Chiral de Rham complex of Calabi-Yau and Hyperkähler Manifolds

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Based on joint work with Bailin Song

1. Outline

1. Overview of vertex algebras
2. Chiral de Rham complex
3. Statement of main results
4. Step 1: Reduction to invariant theory of arc spaces
5. Step 2: Arc space analogue of fundamental theorem of invariant theory for SL_n and Sp_n

2. Vertex operator algebras

Vertex operator algebras (VOAs) were studied by physicists in the 1980s and axiomatized by Borcherds (1986).

A VOA $\mathcal{V}$ is a vector space which is linearly isomorphic to an algebra of formal power series in $\mathrm{End}(\mathcal{V})[[z, z^{-1}]]$.

$$a \leftrightarrow a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \qquad a(n) \in \mathrm{End}(\mathcal{V}).$$

$\mathcal{V}$ has Wick product $:ab:$, generally nonassociative, noncommutative.

Unit 1, derivation $\partial = \frac{d}{dz}$.

Conformal weight grading $\mathcal{V} = \bigoplus_{n \geq 0} \mathcal{V}[n]$, $n \in \mathbb{Z}$ or $\frac{1}{2}\mathbb{Z}$

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3. Operator product expansion

Let $\mathcal{V}$ be a VOA, $a, b \in \mathcal{V}$. Then

$$a(z)b(w) = \sum_{n\geq 0}(a_{(n)}b)(w)(z-w)^{-n-1} + :a(z)b(w):.$$

Expansion of meromorphic function with poles along $z = w$, where

1. $:a(z)b(w):$ is regular part.
2. $(a_{(n)}b)(w)$ is polar part of order $n+1$.

Defines bilinear products $(-_{(n)}-) : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$, where $(a, b) \mapsto a_{(n)}b$.

Also $:a(z)b(w):|_{z=w}$ coincides with Wick product.

Often write

$$a(z)b(w) \sim \sum_{n\geq 0}(a_{(n)}b)(w)(z-w)^{-n-1},$$

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4. $\beta\gamma$ and bc systems

Let $V = \mathbb{C}^d$.

$\beta\gamma$-system $\mathcal{S} = \mathcal{S}(V)$ has even generators $\beta^x, \gamma^{x'}$, linear in $x \in V$, $x' \in V^*$, which satisfy

$$\beta^x(z)\beta^y(w) \sim 0, \qquad \gamma^{x'}(z)\gamma^{y'}(w) \sim 0,$$
$$\beta^x(z)\gamma^{x'}(w) \sim \langle x', x\rangle\,(z-w)^{-1}.$$

Fix a basis $x_1, \dots, x_d$ for V, dual basis $x'_1, \dots, x'_d$ for V^*.

Write $\beta^{x_i} = \beta^i$, $\gamma^{x'_i} = \gamma^i$.

Conformal vector $L^{\mathcal{S}} = \sum_{i=1}^d : \beta^i \partial\gamma^i :$ of central charge $2d$.

β^i, γ^i primary of weights $1, 0$ respectively.

As vector spaces,

$$\mathcal{S} \cong \mathrm{Sym} \bigoplus_{n\geq 0} (V_n \oplus V_n^*), \text{where } V_n \cong V,\ \ V_n^* \cong V^*.$$

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As above, write $b^{x_i} = b^i$, $c^{x_i'} = c^i$.

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6. $\beta\gamma$ and bc systems

Let $\mathcal{W} = \mathcal{S} \otimes \mathcal{E}$. Conformal vector

$$L^{\mathcal{W}} = L^{\mathcal{S}} \otimes 1 + 1 \otimes L^{\mathcal{E}}$$

of central charge 0.

Interpretation:

1. γ^i corresponds to coordinate function $x_i' \in V^*$.
2. c^i corresponds to one-form $dx_i' \in \Omega^1(V)$.
3. β^i corresponds to vector field $\frac{d}{dx_i'} \in \mathrm{Vect}(V)$.
4. b^i corresponds to contraction operator $\iota_{d/dx_i'}$.

