Instantons on ALF spaces and codimension-1 collapse

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§. Motivation

→ behaviour of Yang-Mills instantons on special holonomy manifolds close (in Gromov-Hausdorff sense) to a space of one lower dimension

Kraan – van Baal (1998): charge 1 $SU(2)$ instanton (w/ 0 magnetic charge) on $\mathbb{R}^3 \times S^1_\epsilon$ as a superposition of a monopole & an anti-monopole

→ interesting non-compact hyperkähler geometries as gauge theoretic moduli spaces

classical

"quantum"

§. Background: ALF gravitational instantons

(M^4, g) hyperkähler complete

$\mathrm{Hol}(g) \subseteq SU(2)$

$$S^1 \hookrightarrow \widetilde{M \setminus K}$$
$$\downarrow$$
$$\mathbb{R}^3 \setminus \overline{B_R}$$

$g \approx g_{\mathbb{R}^3} + \epsilon^2 \theta^2$ up to polynom. decaying term

- **cyclic type** (multi Taub NUT)

$$A_k^{\epsilon}(p_1, \dots, p_k)$$

$\epsilon > 0$ $\quad k \in \mathbb{Z}_{\geq 0}$

$p_1 \dots p_k \in \mathbb{R}^3$

$$g_{A_k^m} = \underbrace{\left(1 + \epsilon \sum_{i=1}^{k} \frac{1}{2|x - p_i|}\right)}_{h} g_{\mathbb{R}^3} + \epsilon^2 \left(1 + \epsilon \sum_{i=1}^{k} \frac{1}{2|x - p_i|}\right)^{-1} \theta^2$$

$d\theta = * dh$

$\Delta h = 0$ on $\mathbb{R}^3 \setminus \{p_1, \dots, p_k\}$

(Gibbons-Hawking Ansatz 1978)

- dihedral type

$$D_k^{\epsilon}(\pm p_1, \dots, \pm p_k)$$

$\epsilon > 0$ $\quad k \in \mathbb{Z}_{\geq 0}$

$\pm p_1 \dots \pm p_k \in \mathbb{R}^3$

$\rightarrow$ D_0 = Atiyah-Hitchin mfld $\approx A_{-4}^{\epsilon}(0)/\mathbb{Z}_2$

(Atiyah-Hitchin 1985)

$$h = 1 + \epsilon \frac{-4}{2|x|} \qquad \begin{cases} x \longmapsto -x \\ \theta \longmapsto -\theta \end{cases}$$

§. Background: the structure group

G compact semisimple Lie group of rank rk

$\mathfrak{g} = \mathrm{Lie}(G)$ $\qquad \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} [\![\mathfrak{g}_\alpha]\!]$

$$\underset{\langle \alpha_1^\vee, \dots, \alpha_{rk}^\vee \rangle_{\mathbb{Z}}}{\underset{\parallel}{\Lambda_{cr}}} \subseteq \Lambda \subseteq \underset{\langle \varpi_1^\vee, \dots, \varpi_{rk}^\vee \rangle_{\mathbb{Z}}}{\underset{\parallel}{\Lambda_{cw}}} \quad \text{full rank lattices}$$

maximal torus $T = \mathfrak{h}/\Lambda$

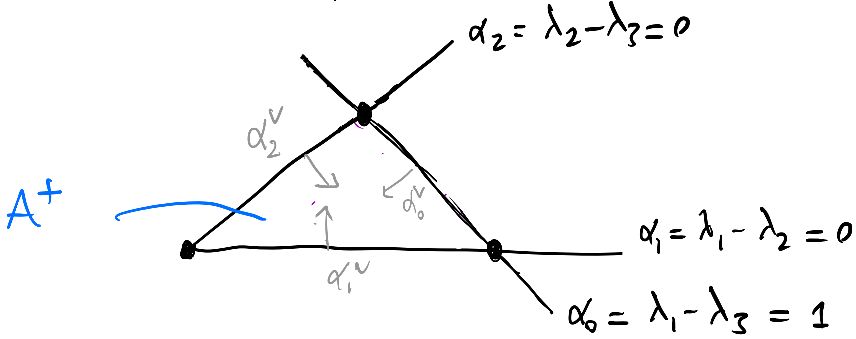
$\alpha_0^\vee = -\sum_{\mu=1}^{rk} m_\mu \alpha_\mu^\vee$ lowest negative coroot

