Amplitude of vector bundles and canonical metrics

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Schneider-Tancredi

Theorem (Schneider and Tancredi, Manuscripta Mathematica, 1985)

Let E be a rank two holomorphic vector bundle over a compact complex surface X. Assume that $c_1(E) > 0$ and that E is semistable with respect to $\det(E)$. Suppose $E\big|_C$ is ample for every closed curve $C \subset X$, and

$$(c_1(E)^2 - 2c_2(E)).X > 0, \quad c_2(E).X > 0.$$

Then E is ample.

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- Griffiths Positivity (GP): $\langle v, \sqrt{-1}Fv\rangle$ is a Kähler form whenever $v \neq 0$ is a vector in E. Equivalent to saying that the induced metric on $\mathcal{O}(1)$ over $\mathbb{P}(E^*)$ is positively curved.
- Nakano Positivity (NP): The form on $TX \otimes E$ induced by $\sqrt{-1}F$ is a positive bilinear form, i.e., $\sqrt{-1}F_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\bar{u}^{j\beta} > 0$ It implies Griffiths positivity. Nakano vanishing: $H^{n,q}(M, E) = 0$ for $q \geq 1$.
- Dual Nakano Positivity (DNP): The dual is Nakano negative. Dual Nakano vanishing: $H^{p,0}(M, E) = 0$ if $p < n$.
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Theorem (Lübke, Indagationes Mathematicae, 1991)

Let (E, h) be a holomorphic Hermitian rank r vector bundle over a compact Kähler manifold (X, ω). Suppose $F_h \wedge \omega^{n-1} = -\sqrt{-1}\lambda\omega^n$, where F_h is the curvature of the Chern connection of h and $\lambda > 0$ is a constant. Assume that

$$c_1(E, h) = \frac{r\lambda}{2\pi}\omega.$$

Suppose there exists a positive function ψ such that either of the following holds:

1. $n = 2$ *and* $c_1^2(E,h) - \frac{2r(r-1)}{r^2-2r+2}c_2(E, h) = \psi\omega^2$, *or*
2. $r = 2$ *and* $c_1^2(E,h) - \frac{4(n-1)^2}{n^2-2n+2}c_2(E, h) = \psi\omega^2$.

Then h is GP.

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where h_0 is a smooth background Hermitian metric, $\mu, \lambda \geq 0$ are fixed constants, $\alpha > 0$ is a large enough constant so that $\Theta_{h_0} + \alpha\omega$ is dual-Nakano positively curved, and $f_t > 0$ are smooth positive functions.

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We prove (P., C.R.Acad. Sci. '21) the following theorem.

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Let E be an ω_0-stable rank$-r$ holomorphic bundle on X. Let H_0 be a Hermitian-Einstein metric on E with respect to ω_0, that is, $\sqrt{-1}F_{H_0}\omega_0^{n-1} = \lambda\omega_0^n$. Let h be a smooth metric on E solving the following cushioned Hermitian-Einstein equation for given parameters $\epsilon \geq 0, \mu \geq 0$.

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The following system might be more reasonable: Let $h_t = e^{-f_t} g_t h_0$ where $\det(g_t) = 1$ and $g_t > 0$. In this setting, the Demailly system boils down to the following set of equations.

$$\det_{TX \otimes E^*} \left(\sqrt{-1} F_{h_t} + (1-t)\alpha\omega_0 \otimes I_{E^*} \right)^{1/r} = f_t \frac{(\det h_0)^{\lambda}}{(\det h_t)^{\lambda}} \omega_0^n, \quad (6)$$

$$\left(\sqrt{-1} F_{h_t} - \frac{\sqrt{-1}}{r} \mathrm{tr} F_{h_t} \right) \omega_t^{n-1} = -\epsilon \frac{(\det h_0)^{\mu}}{(\det h)^{\mu}} \ln \left(\frac{h h_0^{-1}}{\det(h h_0^{-1})^{1/r}} \right) \omega_0^n. \quad (7)$$

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Thank you