Suppose U is a coordinate open set in some complex manifold M with local coordinates $x_1', \dots, x_d'$,

We identify $x_1', \dots, x_d'$ with $\gamma^1, \dots, \gamma^d$, and define

$$\Omega^{\mathrm{ch}}(U) = \mathcal{O}(U) \otimes_{\mathbb{C}[\gamma^1, \dots, \gamma^d]} \mathcal{W}.$$

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7. $\beta\gamma$ and bc systems

This has a natural vertex algebra structure where for each $f \in \mathcal{O}(U)$,

$$\partial f = \sum_{i=1}^{d} :\frac{\partial f}{\partial \gamma^i}\partial\gamma^i:, \qquad fg = :fg:.$$

We have

$$\beta^i(z)f(w) \sim \frac{\partial f}{\partial \gamma^i}(w)(z-w)^{-1},$$
$$b^i(z)f(w) \sim 0, \qquad c^i(z)f(w) \sim 0.$$

Let $\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d$ be another set of coordinates on U, so that

$$\tilde{\gamma}^i = f^i(\gamma^1, \ldots, \gamma^d), \qquad \gamma^i = g^i(\tilde{\gamma}^1, \ldots, \tilde{\gamma}^d).$$

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8. $\beta\gamma$ and bc systems

We have following transformations:

$$\tilde{c}^i = \sum_{j=1}^{d} : \frac{\partial f^i}{\partial \gamma^j}(z) c^j :,$$

$$\partial \tilde{\gamma}^i = \sum_{j=1}^{d} : \frac{\partial f^i}{\partial \gamma^j}(z) \partial \gamma^j :,$$

$$\tilde{b}^i = \sum_{j=1}^{d} : \frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma)) b^j :$$

$$\tilde{\beta}^i = \sum_{j=1}^{d} : \frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma)) \beta^j : + \sum_{j=1}^{d} \sum_{k=1}^{d} : \frac{\partial}{\partial \gamma^k} \left(\frac{\partial g^j}{\partial \tilde{\gamma}^i}(f(\gamma)) \right) c^k b^j : .$$

9. $\beta\gamma$ and bc systems

Thm: (Malikov, Schechtman, Vaintrob 1998) This defines a sheaf of vertex algebras on M.

Weight grading $\Omega^{\mathrm{ch}}_M = \bigoplus_{n\geq 0} \Omega^{\mathrm{ch}}_M[n]$, and $\Omega^{\mathrm{ch}}_M[0] \cong \Omega^*_M$ ordinary de Rham sheaf.

Let $\mathcal{O}_M$ denote the sheaf of holomorphic functions on M.

A subtlety: Ω^{ch}_M is not a sheaf of $\mathcal{O}_M$-modules.

For an open set U, given functions $f, g \in \mathcal{O}(U)$ and $\alpha \in \Omega^{\mathrm{ch}}(U)$, typically $: f(: g\alpha :) : \neq : (fg)\alpha :$.

However, Ω^{ch}_M has a filtration such that associated graded sheaf is sections of the bundle

$$\mathrm{Sym}\Big(\bigoplus_{n=0}^{\infty}(TM_n \oplus T^*M_n)\Big) \bigotimes \bigwedge \Big(\bigoplus_{n=1}^{\infty}(TM_n \oplus T^*M_n)\Big).$$

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10. Some global sections

Consider the following fields given in local coordinates:

$$L = \sum_{i=1}^{d} :\beta^i \partial \gamma^i - :b^i \partial c^i:,$$

$$J = \sum_{i=1}^{d} :b^i c^i:$$

$$Q = \sum_{i=1}^{d} :\beta^i c^i:$$

$$G = \sum_{i=1}^{d} :b^i \partial \gamma^i:.$$

11. Some global sections

Under change of coordinates

$$\tilde{\gamma}^i = f^i(\gamma^1, \dots, \gamma^d), \qquad \gamma^i = g^i(\tilde{\gamma}^1, \dots, \tilde{\gamma}^d),$$

we write down same fields in new variables $\tilde{\beta}^i, \tilde{\gamma}^i, \tilde{c}^i, \tilde{b}^i$ and then rewrite them in terms of old variables.

We get

$$\tilde{L} = L$$

$$\tilde{J} = J + \partial\left(\mathrm{Tr}\,\log \frac{\partial f^i}{\partial \gamma^j}\right),$$

$$\tilde{Q} = Q + \partial\left(\sum_{r=1}^{d} : \frac{\partial}{\partial \tilde{\gamma}^r}\left(\mathrm{Tr}\,\log \frac{\partial g^i}{\partial \tilde{\gamma}^j}\right)\tilde{c}^r :\right),$$

$$\tilde{G} = G.$$

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$$\tilde{G} = G.$$

12. Some global sections

Conclusion:

1. L and G are globally defined.
2. J and Q are globally defined when $c_1(TM) = 0$.
3. Zero modes $J_{(0)}$ and $Q_{(0)}$ are always globally defined.

