Supersymmetric flows and heterotic compactifications

Yann Proto
LPTHE, Paris

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Motivation

Geometric flows are important tools to investigate solutions to a geometric equation

Properties required

- (strong/weak) parabolicity (ensures short time existence and uniqueness)
- geometric meaning of fixed points

Examples

- Ricci flow [Hamilton 1982]
  (and variations with extra structure, e.g. Kähler–Ricci, G_2, ...)

$$\partial_\lambda g_{mn} = -R_{mn}$$

- Donaldson heat flow [Donaldson 1985]
  (gauge equivalent to Yang–Mills flow)

$$h^{-1}\partial_\lambda h = -g^{i\bar{\jmath}}\mathcal{F}_{i\bar{\jmath}}$$

→ The flow that we will consider will generalize both examples!

Motivation

Heuristic motivation: flows arising in string theory expected to behave “nicely”

e.g. **Perelman–Ricci flow** [Perelman 2002]

$$\partial_\lambda g_{mn} = -\left(R_{mn} + 2\nabla_m \nabla_n \varphi\right) \qquad \partial_\lambda(\sqrt{|g|}\,\mathrm{e}^{-2\varphi}) = 0$$

- Ricci flow as a gradient flow (up to diffeormorphisms)
- with this gauge choice: weakly parabolic → strongly parabolic (DeTurck trick)
- functional: string-frame effective action for a metric and dilaton

$$S = \int_X \sqrt{|g|}\,\mathrm{e}^{-2\varphi}\left(R + 4(\nabla\varphi)^2\right)$$

Outline

Goal: set up a heterotic string theory framework for anomaly flows, which are geometric flows on $\mathrm{SU}(3)$ structure manifolds

I) Heterotic supergravity and flux compactifications
II) Anomaly flows
III) Recasting as a gradient flow
IV) Including α' corrections
V) Generalization to G_2 and $\mathrm{Spin}(7)$ structure manifolds

I. Heterotic supergravity and flux compactifications

Heterotic basics

Bosonic fields:
- tetrad $e_{\mathrm{M}}{}^{\mathrm{A}}$
  (metric $g_{\mathrm{MN}} = e_{\mathrm{M}}{}^{\mathrm{A}} e_{\mathrm{NA}}$)
- dilaton φ
- B-field B_{MN}
- gauge connection $\mathcal{A}_{\mathrm{M}}$

Fermionic fields:
- gravitino ψ_{M}
- dilatino λ
- gaugino χ

Heterotic action

$$S = \int_{M_{10}} |e|\, \mathrm{e}^{-2\varphi} \left(R + 4(\nabla\varphi)^2 - \frac{1}{2}H^2 - \frac{\alpha'}{4}\left(\operatorname{tr}\mathcal{F}^2 - \operatorname{tr}\mathcal{R}_+^2\right)\right) + \text{fermions}$$

- curvature $\mathcal{R}_+$ computed from the Hull connection $\Gamma_+ = \Gamma + \frac{1}{2}H$
- two-form B_{MN} only appears through the field strength

$$H = \mathrm{d}B + \tfrac{\alpha'}{4}\left(\omega_{\mathrm{CS}}(\mathcal{A}) - \omega_{\mathrm{CS}}(\Gamma_+)\right) \qquad \omega_{\mathrm{CS}}(\mathcal{A}) = \operatorname{tr}(\mathrm{d}\mathcal{A}\wedge\mathcal{A} + \tfrac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})$$

resulting in the Bianchi identity

$$\mathrm{d}H = \tfrac{\alpha'}{4}\left(\operatorname{tr}\mathcal{F}\wedge\mathcal{F} - \operatorname{tr}\mathcal{R}_+\wedge\mathcal{R}_+\right)$$

Heterotic basics

Bosonic fields:
- tetrad $e_M{}^A$ (metric $g_{MN} = e_M{}^A e_{NA}$)
- dilaton φ
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- gauge connection $\mathcal{A}_M$

Fermionic fields:
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$$\mathrm{d}H = \tfrac{\alpha'}{4}\left(\operatorname{tr}\mathcal{F}\wedge\mathcal{F} - \operatorname{tr}\mathcal{R}_+\wedge\mathcal{R}_+\right)$$

Minkowski heterotic compactifications

Compactify heterotic theory on $\mathbb{R}^{1,3} \times X$ with a compact six-manifold X

- Poincaré invariance
  - → fermionic fields vanish
  - → bosonic fields supported on X

$$g = \eta_{\mathbb{R}^{1,3}} + g_X \qquad H, \varphi, \mathcal{F} \in \Omega^*(X)$$

- equations of motion and Bianchi identity reduce to the internal space
  - → effective six-dimensional bosonic action

$$S = \int_X |e|\, \mathrm{e}^{-2\varphi} \left(R + 4(\mathrm{d}\varphi)^2 - \frac{1}{2} H^2 - \frac{\alpha'}{4} \left(\operatorname{tr} \mathcal{F}^2 - \operatorname{tr} \mathcal{R}_+^2 \right) \right)$$

Heterotic equations of motion (bosonic)

$$\mathrm{eom}[e]_{mn} = R_{mn} + 2\nabla_m \nabla_n \varphi - \tfrac{1}{4} H_m{}^{pq} H_{npq} - \tfrac{\alpha'}{4} \left(\operatorname{tr} \mathcal{F}_m{}^p \mathcal{F}_{np} - \operatorname{tr} \mathcal{R}^+_m{}^p \mathcal{R}^+_{np} \right)$$

$$\mathrm{eom}[\varphi] = R + 4\nabla^2 \varphi - 4(\nabla\varphi)^2 - \tfrac{1}{2} H^2 - \tfrac{\alpha'}{8} \left(\operatorname{tr} \mathcal{F}^2 - \operatorname{tr} \mathcal{R}_+^2 \right)$$

$$\mathrm{eom}[B] = \star\, \mathrm{e}^{2\varphi} \mathrm{d} \left(\mathrm{e}^{-2\varphi} \star H \right)$$

$$\mathrm{eom}[\mathcal{A}] = \star\, \mathrm{e}^{2\varphi} \mathcal{D}_- \left(\mathrm{e}^{-2\varphi} \star \mathcal{F} \right)$$

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Supersymmetric compactifications and Hull–Strominger equations

Preserving $\mathcal{N} = 1$ supersymmetry requires solving the Killing spinor equations

$$D_m \epsilon = 0 \qquad \not{D} \epsilon = 0 \qquad \not{\mathcal{F}} \epsilon = 0$$

with the supersymmetry operators

$$D_m = \nabla_m + \tfrac{1}{8} H_{m n_1 n_2} \gamma^{n_1 n_2}$$
$$\not{D} = \gamma^m \nabla_m + \tfrac{1}{24} H_{m_1 \ldots m_3} \gamma^{m_1 \ldots m_3} - \nabla_m \varphi \, \gamma^m$$
$$\not{\mathcal{F}} = \tfrac{1}{2} \mathcal{F}_{m_1 m_2} \gamma^{m_1 m_2}$$

