- Borrow action from gauged supergravity

  Samtleben+Sezgin+Wimmer, 2011
- Field content:
  - (1,0) tensor multiplet (ϕ, χ^i, B), values in $\mathbb{R}^2$, $\phi = \phi_s + \phi_r$, ...
  - (1,0) vector multiplet (A, λ^i, Y^{ij}), values in $\mathfrak{g} \oplus \mathbb{R}$
  - C-field (3-form), values in $\mathbb{R} \oplus \mathfrak{g}^*$
  - D-field (4-form), values in $\mathfrak{g}^*$
- Lagrangian: D Rist, CS, van der Worp (2020)

$$
\begin{aligned}
\mathcal{L}_{\mathrm{TF}}^{\mathcal{N}=(1,0)} = & -\mathrm{d}\phi_s \wedge \star \mathrm{d}\phi_r - \star 4 \bar{\chi}_s \not{\partial} \chi_r + \star 4 \bar{\chi}_s (\not{F}_t, \lambda_t) - \star 8 \bar{\chi}^i_s (Y_{tij}, \lambda^j_t) \\
& + \phi_s \Big((F_t, \star F_t) - \star 2 (Y_{tij}, Y^{ij}_t) + \star 4 (\bar{\lambda}_t, \not{\nabla} \lambda_t) \Big) + \mathcal{L}_{0,\mathrm{hyper}} \\
& + (\mathrm{d}B_s)^+ \wedge \big(\mathrm{d}B_r + \mathrm{cs}(A)_r + (\bar{\lambda}_t, \gamma_{(3)} \lambda_t) \big) + (\mathrm{d}B_s)^- \wedge C^+_q \\
& - \beth_s \wedge (\mathrm{d}B_r + \mathrm{cs}(A)_r - C_q + (\bar{\lambda}_t, \gamma_{(3)} \lambda_t)) \\
& + (\nabla C_u)(\daleth_t) + 2 B_s \wedge (F_t, \daleth_t) + D_v(\daleth_t) - 2\phi_s(\daleth_t, \star F_t) \\
& - \star 4 \bar{\chi}_s (\not{\daleth}_t, \lambda_t) - B_s \wedge (\daleth_t, \daleth_t) + \phi_s (\daleth_t, \star \daleth_t) \ ,
\end{aligned}
$$

Comments on the action

What properties should a 6d SCFT have?

- Mathematically consistent theory of 2-form potential ✓
- Interactions! ✓
- Self-duality of the curvature $H := \mathrm{d}B + \ldots = \star H$ ✓
- At least some supersymmetry ✓, ideally $\mathcal{N} = 2$ ✗
- Well-defined quantum theory ?
- Suitable reductions to
  - Yang–Mills theory in 4d ✓
  - M2-brane models in 3d ✓
- Non-abelian self-dual strings as BPS states ✓
- Other physics tests ?

Conclusion: perhaps not impossible, but more work needed

III: Applications

Infinitesimal adjustment from alternators

Recall:

2-term EL_∞-algebras Roytenberg (2007)

- 2-term cochain complex $\mathfrak{E} = \mathfrak{E}_{-1} \oplus \mathfrak{E}_0$ with Leibniz bracket
- antisymmetric and Jacobi up to homotopies (alternator, ε_3).

Observation:

In example "het. supergravity", infinitesimal adjustment: alternator.

Idea: Generalize this!

Generalize to EL_∞-algebras

What we have:

- E_2L_∞-algebras
  - Take Roytenberg's 2-term EL_∞-algebras
  - Extract strict part ("hemistrict Lie algebra")
  - Generalize to chain complexes: $h\mathcal{L}ie_2$-algebras
  - Homotopy algebras from E_2L_∞-algebras with products ε_1^0, ε_2^0, ε_3^{00}, ε_3^{10}, ε_3^{01}, $\ldots$ and $\varepsilon_2^1 = \mathsf{alt}$
- Any E_2L_∞-algebra antisymmetrizes to quasi-iso. L_∞-algebra