We have $[Q_{(0)}, G_{(1)}] = L_{(1)}$ and $Q_{(0)}|_{\Omega^*_M} = d$, so $(\Omega^*_M, d_{DR}) \hookrightarrow (\Omega^{ch}_M, Q_{(0)})$ is a quasi-isomorphism.

From now on we assume $c_1(TM) = 0$.

It is convenient to change variables: replace L with

$$T = L + \frac{1}{2}\partial J$$

which is a Virasoro field with $c = 3d$, where $d = \dim M$.

Note: b^i, c^i have weight $\frac{1}{2}$ with respect to T.

T, J, Q, G generate simple $\mathcal{N} = 2$ superconformal algebra with $c = 3d$.

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13. Calabi-Yau manifolds

Suppose that M is a compact Calabi-Yau manifold, with holomorphic volume form ω_0.

Choose a local coordinate system U with coordinates $\gamma^1, \ldots, \gamma^d$, so that locally

$$\omega_0|_U = d\gamma^1 \wedge \cdots \wedge d\gamma^d.$$

Corresponding to ω_0 are the following additional fields:

1. $D = : b^1 \cdots b^d :$,
2. $E = : c^1 \cdots c^d :$,
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14. Calabi-Yau manifolds

The eight fields T, J, Q, G, D, E, B, C close (nonlinearly) under OPE, and generate a vertex algebra $\mathcal{V}_0$.

D, E have weight $\frac{d}{2}$ and B, C have weight $\frac{d+1}{2}$.

$\mathcal{V}_0$ was originally introduced by Odake in 1986.

Thm: (Heluani, Ekstrand, Kallen and Zabzine, 2013)
T, J, Q, G, D, E, B, C give rise to global sections, so that $\Omega^{\mathrm{ch}}(M)$ contains $\mathcal{V}_0$.

Case $d = 3$ was studied in detail.

Thm: (Song, 2021) $\mathcal{V}_0$ is a simple vertex algebra.

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15. Hyperkähler manifolds

Suppose next that $d = 2\ell$ is even, and M is a hyperkähler manifold, with holomorphic symplectic form ω_1.

Choose a local coordinate system U with coordinates $\gamma^1, \dots, \gamma^d$, so

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16. Main result

Thm: (L., Song, 2021)

1. If M is a compact d-dimensional Calabi-Yau manifold with holonomy group SU_d, then the algebra of global sections $\Omega^{\mathrm{ch}}(M) = \mathcal{V}_0$.
2. If M is a compact $d = 2\ell$-dimensional hyperkähler manifold with holonomy group Sp_ℓ, then the algebra of global sections $\Omega^{\mathrm{ch}}(M) = \mathcal{V}_1$.

In particular, $\Omega^{\mathrm{ch}}(M)$ only depends on dim M and the holonomy group.

There are two steps to the proof.

1. $\Omega^{\mathrm{ch}}(M)$ is isomorphic to the subalgebra of $\mathcal{W}$ which is invariant under a Lie algebra of Cartan type.
2. This algebra can be described using the arc space analogue of Weyl's first fundamental theorem of invariant theory.

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17. **Step 1**

Step 1 is carried out in the following paper:

B. Song, *The global sections of Chiral de Rham complexes on compact Ricci-flat Kähler manifolds*, Comm Math. Phys 382, 351-397 (2021).

Sheaf cohomology $H^*(M, \Omega_M^{\mathrm{ch}})$ is called **chiral Hodge cohomology** of M, global section algebra $\Omega^{\mathrm{ch}}(M)$ is just $H^0(M, \Omega_M^{\mathrm{ch}})$.

Regarding M as a smooth $2d$-dimensional real manifold, we have smooth chiral de Rham complex $\Omega_M^{\mathrm{ch,\ sm}}$ which contains Ω_M^{ch} as a subsheaf, as well as complex conjugate.

We have antiholomorphic fields $\bar{L}, \bar{G}, \bar{Q}, \bar{J}$ which commute with L, G, Q, J.