→ very restrictive! e.g. $\partial_m(\epsilon^\dagger \epsilon) = 0$

Internal space should be endowed with globally defined forms

$$J_{m_1 m_2} = -i \epsilon^\dagger \gamma_{m_1 m_2} \gamma_\star \epsilon \qquad \Omega_{m_1 \ldots m_3} = -i \epsilon^\dagger \gamma_{m_1 \ldots m_3} (I + \gamma_*) \epsilon$$

defining a SU(3) structure on X

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defining a SU(3) structure on X

Supersymmetric compactifications and Hull–Strominger equations

Hull–Strominger system [Hull 1986, Strominger 1986]

Conditions for a supersymmetric solution:

- X is a complex manifold
- X has $\mathrm{SU}(3)$ structure: $(1,1)$-form J and $(3,0)$-form Ω satisfying

$$J \wedge \Omega = 0 \qquad J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$$

  as well as the differential conditions

$$\mathrm{d}\left(e^{-2\varphi} J \wedge J\right) = 0 \qquad \mathrm{d}\left(e^{-2\varphi} \Omega\right) = 0$$

- gauge bundle satisfies the Hermitian Yang–Mills equations

$$J \wedge J \wedge \mathcal{F} = 0 \qquad \Omega \wedge \mathcal{F} = \bar{\Omega} \wedge \mathcal{F} = 0$$

- H-flux is defined from $H = -\mathrm{i}(\partial - \bar{\partial})J$ and constrained by the Bianchi identity

$$2\mathrm{i}\, \partial\bar{\partial} J = \frac{\alpha'}{4} \left(\mathrm{tr}\, \mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\, \mathcal{R}_+ \wedge \mathcal{R}_+\right)$$

Flux solutions

$$H = -\mathrm{i}(\partial - \bar{\partial})J$$

non-vanishing H-flux $\Leftrightarrow$ non-Kähler background

Numerous works on non-Kähler geometry in both `math.DG` and `hep-th`

[Adams, Becker, Curio, Dall'Agata, Dasgupta, Ernebjerg, Fei, Fernandez, Fino, García-Fernández, Grantcharov, Huang, Israël, Ivanov, Lapan, Lopes Cardoso, Lust, Manousselis, Melnikov, Minasian, Otal, Petrini, Picard, Sethi, Tseng, Ugarte, Vassilev, Vezzoni, Villacampa, Yau, Zoupanos, ..., ...]

Fu–Yau backgrounds [Dasgupta–Rajesh–Sethi 1999, Goldstein–Prokushkin 2004]

- well motivated from the physics side (M-theory dual)
- X constructed as a principal torus fibration over a $\mathrm{K3}$ surface

$$\begin{array}{ccc} T^2 & \hookrightarrow & X \\ & & \downarrow \\ & & \mathrm{K3} \end{array}$$

- gauge connection pulled back from a HYM connection on $\mathrm{K3}$
- Bianchi identity is a top form on $\mathrm{K3}$ and admits solutions! [Fu–Yau 2008]

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II. Anomaly flows

Anomaly flows [Phong–Picard–Zhang 2015]

Anomaly flows are a coupled flow on a complex manifold X for

- a $SU(3)$ structure on $X \to J_\lambda$, Ω_λ, φ_λ,
- a hermitian gauge bundle over $X \to h_\lambda$

Anomaly flow equations

$$\partial_\lambda \left(e^{-2\varphi} J \wedge J\right) = 2i\,\partial\bar{\partial} J - \frac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\,\mathcal{R}_{[\Gamma]} \wedge \mathcal{R}_{[\Gamma]}\right)$$
$$\partial_\lambda \left(e^{-2\varphi}\Omega\right) = 0$$
$$h^{-1}\partial_\lambda h = -g^{i\bar{j}}\mathcal{F}_{i\bar{j}}$$

- preserves supersymmetry: $\partial_\lambda \mathrm{d}\left(e^{-2\varphi} J \wedge J\right) = 0 \quad \partial_\lambda \mathrm{d}\left(e^{-2\varphi}\Omega\right) = 0$
- fixed points solve Bianchi and HYM!
- weakly parabolic $\to$ short-time existence

Anomaly flow on Fu–Yau backgrounds

On Fu–Yau manifolds $T^2 \hookrightarrow X \to \mathrm{K3}$, the anomaly flow becomes a flow for a scalar field on $\mathrm{K3}$ [Phong–Picard–Zhang 2016]

$$\partial_\lambda e^{2\varphi} = \frac{1}{2}\Delta_{\mathrm{K3}} e^{2\varphi} - \mu[\varphi]$$

with $\mu \, \mathrm{vol}_{\mathrm{K3}} = \mathcal{G}_{IJ} F^I \wedge F^J + \frac{\alpha'}{4}(\mathrm{tr}\,\mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\,\mathcal{R} \wedge \mathcal{R})$
(topological requirement $\int_X \mu \, \mathrm{vol}_{\mathrm{K3}} = 0$)

Properties

- parabolic complex Monge-Ampère type equation
- long time existence
- convergence

$\to$ alternative proof of existence of the Fu–Yau solution!

Understanding anomaly flows

Embedding anomaly flow in heterotic?

$$\partial_\lambda \left(e^{-2\varphi} J \wedge J\right) = \mathrm{d}H - \frac{\alpha'}{4}(\mathrm{tr}\,\mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\,\mathcal{R}^{\mathrm{C}} \wedge \mathcal{R}^{\mathrm{C}})$$

$$\partial_\lambda \left(e^{-2\varphi}\Omega\right) = 0$$

$$h^{-1}\partial_\lambda h = -g^{i\bar{j}}\mathcal{F}_{i\bar{j}}$$

Heterotic formulation

(1) Connection Γ appearing in Bianchi:
- → change of connection doesn't affect topology
- → $\mathrm{tr}\,\mathcal{R} \wedge \mathcal{R}$ should be a $(2,2)$-form (?)
- → torsional connection Γ_+ singled out by supersymmetry

(2) expect corrections at higher orders in α'
- → α'^2-corrected flow equations?

Finding how anomaly flows emerge in the heterotic theory would give insight on both issues!

Understanding anomaly flows

Embedding anomaly flow in heterotic:

$$\partial_\lambda \left(e^{-2\varphi} J \wedge J\right) = \mathrm{d}H - \frac{\alpha'}{4}(\operatorname{tr} \mathcal{F} \wedge \mathcal{F} - \operatorname{tr} \mathcal{R}^+ \wedge \mathcal{R}^+)$$

$$\partial_\lambda \left(e^{-2\varphi} \Omega\right) = 0$$

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$$\partial_\lambda \left(e^{-2\varphi}\Omega\right) = \mathcal{O}(\alpha'^2)$$

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Heterotic formulation

(1) Connection Γ appearing in Bianchi:
- $\rightarrow$ change of connection doesn't affect topology
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- $\rightarrow$ torsional connection Γ_+ singled out by supersymmetry

(2) expect corrections at higher orders in α'
- $\rightarrow$ α'^2-corrected flow equations?