What we almost have:

- EL_∞-algebras:
  - Extend $h\mathcal{L}ie_2$-algebras to $h\mathcal{L}ie_\infty$-algebras (also ε_2^2, ε_2^3, $\ldots$ controlling symmetry of alternators)
  - Construct their homotopy algebra
- Subsume Dehling's 3-term EL_∞-algebras
- Any EL_∞-algebra antisymmetrizes to quasi-iso. L_∞-algebra

Useful algebras

The questions

1) Algebraic structure underlying symplectic L_∞-algebroids?
2) Algebraic structure underlying multisymplectic manifolds?
3) Algebraic structure underlying higher curvature forms?
4) Cofibrant replacement of $\mathcal{L}ie$?
5) How do you integrate Leibniz algebras?

have a simple, unifying answer:

$$EL_\infty\text{-algebras}$$

Back to adjusted higher gauge theory

First example:

There is a family of quasi-isomorphic weak Lie 2-algebras

$$\mathfrak{string}_{\mathrm{sk}}^{\mathrm{wk},\alpha}(\mathfrak{g}) \;:=\; (\mathbb{R} \xrightarrow{0} \mathfrak{g}) \;,$$
$$\varepsilon_1(r) = 0 \;,$$
$$\varepsilon_2(x_1, x_2) = [x_1, x_2] \;, \quad \varepsilon_2(x_1, r) = 0 \;,$$
$$\varepsilon_3(x_1, x_2, x_3) = (1-\alpha)(x_1, [x_2, x_3]) \;,$$
$$\mathsf{alt}(x_1, x_2) = -2\alpha(x_1, x_2)$$

Corresponding picture exists for strict 2-group version.

But let's rather turn to more complicated cases...

Why is this interesting?

Mostly, because we have the following refinement of

Theorem Fiorenza/Manetti (2006), Getzler (2009), …
Differential graded Lie algebras carry L_∞-algebra structures on shifted, truncated complex via derived brackets.

$$
\begin{tikzcd}
\text{dg Lie algebra} \arrow[r, bend right, "\text{shift/truncate}"'] \arrow[rr, bend left, "\text{shift/truncate, antisymmetrize}"] & h\mathcal{L}ie_2\text{-algebra} \arrow[r, bend right, "\text{antisymmetrize}"'] & L_\infty\text{-algebra}
\end{tikzcd}
$$

Theorem Borsten/Kim/CS (2022)
Differential graded Lie algebras carry $h\mathcal{L}ie_2$-algebra structures on shifted, truncated complex via derived brackets.

Why is this interesting?

- Start from a dg Lie algebra
- Construct $h\mathcal{L}ie_2$-algebra (easy to handle)
- If necessary, antisymmetrize to L_∞-algebra

This construction is very common in generalized geometry:

- Courant algebroid is symplectic Lie 2-algebroid
  Roytenberg (2002)
- CE-algebra + symplectic form $\Rightarrow$ dg Poisson algebra
- $h\mathcal{L}ie_2$-algebra contains Dorfman bracket
- antisymmetrization to L_∞-algebra: Courant bracket

General adjustment

Theorem (Borsten, Kim, CS)

Any L_∞-algebra obtained by shift/truncation from a differential graded Lie algebra and subsequent antisymmetrization admits a mathematically natural adjustment of the definition of the resulting curvatures.

Note:

- Have explicit formulas for adjustment/curvatures
- String 2-algebra from dgLA $\Rightarrow$ adjusted higher gauge theory
- Tensor hierarchies from dgLA $\Rightarrow$ adjusted higher gauge theory

Tensor Hierarchies in Gauged Supergravities

Application:

- There's a unique theory of supergravity in 11 dimensions.
  - "Fields": representations of supergroup containing $\mathsf{ISO}(1,10)$
  - "Action": functional of fields, invariant under this supergroup
- Place this theory on $\mathbb{R}^{1,10-n} \times T^n$, Fourier expand on T^n
- Modes contain differential forms and "S-duals"
- Modes arrange into representations of $E_{n(n)}$
- Gauge subgroup $\mathsf{G} \hookrightarrow E_{n(n)}$
- Data encoding subgroup G and reps. yield dgLA structure $\mathfrak{g}$
- Construct adjusted connections on higher principal bundles

Example: 5d max. supersymmetric Tensor Hierarchy

Differential graded Lie algebra (reps. of $\mathfrak{e}_{6(6)}$)

$$\begin{array}{cccccccccccccccc} V_{\mathfrak{e}_{6(6)}} & = & V_{-5} & \oplus & V_{-4} & \oplus & V_{-3} & \oplus & V_{-2} & \oplus & V_{-1} & \oplus & V_0 & \oplus & V_1 \\ \rho_{(k)} & & \mathbf{27}\oplus\mathbf{1728} & & \mathbf{351}_c & & \mathbf{78} & & \mathbf{27} & & \mathbf{27}_c & & \mathbf{78} & & \mathbf{351} \end{array}$$

$h\mathcal{L}ie_2$-algebra:

$$\begin{array}{ccccccccccc} \mathfrak{E}_{\mathfrak{e}_{6(6)}} = & \mathfrak{E}_{-4} & \oplus & \mathfrak{E}_{-3} & \oplus & \mathfrak{E}_{-2} & \oplus & \mathfrak{E}_{-1} & \oplus & \mathfrak{E}_0 \\ & \mathbf{27}\oplus\mathbf{1728} & & \mathbf{351}_c & & \mathbf{78} & & \mathbf{27} & & \mathbf{27}_c \end{array}$$

Curvatures:

$$\begin{aligned} F^a &= \mathrm{d}A^a + \tfrac{1}{2}X_{bc}{}^a A^b \wedge A^c + Z^{ab}B_b \\ H_a &= \mathrm{d}B_a - \tfrac{1}{2}X_{ba}{}^c A^b \wedge B_c - \tfrac{1}{6}d_{abc}X_{de}{}^b A^c \wedge A^d \wedge A^e + d_{abc}A^b \wedge F^c + \Theta_a{}^\alpha C_\alpha \\ G_\alpha &= \mathrm{d}C_\alpha - \tfrac{1}{2}X_{a\alpha}{}^\beta A^a \wedge C_\gamma + (\tfrac{1}{4}X_{a\alpha}{}^\beta t_{\beta b}{}^c + \tfrac{1}{3}t_{\alpha a}{}^d X_{(db)}{}^c)A^a \wedge A^b \wedge B_c \\ &\quad + \tfrac{1}{2}t_{\alpha a}{}^b F^a \wedge B_b - \tfrac{1}{2}t_{\alpha a}{}^b H_b \wedge A^a - \tfrac{1}{6}t_{\alpha a}{}^b d_{bcd}A^a \wedge A^c \wedge F^d - Y_{a\alpha}{}^\beta D_\beta{}^a \end{aligned}$$

Note:

- Adjustments are given by alternators of $h\mathcal{L}ie_2$-algebra
- Invisible at level of gauge L_∞-algebra

Summary

- Saw concrete examples of applications of higher gauge theory.
- T-duality
  - Geometric T-duality: span of principal 2-groupoid bundles
  - Buscher rules and compatibility implied
- Penrose–Ward transform
  - Extends to higher bundles
  - Resulting field equations not that interesting
- 6d superconformal field theories
  - Classical descriptions of $\mathcal{N} = (1,0)$-theory
  - Many perceived obstacles can be overcome
  - Still (2,0)-theory seems very difficult
- Tensor hierarchies
  - Gauged supergravities are higher gauge theories
  - New algebraic structure: $h\mathcal{L}ie_2$-algebras
  - dg Lie algebras come with $h\mathcal{L}ie_2$-algebras
  - $h\mathcal{L}ie_2$-algebras come with adjustment
  - Field strengths of gauged supergravity: adjusted curvatures

Thank You!