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Bigrading

$$\Omega_M^{\text{ch, sm}} = \bigoplus_{k\geq 0} \bigoplus_{\ell\in\mathbb{Z}} \Omega_M^{\text{ch, sm}}[k,\ell],$$

where $\bar{L}_{(1)}$ acts by $k \cdot \mathrm{Id}$ and $\bar{J}_{(0)}$ acts by $\ell \cdot \mathrm{Id}$ on $\Omega_M^{\text{ch, sm}}[k,\ell]$.

Define $\Omega_M^{\text{ch},\ell} = \Omega_M^{\text{ch, sm}}[0,\ell]$ and $\Omega_M^{\text{ch},*} = \bigoplus_{\ell=0}^{d} \Omega_M^{\text{ch, sm}}[0,\ell]$.

Set $\bar{\partial} = \bar{Q}_{(0)}|_{\Omega_M^{\text{ch},*}}$, so $(\Omega_M^{\text{ch},*}, \bar{\partial}) \hookrightarrow (\Omega_M^{\text{ch, sm}}, \bar{Q}_{(0)})$ is a subcomplex.

We have

$$[\bar{Q}_{(0)}, \bar{G}_{(1)}] = \bar{L}_{(1)},$$

so $\bar{G}_{(1)}$ is a contracting homotopy for $\bar{Q}_{(0)}$, and above inclusion is a quasi-isomorphism.

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Given a vector space $V = \mathbb{C}^d$, let $\mathrm{Vect}(V)$ be the Lie algebra of algebraic vector fields $\sum_{i=1}^{d} P_i \frac{\partial}{\partial x_i'}$, where P_i is a polynomial in linear functions $x_1', \dots, x_d'$ on V.

Given a k-form $\omega \in \wedge^k V^*$, we have Lie subalgebra $\mathrm{Vect}(V, \omega)$ annihilating ω.

Given a global section $\alpha \in H^0(M, \Omega_M^{\mathrm{ch}})$ and a point $x \in M$, one can restrict α to x, obtaining an element of $\mathcal{W} = \mathcal{W}(T_x M)$ the $bc\beta\gamma$-system of rank d.

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1. If M is a compact d-dimensional Calabi-Yau manifold with holonomy group SU_d, then $H^0(M, \Omega_M^{\mathrm{ch}}) \cong \mathcal{W}^{\mathrm{Vect}(T_x M, \omega_0|_x)}$.
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$\mathcal{W}^{\mathrm{Vect}(T_xM,\omega_0|_x)}$ and $\mathcal{W}^{\mathrm{Vect}(T_xM,\omega_0|_x)}$ are contained in the subalgebra $\mathcal{W}_+$ generated by $\beta^i, \partial\gamma^i, b^i, c^i$.

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Associated graded algebra

$$\mathrm{gr}(\mathcal{W}_+) \cong \mathrm{Sym}\left(\bigoplus_{n\geq 0} \mathbb{C}^d_n \oplus (\mathbb{C}^d_n)^*\right) \bigotimes \bigwedge \left(\bigoplus_{n\geq 0} \mathbb{C}^d_n \oplus (\mathbb{C}^d_n)^*\right).$$

This has natural action of arc space G_∞ where $G = SL_d$ or Sp_ℓ, and invariants under G_∞ are same as invariants under $\mathfrak{g}[t]$.

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21. Arc spaces

Let X be a scheme of finite type over $\mathbb{C}$.

There is another scheme X_∞ called the **arc space** of X.

It is characterized by its **functor of points**.

For a commutative ring A, the A-valued points of X_∞ are in bijection with the $A[[t]]$-valued points of X.

In other words,

$$\mathrm{Hom}(\mathrm{Spec}\ A, X_\infty) \cong \mathrm{Hom}(\mathrm{Spec}\ A[[t]], X).$$

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Suppose $X = \text{Spec } R$ for

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$\mathbb{C}$-valued points of $X_\infty \leftrightarrow \mathbb{C}[[t]]$-valued points of X.

Morphism $\text{Spec } \mathbb{C}[[t]] \to \text{Spec } \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_k \rangle$ corresponds to a ring homomorphism

$$\phi : \mathbb{C}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_k \rangle \to \mathbb{C}[[t]].$$

ϕ is determined by its **values on the generators** $x_1, \ldots, x_n$.