Finding how anomaly flows emerge in the heterotic theory would give insight on both issues!

Understanding anomaly flows

For now:

- focus on the geometric part (existence results for HYM)
- set α' to zero $\Rightarrow$ α' corrections later ($\alpha' \to 0$ limit is only formal!)

Simplified "anomaly flow" on a $\mathrm{SU}(3)$ structure manifold X

$$\partial_\lambda \left(e^{-2\varphi} J \wedge J\right) = \mathrm{d}H$$
$$\partial_\lambda \left(e^{-2\varphi} \Omega\right) = 0$$

- initial data: J_0, Ω_0, φ_0 with

$$\mathrm{d}\left(e^{-2\varphi} J \wedge J\right)\big|_{\lambda=0} = 0 \qquad \mathrm{d}\left(e^{-2\varphi} \Omega\right)\big|_{\lambda=0} = 0$$

- X has to be Kähler for convergence...
- non-trivial fixed points (astheno-Kähler metrics) [Phong–Picard–Zhang 2018]

Understanding anomaly flows

The flow of the metric can be integrated from the flow of $J \wedge J$ as

$$\partial_\lambda g_{mn} = \frac{1}{4} \mathrm{e}^{2\varphi} J^{p_1 p_2} J_m{}^q \mathrm{d}H_{p_1 p_2 q n}$$

For supersymmetric configurations, using identities of conformally balanced manifolds, this flow can be recast in the form

$$\partial_\lambda g_{mn} = -\mathrm{e}^{2\varphi} \left(R_{mn} + 2\nabla_m \nabla_n \varphi - \frac{1}{4} H_m{}^{pq} H_{npq} \right)$$

→ flow by the equation of motion!

Similar structure for the dilaton:

$$\partial_\lambda \varphi = -\frac{1}{4} \mathrm{e}^{2\varphi} \left(R + 4\nabla^2 \varphi - 4(\nabla\varphi)^2 - \frac{1}{2} H^2 \right)$$

Questions

- derive from an action?
- what about the B-field?

III. Recasting as a gradient flow

Flow and supergravity fields

Recall for a $\mathrm{SU}(3)$ structure: $\{J, \Omega, \varphi\} \leftrightarrow \{e^a, \epsilon, \varphi\}$

Flow of the supergravity fields

$$\partial_\lambda \epsilon = -\tfrac{1}{4}\mathrm{e}^{2\varphi}(I - \epsilon\epsilon^\dagger)\mathrm{d}\not{H}\epsilon$$
$$\partial_\lambda e_m{}^a = -\tfrac{1}{4}\mathrm{e}^{2\varphi}\tfrac{1}{3!}(\mathrm{d}H)_{mn_1\dots n_3}\epsilon^\dagger\gamma^{an_1\dots n_3}\epsilon$$
$$\partial_\lambda \varphi = -\tfrac{1}{4}\mathrm{e}^{2\varphi}\epsilon^\dagger\mathrm{d}\not{H}\epsilon \qquad \mathrm{d}\not{H} = \tfrac{1}{4!}(\mathrm{d}H)_{m_1\dots m_4}\gamma^{m_1\dots m_4}$$

Correspond to "functional derivatives" of

$$I = \int_X |e|\,\mathrm{e}^{-2\varphi}\epsilon^\dagger\mathrm{d}\not{H}\epsilon = \int_X \mathrm{e}^{-2\varphi} J \wedge \mathrm{d}H$$

with $\mathrm{d}H$ kept fixed. . . (using $(J \wedge J)_{m_1\dots m_4} = \epsilon^\dagger\gamma_{m_1\dots m_4}\epsilon$)

Where is the B-field?

Missing degree of freedom corresponding to the B-field

Supersymmetric configurations satisfy $\boxed{H = -\mathrm{i}(\partial - \bar{\partial})J = \star\, \mathrm{e}^{2\varphi}\mathrm{d}\left(\mathrm{e}^{-2\varphi}J\right)}$

- defining B-field from H as $H = \mathrm{d}B$ is inconsistent with $\mathrm{d}H = 2\mathrm{i}\partial\bar{\partial}J$
- however $\mathrm{d}\left(\mathrm{e}^{-2\varphi}\star H\right) = \mathrm{d}^2\left(-\mathrm{e}^{-2\varphi}J\right) = 0$

$\rightarrow$ possible to dualize!

Dual $\widetilde{B}$-field

$$\mathrm{e}^{-2\varphi}\star H = \mathrm{d}\widetilde{B}$$

- constrained by supersymmetry to be (up to gauge transformations)

$$\boxed{\widetilde{B} = -\mathrm{e}^{-2\varphi}J}$$

- for consistency with anomaly flow

$$\begin{aligned}\partial_\lambda \widetilde{B} &= -\partial_\lambda\left(\mathrm{e}^{-2\varphi}J\right)\\ &= -\tfrac{1}{2}\star\mathrm{d}H\end{aligned}$$

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Missing degree of freedom corresponding to the B-field

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- for consistency with anomaly flow

$$\begin{aligned} \partial_\lambda \widetilde{B} &= -\partial_\lambda\left(\mathrm{e}^{-2\varphi} J\right) \\ &= -\tfrac{1}{2} \star \mathrm{d}H \end{aligned}$$

A functional for anomaly flows

Define the anomaly flow functional as

$$\mathcal{I} = \int_X \left(\widetilde{B} + \mathrm{e}^{-2\varphi} J\right) \wedge \mathrm{d}H$$

The equations of motion of $\mathcal{I}$ reproduce the (simplified) anomaly flow equations

Explicitly

$$\partial_\lambda e^a = \tfrac{1}{4}\mathrm{e}^{2\varphi}\frac{\delta\mathcal{I}}{\delta e_a} + \tfrac{1}{8}\mathrm{e}^{2\varphi}\frac{\delta\mathcal{I}}{\delta\varphi}e^a \qquad \partial_\lambda \epsilon = -\tfrac{1}{4}\mathrm{e}^{2\varphi}\frac{\delta\mathcal{I}}{\delta\epsilon}$$

$$\partial_\lambda \varphi = \tfrac{1}{8}\mathrm{e}^{2\varphi}\frac{\delta\mathcal{I}}{\delta\varphi} \qquad \partial_\lambda \widetilde{B} = -\tfrac{1}{2}\mathrm{e}^{-2\varphi}\frac{\delta\mathcal{I}}{\delta\widetilde{B}}$$

$\rightarrow$ Does $\mathcal{I}$ appear in the heterotic theory?

Bismut–Lichnerowicz formula

Lichnerowicz formula: $(\nabla^m \nabla_m - \nabla\!\!\!/^2)\epsilon = \frac{1}{4} R\epsilon$ [Lichnerowicz 1963]

Coupling to H-flux: [Bismut 1989]

$$\begin{aligned} D_m &= \nabla_m + \alpha\, \tfrac{1}{2!} H_{m n_1 n_2} \gamma^{n_1 n_2} \\ D\!\!\!/ &= \gamma^m \nabla_m + \beta\, \tfrac{1}{3!} H_{m_1 \dots m_3} \gamma^{m_1 \dots m_3} \end{aligned}$$

Difference of squares:

$$\begin{aligned} (D^m D_m - D\!\!\!/^2)\epsilon =& \tfrac{1}{4}(R - 12(\alpha^2 - \tfrac{1}{3}\beta^2) H^2)\epsilon - \beta\, \tfrac{1}{4!} \mathrm{d}H_{m_1 \dots m_4} \gamma^{m_1 \dots m_4} \epsilon \\ &+ \tfrac{1}{2}(\alpha - \beta)(\star \mathrm{d} \star H)_{m_1 m_2} \gamma^{m_1 m_2} \epsilon \\ &+ \tfrac{1}{4}(\alpha^2 - \beta^2) H_{m_1 m_2}{}^{n} H_{m_3 m_4 n} \gamma^{m_1 \dots m_4} \epsilon \\ &+ (\alpha - \beta) H_{m_1 m_2}{}^{n} \gamma^{m_1 m_2} \nabla_n \epsilon \end{aligned}$$

- setting $\alpha = \beta \to$ **Bismut–Lichnerowicz** ($\alpha = \frac{1}{4}$ for correct normalization of H^2)
- coupling to dilaton?

Bismut–Lichnerowicz formula

Lichnerowicz formula: $(\nabla^m \nabla_m - \nabla\!\!\!/^2)\epsilon = \frac{1}{4} R\epsilon$ [Lichnerowicz 1963]

Coupling to H-flux: [Bismut 1989]

$$D_m = \nabla_m + \alpha\, \tfrac{1}{2!} H_{mn_1n_2}\gamma^{n_1n_2}$$
$$D\!\!\!/ = \gamma^m \nabla_m + \beta\, \tfrac{1}{3!} H_{m_1\dots m_3}\gamma^{m_1\dots m_3}$$

Difference of squares:

$$\begin{aligned}(D^m D_m - D\!\!\!/^2)\epsilon =& \tfrac{1}{4}(R - 12(\alpha^2 - \tfrac{1}{3}\beta^2)H^2)\epsilon - \beta\, \tfrac{1}{4!} \mathrm{d}H_{m_1\dots m_4}\gamma^{m_1\dots m_4}\epsilon \\ &+ \tfrac{1}{2}(\alpha - \beta)(\star \mathrm{d} \star H)_{m_1m_2}\gamma^{m_1m_2}\epsilon \\ &+ \tfrac{1}{4}(\alpha^2 - \beta^2)H_{m_1m_2}{}^{n}H_{m_3m_4n}\gamma^{m_1\dots m_4}\epsilon \\ &+ (\alpha - \beta)H_{m_1m_2}{}^{n}\gamma^{m_1m_2}\nabla_n\epsilon\end{aligned}$$

- setting $\alpha = \beta \rightarrow$ **Bismut–Lichnerowicz** ($\alpha = \frac{1}{4}$ for correct normalization of H^2)
- coupling to dilaton?

Bismut–Lichnerowicz formula

With dilaton coupling: [Minasian–Petrini–Svanes 2017]

$$\begin{aligned} D_m &= \nabla_m + \tfrac{1}{8} H_{m n_1 n_2} \gamma^{n_1 n_2} \\ \not{D} &= \gamma^m \nabla_m + \tfrac{1}{24} H_{m_1 \ldots m_3} \gamma^{m_1 \ldots m_3} - \nabla_m \varphi \, \gamma^m \end{aligned}$$

→ supersymmetry operators!

- still non-tensorial...

$$\begin{aligned} (D^m D_m - \not{D}^2)\epsilon = {} & \tfrac{1}{4}(R - \tfrac{1}{2} H^2 - 4(\nabla\varphi)^2 + 4\nabla^2\varphi)\epsilon \\ & - \tfrac{1}{4}\,\tfrac{1}{4!} \mathrm{d}H_{m_1 \ldots m_4} \gamma^{m_1 \ldots m_4} \epsilon \\ & + 2\, \nabla^m \varphi \, D_m \epsilon \end{aligned}$$

- contract with ϵ and integrate by part:

$$\int_X |e|\, \mathrm{e}^{-2\varphi} (R + 4(\nabla\varphi)^2 - \tfrac{1}{2} H^2)\, \epsilon^\dagger \epsilon = 4 \int_X |e|\, \mathrm{e}^{-2\varphi} \left(|\not{D}\epsilon|^2 - |D\epsilon|^2 \right) + \int_X |e|\, \mathrm{e}^{-2\varphi} \epsilon^\dagger \mathrm{d}\not{H} \epsilon$$

Bismut–Lichnerowicz formula

With dilaton coupling: [Minasian–Petrini–Svanes 2017]

$$
\begin{aligned}
D_m &= \nabla_m + \tfrac{1}{8} H_{m n_1 n_2} \gamma^{n_1 n_2} \\
\not{D} &= \gamma^m \nabla_m + \tfrac{1}{24} H_{m_1 \dots m_3} \gamma^{m_1 \dots m_3} - \nabla_m \varphi \, \gamma^m
\end{aligned}
$$

→ supersymmetry operators!

- still non-tensorial…

$$
\begin{aligned}
(D^m D_m - \not{D}^2)\epsilon = {} & \tfrac{1}{4}\big(R - \tfrac{1}{2} H^2 - 4(\nabla\varphi)^2 + 4\nabla^2\varphi\big)\epsilon \\
& - \tfrac{1}{4}\,\tfrac{1}{4!} \mathrm{d}H_{m_1 \dots m_4} \gamma^{m_1 \dots m_4} \epsilon \\
& + 2\, \nabla^m \varphi \, D_m \epsilon
\end{aligned}
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- contract with ϵ and integrate by part:

$$
\int_X |e| \, \mathrm{e}^{-2\varphi} \big(R + 4(\nabla\varphi)^2 - \tfrac{1}{2} H^2\big)\, \epsilon^\dagger \epsilon = 4 \int_X |e| \, \mathrm{e}^{-2\varphi} \left(|\not{D}\epsilon|^2 - |D\epsilon|^2 \right) + \int_X |e| \, \mathrm{e}^{-2\varphi} \epsilon^\dagger \mathrm{d}\not{H} \epsilon
$$

Bismut–Lichnerowicz formula

With dilaton coupling: [Minasian–Petrini–Svanes 2017]

$$D_m = \nabla_m + \tfrac{1}{8} H_{mn_1n_2}\gamma^{n_1n_2}$$
$$\not{D} = \gamma^m\nabla_m + \tfrac{1}{24} H_{m_1\dots m_3}\gamma^{m_1\dots m_3} - \nabla_m\varphi\,\gamma^m$$

$\rightarrow$ supersymmetry operators!