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$$\phi(x_i) = x_i^{(0)} + x_i^{(1)} t + x_i^{(2)} t^2 + \cdots, \qquad i = 1, \ldots, n.$$

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Since ϕ is a ring homomorphism, for each $\ell = 1, \dots, k$ we have

$$f_\ell\big(\phi(x_1), \dots, \phi(x_n)\big)$$
$$= f_\ell\left((x_1^{(0)} + x_1^{(1)} t + x_1^{(2)} t^2 + \cdots), \dots, (x_n^{(0)} + x_n^{(1)} t + x_n^{(2)} t^2 + \cdots)\right)$$
$$= 0.$$

We regard $x_i^{(j)}$ for $j = 0, 1, 2, \dots,$ as **coordinate functions** on X_∞.

Polynomial ring $\mathbb{C}[x_i^{(j)}]$ has a derivation D defined as follows:

1. $D(x_i^{(j)}) = x_i^{(j+1)}$.
2. Extend by the Leibniz rule to monomials.
3. Extend by $\mathbb{C}$-linearity to all of $\mathbb{C}[x_i^{(j)}]$

23. Arc spaces

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24. Arc spaces

Ex:

$$D\left(x_1^{(3)}(x_2^{(4)})^3 + 2(x_3^{(5)})^7\right)$$
$$= x_1^{(4)}(x_2^{(4)})^3 + x_1^{(3)}3(x_2^{(4)})^2x_2^{(5)} + 14(x_3^{(5)})^6x_3^{(6)}.$$

In particular, $f_\ell^{(r)} = D^r(f_\ell)$ is a well-defined polynomial in $\mathbb{C}[x_i^{(j)}]$.

The requirement

$$f_\ell(\phi(x_1), \dots, \phi(x_n)) = 0, \text{ for all } \ell = 1, \dots, k$$

translates to the following condition:

For all $\ell = 1, \dots, k$ and $r \geq 0$,

$$f_\ell^{(r)}(x_1^{(0)}, \dots, x_n^{(0)}, x_1^{(1)}, \dots, x_n^{(1)}, \dots, x_1^{(r)}, \dots, x_n^{(r)}) = 0.$$

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25. **Arc spaces**

Therefore $X_\infty = \text{Spec } R_\infty$, where

$$R_\infty = \mathbb{C}[x_1^{(j)}, \dots, x_n^{(j)} |\ j \geq 0] / \langle f_1^{(\ell)}, \dots, f_k^{(\ell)} |\ \ell \geq 0 \rangle.$$

Identifying x_i with $x_i^{(0)}$, we have embedding $R \hookrightarrow R_\infty$.

By construction,

1. R generates R_∞ as a differential algebra.
2. Ideal of relations among $x_i^{(j)}$ is a differential ideal, and is generated by relations in R.

R_∞ satisfies a **universal property**:

If S is any differential commutative ring containing R which is generated by R as a differential ring, then S is a quotient of R_∞.

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26. Invariant theory of arc spaces

G an algebraic group, V a finite-dimensional G-module.

Arc space G_∞ is an algebraic group, and G_∞ acts on V_∞.

Consider categorical quotient $V /\!\!/ G = \text{Spec } \mathbb{C}[V]^G$.

Quotient morphism $V \to V /\!\!/ G$ induces a morphism $V_\infty \to (V /\!\!/ G)_\infty$, so we have a morphism $V_\infty /\!\!/ G_\infty \to (V /\!\!/ G)_\infty$.

Induced ring homomorphism

$$\psi_{G,V} : \mathbb{C}[(V /\!\!/ G)_\infty] \to \mathbb{C}[V_\infty]^{G_\infty}.$$

In general, $\psi_{G,V}$ is neither injective nor surjective.

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27. Invariant theory of arc spaces

Thm: (L., Song, 2021) For $n \geq 1$, let $G = GL_n$ and $W = \mathbb{C}^n$ be the standard representation.

Let $V = W^{\oplus p} \oplus (W^*)^{\oplus q}$ be the sum of p copies of W and q copies of the dual module W^*.

Then for all p, q, $\psi_{G,V}$ is an isomorphism.

Thm: (L., Song, 2021) For $n \geq 1$, let $G = Sp_n$ and $W = \mathbb{C}^{2n}$ be the standard representation.

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Thm: (L., Song, 2021) For $n \geq 2$, let $G = SL_n$ and $W = \mathbb{C}^n$ be the standard representation.

Let $V = W^{\oplus p} \oplus (W^*)^{\oplus q}$ be the sum of p copies of W and q copies of the dual module W^*. Then

1. If $p, q \leq n+2$, $\psi_{G,V}$ is an isomorphism.
2. If $\max\{p, q\} > n+2$, $\psi_{G,V}$ is surjective but not injective.
3. Its kernel coincides with the nilradical $\mathcal{N} \subseteq \mathbb{C}[(V /\!\!/ G)_\infty]$, finitely generated as a differential ideal.