- still non-tensorial...

$$(D^m D_m - \not{D}^2)\epsilon = \tfrac{1}{4}(R - \tfrac{1}{2}H^2 - 4(\nabla\varphi)^2 + 4\nabla^2\varphi)\epsilon - \tfrac{1}{4}\,\tfrac{1}{4!}\mathrm{d}H_{m_1\dots m_4}\gamma^{m_1\dots m_4}\epsilon + 2\,\nabla^m\varphi\,D_m\epsilon$$

- contract with ϵ and integrate by part:

$$\int_X |e|\,\mathrm{e}^{-2\varphi}(R + 4(\nabla\varphi)^2 - \tfrac{1}{2}H^2)\,\epsilon^\dagger\epsilon = 4\int_X |e|\,\mathrm{e}^{-2\varphi}\left(|\not{D}\epsilon|^2 - |D\epsilon|^2\right) + \boxed{\int_X |e|\,\mathrm{e}^{-2\varphi}\epsilon^\dagger\mathrm{d}\not{H}\epsilon}$$

$\leftarrow$ appears in $\mathcal{I}$

Recognizing the anomaly flow functional

For a supersymmetric background ($\epsilon^\dagger\epsilon = 1$ and $D\epsilon = \not{D}\epsilon = 0$)

$$\mathcal{I} = \int_X |e|\, \mathrm{e}^{-2\varphi}(R + 4(\nabla\varphi)^2 - \tfrac{1}{2}H^2) + \int_X \widetilde{B} \wedge \mathrm{d}H$$

The anomaly flow functional reproduces the dualized heterotic bosonic action

So far

- rephrased (simplified) anomaly flow as a flow for supergravity fields
- defined a functional $\mathcal{I}$ for the flow
- identified $\mathcal{I}$ with the heterotic bosonic action

IV. Including α' corrections

Choice of connection

Original formulation of anomaly flows uses the Chern connection Γ^{C}

- $\operatorname{tr} \mathcal{R}^{\mathrm{C}} \wedge \mathcal{R}^{\mathrm{C}}$ in Bianchi is a $(2,2)$-form
- usual choice in (part of) the literature

Changing connection on TX

- does not affect topological properties
- is correlated with the local form of supersymmetry equations
- corresponds to changing regularization scheme in the effective action

Choice singled out by supersymmetry: Hull connection Γ^{+}
(not a new degree of freedom! $\Gamma^{+} = \Gamma^{+}[J]$)

Choice of connection

How Γ^+ appears in heterotic supergravity

(1) Hull connection fits in a composite Yang–Mills multiplet with the gravitino curvature ψ_{ab} [Bergshoeff–de Roo 1989]

$$\delta\psi_{ab} = \tfrac{1}{8} R^+_{abcd}\gamma^{cd}\epsilon \qquad \tfrac{1}{2}\delta\Gamma^+_{mab} = -\epsilon^\dagger\gamma_m\psi_{ab}$$

(2) compatibility between susy and eoms requires an instanton condition on the curvature [Ivanov 2009, de la Ossa–Svanes 2014]

$$\boxed{\mathcal{R}\epsilon = \mathcal{O}(\alpha')}$$

which distinguishes the Hull connection

$$\mathcal{R}^+_{a_1a_2}\epsilon = 2[D_{a_1}, D_{a_2}]\epsilon - \tfrac{1}{4}\mathrm{d}H_{m_1m_2a_1a_2}\gamma^{m_1m_2}\epsilon$$

$$= \mathcal{O}(\alpha') \text{ for solutions of the Bianchi identity}$$

(can be computed from $R^-_{m_1m_2n_1n_2} = R^+_{n_1n_2m_1m_2} + \tfrac{1}{2}\mathrm{d}H_{m_1m_2n_1n_2}$ where Γ^- is the connection associated to D_m)

→ The choice of Γ^+ is also singled out by anomaly flows!

Anomaly flow at first order in α'

Consider the anomaly flow with couplings to an arbitrary connection $\check{\Gamma}$

Flow of the metric

- integrate $\partial_\lambda g$ from the $\mathrm{SU}(3)$ structure flow

$$\partial_\lambda g_{m_1 m_2} = \tfrac{1}{4}\mathrm{e}^{2\varphi} J^{n_1 n_2} J_{m_1}{}^{p} \partial_\lambda \left(\mathrm{e}^{-2\varphi} J \wedge J\right)_{n_1 n_2 p m_2}$$

- rewrite using susy operators D and $\not{D}$

At zeroth order in α'

$$\begin{aligned}\partial_\lambda g_{mn} = &-\mathrm{e}^{2\varphi}\left(R_{mn} + 2\nabla_m\nabla_n\varphi - \tfrac{1}{4}H_m{}^{p_1p_2}H_{np_1p_2}\right)\\ &+\mathrm{e}^{2\varphi}\left(\epsilon^\dagger\gamma_{(m}\not{D}D_{n)}\epsilon - \epsilon^\dagger\gamma_{(m}D_{n)}\not{D}\epsilon + \tfrac{1}{2}H_{(m}{}^{pq}\epsilon^\dagger\gamma_{n)}\gamma_p D_q\epsilon + \text{c.c.}\right)\\ &+\mathcal{O}(\alpha')\end{aligned}$$

$$= \boxed{-\mathrm{e}^{2\varphi}\mathrm{eom}[g]_{mn} + [D, \not{D}\ \text{bilinears}] + \mathcal{O}(\alpha')}$$

→ for an initial susy configuration, flow by the equation of motion

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$$\begin{aligned}\partial_\lambda g_{mn} = &-\mathrm{e}^{2\varphi}\left(R_{mn} + 2\nabla_m\nabla_n\varphi - \tfrac{1}{4} H_m{}^{p_1 p_2} H_{n p_1 p_2}\right) \\ &+\mathrm{e}^{2\varphi}\left(\epsilon^\dagger \gamma_{(m} \not{D} D_{n)}\epsilon - \epsilon^\dagger \gamma_{(m} D_{n)} \not{D}\epsilon + \tfrac{1}{2} H_{(m}{}^{pq}\, \epsilon^\dagger \gamma_{n)} \gamma_p D_q \epsilon + \text{c.c.}\right) \\ &+\mathcal{O}(\alpha')\end{aligned}$$

$$= \boxed{-\mathrm{e}^{2\varphi}\mathrm{eom}[g]_{mn} + [D, \not{D} \text{ bilinears}] + \mathcal{O}(\alpha')}$$

$\rightarrow$ for an initial susy configuration, flow by the equation of motion

Anomaly flow at first order in α'

At first order in α' this structure breaks down

$$\partial_\lambda g_{mn} = -\mathrm{e}^{2\varphi}\mathrm{eom}[g]_{mn} + [D, \not{D} \text{ bilinears}]$$
$$+\frac{\alpha'}{4}\mathrm{e}^{2\varphi}\left(\mathrm{tr}\,\epsilon^\dagger \mathcal{F}_{mp}\gamma_n{}^p \not{\mathcal{F}}\epsilon - \mathrm{tr}\,\epsilon^\dagger \check{\mathcal{R}}_{mp}\gamma_n{}^p \check{\not{\mathcal{R}}}\epsilon\right)$$

- $\not{\mathcal{F}}\epsilon$ and $\check{\not{\mathcal{R}}}\epsilon$ should vanish at fixed points of the flow up to $\mathcal{O}(\alpha'^2)$ terms
  - → $\not{\mathcal{F}}\epsilon = 0$ by HYM
  - → recover instanton condition $\check{\not{\mathcal{R}}}\epsilon = \mathcal{O}(\alpha')$

Chern connection is generically not an $\mathrm{SU}(3)$ instanton [Martelli–Sparks 2011]

Anomaly flow and α' expansion

Employing the Hull connection without an α' expansion is inconsistent

- $\operatorname{tr}\mathcal{R}^+ \wedge \mathcal{R}^+$ is $(2,2)$ only up to $\mathcal{O}(\alpha')$
- $\not{\mathcal{R}}^+\epsilon = 0$ at fixed points $\Rightarrow$ X is Calabi–Yau [Ivanov–Papadopoulos 2000]

Higher order α' corrections

- to the equations of motion
- to the Bianchi identity
- to the supersymmetry operators
  e.g. at order α'^2

$$D_m\epsilon = \nabla_m\epsilon + \frac{1}{8}H_{mn_1n_2}\gamma^{n_1n_2}\epsilon - \frac{3}{2}\alpha'\mathrm{e}^{2\varphi}\nabla_-^n(\mathrm{e}^{-2\varphi}\mathrm{d}H_{nmp_1p_2})\gamma^{p_1p_2}\epsilon$$

[de la Ossa–Svanes 2015]

Anomaly flow and α' expansion

At order α', the functional driving the flow becomes

$$\mathcal{I} = \int_X (\widetilde{B} + \mathrm{e}^{-2\varphi} J) \wedge \Big(\mathrm{d}H + \tfrac{\alpha'}{4}\left(\operatorname{tr}\mathcal{F}\wedge\mathcal{F} - \operatorname{tr}\mathcal{R}_+ \wedge \mathcal{R}_+\right)\Big)$$

- functional derivatives of $\mathcal{I}$ reproduce the anomaly flow equations
- Lichnerowicz structure

$$\mathcal{I} = \underbrace{\int_X \mathrm{e}^{-2\varphi}\mathscr{L}}_{\text{(dualized) bosonic action}} - \underbrace{\int_X |e|\,\mathrm{e}^{-2\varphi}\Big(4(|\not{D}\epsilon|^2 - |D\epsilon|^2) + \tfrac{\alpha'}{4}(\operatorname{tr}|\not{\mathcal{F}}\epsilon|^2 - \operatorname{tr}|\not{\mathcal{R}}^+\epsilon|^2)\Big)}_{=\mathcal{O}(\alpha'^2)\text{ along the flow or at fixed points}}$$

Guiding principle for α' expansion of the flow (schematically)

- expect the flow to be corrected order by order in α'
- construct $\mathcal{I}(\alpha')$ by maintaining Lichnerowicz structure at every order in α' (with α'-corrected functional, action and susy operators)

Anomaly flow and α' expansion

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$$\mathcal{I} = \underbrace{\int_X \mathrm{e}^{-2\varphi}\mathscr{L}}_{\text{(dualized) bosonic action}} \quad - \underbrace{\int_X |e|\,\mathrm{e}^{-2\varphi}\left(4(|\not{D}\epsilon|^2 - |D\epsilon|^2) + \frac{\alpha'}{4}(\operatorname{tr}|\not{\mathcal{F}}\epsilon|^2 - \operatorname{tr}|\not{\mathcal{R}}^+\epsilon|^2)\right)}_{=\mathcal{O}(\alpha'^2)\text{ along the flow or at fixed points}}$$

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V. Generalization to G_2 and $\mathrm{Spin}(7)$ structure manifolds

Other heterotic flows

Generalizations of anomaly flows

- $\mathrm{SU}(n)$ structure manifolds [Phong–Picard–Zhang 2018]

$$\partial_\lambda \left(e^{-2\varphi} J^{n-1}\right) = 2i\, \partial\bar{\partial} J^{n-2} + \ldots \qquad \partial_\lambda \left(e^{-2\varphi}\Omega\right) = 0$$

- other spaces?
  - → supergravity reformulation of the flow is dimension-agnostic
  - → Lichnerowicz identity exists on any (spin) manifold
- should extend to any manifold with a covariantly constant spinor
  - → properties? (e.g. supersymmetry)

Two examples

- G_2 compactifications
- $\mathrm{Spin}(7)$ compactifications

G_2 heterotic flow

G_2 structure manifolds

- G_2 structure defined from a nowhere vanishing spinor in seven dimensions
- associative three-form ϕ and coassociative four-form $\star\phi$

$$\boxed{\phi_{m_1\dots m_3} = -\mathrm{i}\epsilon^\dagger\gamma_{m_1\dots m_3}\epsilon} \qquad \boxed{\star\phi_{m_1\dots m_4} = \epsilon^\dagger\gamma_{m_1\dots m_4}\epsilon}$$

Supersymmetric geometries

- supersymmetry conditions for Minkowski $D = 3$ compactifications

$$\boxed{\mathrm{d}\left(\mathrm{e}^{-2\varphi}\star\phi\right) = 0} \qquad \boxed{\phi\wedge\mathrm{d}\phi = 0}$$

with H-flux is defined as $H = -\mathrm{e}^{2\varphi}\star\mathrm{d}\left(\mathrm{e}^{-2\varphi}\phi\right)$

- dual three-form field $\boxed{\widetilde{B} = -\mathrm{e}^{-2\varphi}\phi}$

G_2 heterotic flow

Define a flow for the supergravity fields — inspired by anomaly flows — as

$$\partial_\lambda \epsilon = \alpha_1 \, \mathrm{e}^{2\varphi} (I - \epsilon\epsilon^\dagger) \mathrm{d}\not{H} \epsilon$$
$$\partial_\lambda e_m{}^a = \alpha_2 \, \mathrm{e}^{2\varphi} \tfrac{1}{3!} \epsilon^\dagger \gamma_m{}^{n_1 \dots n_3} \epsilon \, \mathrm{d}H^a{}_{n_1 \dots n_3}$$
$$\partial_\lambda \varphi = \alpha_3 \, \mathrm{e}^{2\varphi} \epsilon^\dagger \mathrm{d}\not{H} \epsilon$$

Flow of the G_2 form

$$\partial_\lambda \phi = \mathrm{e}^{2\varphi} \left(12\alpha_2 \, \mathbb{P}_{\mathbf{1}} + (8\alpha_1 - 6\alpha_2)\mathbb{P}_{\mathbf{7}} - 2\alpha_2 \, \mathbb{P}_{\mathbf{27}}\right) \star \mathrm{d}H$$