Note: Similar results for the orthogonal groups are also expected but cannot be proven at the moment.

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29. Extension to odd variables

Suppose G is an algebraic group, $\tilde{U}$ and U finite-dimensional G-modules.

For each $j \geq 0$, let $\tilde{U}_j \cong \tilde{U}^*$.

Fix a basis $\{x_{1,j}, \dots, x_{m,j}\}$ for $\tilde{U}_j$.

Let $S^{\tilde{U}} = \mathbb{C}[\bigoplus_{j \geq 0} \tilde{U}_j]$, with differential D defined by $D(x_{i,j}) = x_{i,j+1}$.

The map $\mathbb{C}[\tilde{U}_\infty] \to S^{\tilde{U}}$ sending $x_i^{(j)} \mapsto x_{i,j}$ is an isomorphism of differential algebras.

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30. **Extension to odd variables**

For $j \geq 0$, let $U_j \cong U^*$ and let $L^U = \bigwedge \bigoplus_{j \geq 0} U_j$.

Fix a basis $\{y_{1,j}, \dots, y_{n,j}\}$ for U_j.

Extend the differential on $S^{\tilde{U}}$ to an even differential D on $S^{\tilde{U}} \otimes L^U$, defined on generators by $D(y_{i,j}) = y_{i,j+1}$.

Then G_∞ acts on $S^{\tilde{U}} \otimes L^U$, and we may consider the invariant ring $(S^{\tilde{U}} \otimes L^U)^{G_\infty}$.

Let $S_0^{\tilde{U}} = \mathbb{C}[\tilde{U}_0] \subseteq S^{\tilde{U}}$ and $L_0^U = \bigwedge(\tilde{U}_0) \subseteq L$.

Let $\langle (S_0^{\tilde{U}} \otimes L_0^U)^G \rangle$ be the differential algebra generated by $(S_0^{\tilde{U}} \otimes L_0^U)^G$, which lies in $(S^{\tilde{U}} \otimes L^U)^{G_\infty}$.

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Extend the differential on $S^{\tilde{U}}$ to an even differential D on $S^{\tilde{U}} \otimes L^U$, defined on generators by $D(y_{i,j}) = y_{i,j+1}$.

Then G_∞ acts on $S^{\tilde{U}} \otimes L^U$, and we may consider the invariant ring $(S^{\tilde{U}} \otimes L^U)^{G_\infty}$.

Let $S_0^{\tilde{U}} = \mathbb{C}[\tilde{U}_0] \subseteq S^{\tilde{U}}$ and $L_0^U = \bigwedge(\tilde{U}_0) \subseteq L$.

Let $\langle (S_0^{\tilde{U}} \otimes L_0^U)^G \rangle$ be the differential algebra generated by $(S_0^{\tilde{U}} \otimes L_0^U)^G$, which lies in $(S^{\tilde{U}} \otimes L^U)^{G_\infty}$.

30. Extension to odd variables

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31. Extension to odd variables

Since G acts on the direct sum $\tilde{U} \oplus U^{\oplus k}$, we have a map

$$\mathbb{C}[((\tilde{U} \oplus U^{\oplus k}) /\!\!/ G)_\infty] \to \mathbb{C}[(\tilde{U} \oplus U^{\oplus k})_\infty]^{G_\infty}. \tag{1}$$

Thm: (L., Song, 2021). Suppose that (1) is surjective for all $k \geq 0$. Then

$$(S^{\tilde{U}} \otimes L^{U})^{G_\infty} = \langle (S_0^{\tilde{U}} \otimes L_0^{U})^{G} \rangle.$$

In particular, if we fix a generating set $\{\alpha_1, \dots, \alpha_k\}$ for $(S_0^{\tilde{U}} \otimes L_0^{U})^{G}$, then $\{\alpha_1, \dots, \alpha_k\}$ generates $(S^{\tilde{U}} \otimes L^{U})^{G_\infty}$ as a differential algebra.

Main result follows from cases

1. $G = SU_d$ and $U = \mathbb{C}^d = \tilde{U}$,
2. $G = Sp_\ell$ and $U = \mathbb{C}^{2\ell} = \tilde{U}$.

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