In particular

$$\partial_\lambda \left(\mathrm{e}^{-2\varphi} \star \phi\right) = \left((16\alpha_2 - 14\alpha_3)\mathbb{P}_{\mathbf{1}} + (8\alpha_1 - 6\alpha_2)\mathbb{P}_{\mathbf{7}} + 2\alpha_2 \, \mathbb{P}_{\mathbf{27}}\right) \mathrm{d}H$$

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\partial_\lambda \varphi &= \alpha_3\, \mathrm{e}^{2\varphi} \epsilon^\dagger \mathrm{d}\not{H} \epsilon
\end{aligned}$$

Flow of the G_2 form

$$\boxed{\partial_\lambda \phi = \mathrm{e}^{2\varphi} \left(12\alpha_2\, \mathbb{P}_{\mathbf{1}} + (8\alpha_1 - 6\alpha_2)\mathbb{P}_{\mathbf{7}} - 2\alpha_2\, \mathbb{P}_{\mathbf{27}}\right) \star \mathrm{d}H}$$

In particular

$$\begin{aligned}
\partial_\lambda \left(\mathrm{e}^{-2\varphi} \star \phi\right) &= \left((16\alpha_2 - 14\alpha_3)\mathbb{P}_{\mathbf{1}} + (8\alpha_1 - 6\alpha_2)\mathbb{P}_{\mathbf{7}} + 2\alpha_2\, \mathbb{P}_{\mathbf{27}}\right) \mathrm{d}H \\
&= \boxed{2\alpha_1\, \mathrm{d}H} \quad \text{for } \alpha_1 = \alpha_2 = \alpha_3
\end{aligned}$$

G_2 heterotic flow

G_2 version of anomaly flows

$$\partial_\lambda \left(e^{-2\varphi} \star \phi\right) = -\frac{1}{2}\left(dH + \frac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F} - \mathrm{tr}\,\mathcal{R}_+\wedge\mathcal{R}_+\right)\right)$$

- preserves the supersymmetry condition $d\left(e^{-2\varphi}\star\phi\right) = 0$
- fixed points solve Bianchi identity
- reproduces $\mathrm{SU}(3)$ anomaly flow on $X_7 = X_6 \times S^1$
- gradient flow formulation with

$$\mathcal{I} = \int_X (\widetilde{B} + e^{-2\varphi}\phi) \wedge \left(dH + \frac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F} - \mathrm{tr}\,\mathcal{R}_+\wedge\mathcal{R}_+\right)\right)$$

Open questions

- supersymmetry condition $\phi\wedge d\phi = 0$ not generically preserved (study torsion classes and G_2 cohomology?)
- flow of the gauge bundle?
- weak parabolicity?

Spin(7) heterotic flow

Spin(7) **structure manifolds**

- Spin(7) structure defined from a nowhere vanishing spinor in eight dimensions
- Cayley four-form Φ (self-dual)

$$\boxed{\Phi_{m_1 \dots m_4} = \epsilon^\dagger \gamma_{m_1 \dots m_4} \epsilon}$$

Supersymmetric geometries

- supersymmetry conditions for Minkowski $D = 2$ compactifications

$$\boxed{\Phi \wedge \star \mathrm{d}\Phi = 12 \star \mathrm{d}\varphi}$$

with H-flux is defined as $H = \star\, \mathrm{e}^{2\varphi} \mathrm{d} \left(\mathrm{e}^{-2\varphi} \Phi \right)$

- dual four-form field $\boxed{\widetilde{B} = -\mathrm{e}^{-2\varphi} \Phi}$

Spin(7) heterotic flow

Similarly

$$\partial_\lambda \epsilon = \alpha_1\, e^{2\varphi}(I - \epsilon\epsilon^\dagger)\mathrm{d}\not{H}\epsilon$$
$$\partial_\lambda e_m{}^a = \alpha_2\, e^{2\varphi}\, \tfrac{1}{3!}\epsilon_m{}^{n_1\dots n_3}\mathrm{d}H^a{}_{n_1\dots n_3}$$
$$\partial_\lambda \varphi = \alpha_3\, e^{2\varphi}\epsilon^\dagger \mathrm{d}\not{H}\epsilon$$

Flow of the $\mathrm{Spin}(7)$ **form**

$$\boxed{\partial_\lambda \Phi = \tfrac{1}{3}e^{2\varphi}\left(84\alpha_2\,\mathbb{P}_{\mathbf{1}} + 48(\alpha_1 - \alpha_2)\mathbb{P}_{\mathbf{7}} + 12\alpha_2\,\mathbb{P}_{\mathbf{35}}\right)\star\mathrm{d}H}$$

simplifies for $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{4}$

$$\partial_\lambda\left(e^{-2\varphi}\Phi\right) = -\tfrac{1}{2}(I - \star)\,\mathrm{d}H$$

$\mathrm{Spin}(7)$ heterotic flow

$\mathrm{Spin}(7)$ **version of anomaly flows**

$$\partial_\lambda \left(\mathrm{e}^{-2\varphi}\Phi\right) = -\tfrac{1}{2}(I-\star)\left(\mathrm{d}H + \tfrac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F} - \mathrm{tr}\,\mathcal{R}_+\wedge\mathcal{R}_+\right)\right)$$

- fixed points solve Bianchi identity
- reproduces $\begin{cases} \mathrm{SU}(3)\,\text{anomaly flows on } X_8 = X_6\times T^2 \\ \mathrm{G}_2\,\text{anomaly flows on } X_8 = X_7\times S^1 \end{cases}$
- gradient flow formulation with

$$\mathcal{I} = \int_X (\widetilde{B} + \mathrm{e}^{-2\varphi}\Phi)\wedge\left(\mathrm{d}H + \tfrac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F} - \mathrm{tr}\,\mathcal{R}_+\wedge\mathcal{R}_+\right)\right)$$

Open questions: supersymmetry conditions? gauge bundle?

Summary and outlook

Conclusions

Anomaly flows from a heterotic perspective

- reframing of the flow equations as the heterotic equations of motion
- gradient flow formulation, $\mathcal{I} \sim$ heterotic action restricted to a susy locus
- generalization to manifolds with parallel spinors, e.g. $\mathrm{G}_2/\mathrm{Spin}(7)$

Outlook

- understand Yang-Mills part of the flow
  - → $\mathrm{SU}(3)$: no gradient flow description
  - → $\mathrm{G}_2/\mathrm{Spin}(7)$: canonical flow to couple to anomaly flows?
- study stability (and relate to α' corrections?) [Bedulli–Vezzoni 2020]
- embed the flow in generalized geometry?
- numerical implementation?
- relate to other geometric flows, e.g. spinor flows with flux [Collins–Phong 2021]

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Thank you!

A little more on anomaly flows. . .

$\mathrm{SU}(3)$ flows on T^2 fibrations over $\mathrm{K3}$

Consider a Fu–Yau background $T^2 \hookrightarrow X \to \mathrm{K3}$

- $\mathrm{SU}(2)$ structure of $\mathrm{K3}$ $\quad \frac{1}{2} j \wedge j = \frac{1}{4} \omega \wedge \omega = \mathrm{vol}_{\mathrm{K3}}$
- one-forms $\Theta^I = \mathrm{d}\theta^I + A^I$ associated to $U(1)^2$ isometries $\quad \boxed{\mathrm{d}\Theta^I = F^I}$
  $\rightarrow$ complexified to $\Theta = \Theta^2 + \mathrm{i}\Theta^1$ and $F = F^2 + \mathrm{i}F^1$
- $\mathrm{SU}(3)$ structure

$$\boxed{J = \mathrm{e}^{2\varphi} j + \tfrac{\mathrm{i}}{2} a\, \Theta \wedge \bar{\Theta}}$$

$$\boxed{\Omega = \mathrm{e}^{2\varphi} \sqrt{a}\, \omega \wedge \Theta}$$

- supersymmetry conditions require $F^{(0,2)} = 0$

$\mathrm{SU}(3)$ **flow**

$$\partial_\lambda e^{2\varphi} = \tfrac{1}{2} \Delta_{\mathrm{K3}} e^{2\varphi} - \mu[\varphi]$$
$$\partial_\lambda A = \star_{\mathrm{K3}} \mathrm{d} F_{(2,0)}$$

with

$$\mu[\varphi]\, \mathrm{vol}_{\mathrm{K3}} = \tfrac{1}{2} a \left(F_{(1,1)} \wedge \bar{F}_{(1,1)} - F_{(2,0)} \wedge \bar{F}_{(0,2)} \right) + \tfrac{\alpha'}{8} \left(\mathrm{tr}\, \mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\, \mathcal{R}^+ \wedge \mathcal{R}^+ \right)$$

G_2 flows on T^3 fibrations over K3

Consider a Fu–Yau like background $T^3 \hookrightarrow X \to \mathrm{K3}$

- hyper-Kähler structure of K3 $\frac{1}{2} j_I \wedge j_J = \delta_{IJ} \mathrm{vol}_{\mathrm{K3}}$
- one-forms $\Theta^I = \mathrm{d}\theta^I + A^I$ associated to $U(1)^3$ isometries $\boxed{\mathrm{d}\Theta^I = F^I}$
- G_2 structure

$$\boxed{\phi = a^{1/2} \mathrm{e}^{2\varphi} j_I \wedge \Theta^I - \tfrac{1}{6} a^{3/2} \varepsilon_{IJK} \Theta^I \wedge \Theta^J \wedge \Theta^K}$$

$$\boxed{\star\phi = \tfrac{1}{2} a\, \mathrm{e}^{2\varphi} \epsilon_{IJK} j^I \wedge \Theta^J \wedge \Theta^K - \mathrm{e}^{4\varphi} \mathrm{vol}_{\mathrm{K3}}}$$

- supersymmetry conditions require $F^I = f^I + \frac{1}{2} \lambda^{IJ} j_J$
  (f^I anti-self dual, λ^{IJ} symmetric)

G_2 **flow**

$$\partial_\lambda e^{2\varphi} = \tfrac{1}{2} \Delta_{\mathrm{K3}} e^{2\varphi} + \tfrac{1}{4} a\, \lambda^{IJ} \lambda_{IJ} - \mu[\varphi]$$

$$\partial_\lambda A^I = \tfrac{1}{2} \star_{\mathrm{K3}} (\mathrm{d}\lambda^{IJ} \wedge j_J)$$

with $\mu[\varphi]\, \mathrm{vol}_{\mathrm{K3}} = \frac{1}{2} a\, f^I \wedge f_I + \frac{\alpha'}{8} \left(\mathrm{tr}\, \mathcal{F} \wedge \mathcal{F} - \mathrm{tr}\, \mathcal{R}^+ \wedge \mathcal{R}^+ \right)$

G_2 flow and anti-de Sitter compactifications

Supersymmetric $D = 3$ AdS compactifications allowed with external H-flux given by $\frac{2}{\ell}\mathrm{vol}_{\mathrm{AdS}_3}$ (ℓ: AdS radius)

Supersymmetric geometries

- supersymmetry conditions for AdS_3 backgrounds

$$\mathrm{d}\left(\mathrm{e}^{-2\varphi}\star\phi\right)=0 \qquad \phi\wedge\mathrm{d}\phi=-\frac{12}{7\ell}\phi\wedge\star\phi$$

with H-flux is defined as $H=-\mathrm{e}^{2\varphi}\star\mathrm{d}\left(\mathrm{e}^{-2\varphi}\phi\right)-\frac{2}{\ell}\phi$

(Minkowski limit $\ell\to\infty$)

- G_2 flow takes the same form

$$\partial_\lambda\left(\mathrm{e}^{-2\varphi}\star\phi\right)=-\frac{1}{2}\left(\mathrm{d}H+\frac{\alpha'}{4}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F}-\mathrm{tr}\,\mathcal{R}_+\wedge\mathcal{R}_+\right)\right)$$

G_2 flow and anti-de Sitter compactifications

Example: flow on $\mathrm{K3} \times S^3$

$$g_X = \mathrm{e}^{2\varphi} g_{\mathrm{K3}} + \tfrac{1}{4}\ell^2 g_{S^3}$$

- hyper-Kähler structure of $\mathrm{K3}$ $\quad \tfrac{1}{2} j_I \wedge j_J = \delta_{IJ} \mathrm{vol}_{\mathrm{K3}}$
- Maurer–Cartan triplet $\quad \mathrm{d}\vartheta^I + \tfrac{1}{2}\varepsilon_{IJK}\vartheta^J \wedge \vartheta^K = 0$
- G_2 structure

$$\phi = \tfrac{1}{2}\mathrm{e}^{2\varphi}\ell\, j_I \wedge \vartheta^I - \tfrac{1}{8}\ell^3 \mathrm{vol}_{S^3}$$

$$\star\phi = \tfrac{1}{8}\mathrm{e}^{2\varphi}\ell^2 \varepsilon_{IJK} j^I \wedge \vartheta^J \wedge \vartheta^K - \mathrm{e}^{4\varphi}\mathrm{vol}_{\mathrm{K3}}$$

As $\mathrm{d}H = \Delta_{\mathrm{K3}}\mathrm{e}^{2\varphi}\,\mathrm{vol}_{\mathrm{K3}}$, the G_2 flow becomes a flow for the warp factor

$$\partial_\lambda e^{2\varphi} = \frac{1}{2}\Delta_{\mathrm{K3}} e^{2\varphi} - \mu[\varphi]$$

with $\mu[\varphi]\,\mathrm{vol}_{\mathrm{K3}} = \frac{\alpha'}{8}\left(\mathrm{tr}\,\mathcal{F}\wedge\mathcal{F} - \mathrm{tr}\,\mathcal{R}\wedge\mathcal{R}\right)$