$\pi_1(G) = \Lambda/\Lambda_{cr}$ $\qquad \{\underbrace{\varpi_i^\vee \mid m_i = 1}_{\text{cominuscule coweights, i.e. } \alpha(\varpi^\vee) \in \{0,1\}\ \forall \alpha \in R^+}\} \bmod \Lambda_{cr} = \pi_1(G_{ad})$

$\mathfrak{h}/W \ltimes \Lambda_{cr} \simeq A^+$ fundamental alcove

Example: $\mathfrak{g} = \mathfrak{su}(3)$

$$\mathfrak{h} = \left\{ \begin{pmatrix} i\lambda_1 & & \\ & i\lambda_2 & \\ & & i\lambda_3 \end{pmatrix} \,\middle|\, \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$$



§. Instantons on multi-Taub-NUT spaces

$G = G_{ad}$

$P \to TN_k$ principal G-bdle $\qquad \mathbb{A} \in \mathcal{A}(P)$

$g_\epsilon = h g_{\mathbb{R}^3} + \epsilon^2 h^{-1} \theta^2 \qquad h = 1 + \epsilon \sum_{p \in NUT} \frac{1}{2|x-p|}$

- Boundary conditions

Cherkis – Larraín-Hubach – Stern (2021):

$\mathbb{A}$ ASD, $\|F_{\mathbb{A}}\|_{L^2} < +\infty$, generic holonomy @ ∞

$$\Rightarrow \mathbb{A}|_{TN_k \setminus K} = \underbrace{A_{\gamma_m}}_{\mathbb{R}^3 \setminus B_R} + \epsilon \left(\frac{\omega}{\epsilon} - \gamma_m \frac{1}{2|x|} \right) \otimes h^{-1} \theta + O\left(|x|^{-1-\delta}\right)$$

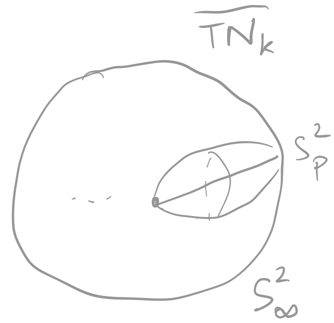
$\omega \in \mathring{A}^+$ holonomy $\qquad dA_{\gamma_m} = \frac{1}{2} \gamma_m \, dv_{S^2}$

$\gamma_m \in \Lambda$ magnetic charge

- Topological data

$$(\overline{P}, \overline{A}) \longrightarrow \overline{TN_k} = TN_k \cup S^2_\infty \cong \#_k \mathbb{CP}^2$$

singular along S^2_∞



$$w \in H^2(\overline{TN_k}; \Lambda/\Lambda_{cr})$$

γ_p cominuscule coweight $\forall\, p \in NUT$

$$\gamma_m + \sum_{p \in NUT} \gamma_p \equiv 0 \mod \Lambda_{cr}$$

$$n_0 = \langle p_1(\overline{P}), [\overline{TN_k}] \rangle$$ instanton number

$$\rightsquigarrow \gamma_m = \sum_{\mu=0}^{rk} n_\mu \alpha_\mu^\vee - \sum_{p \in NUT} \gamma_p = \sum_{\mu=1}^{rk} (n_\mu - m_\mu n_0) \alpha_\mu^\vee - \sum_{p \in NUT} \gamma_p$$

$n_1, \dots, n_{rk}$ monopole numbers

§. A gluing construction

$\mathbb{A}_\epsilon$ as $\epsilon \to 0$ ASD $*_{g_\epsilon} F_{\mathbb{A}_\epsilon} = -F_{\mathbb{A}_\epsilon}$

- $\mathbb{A}_\epsilon \approx \mathbb{A}_{sing} = A_{sing} + \epsilon \Phi_{sing} \otimes h^{-1}\theta$ abelian S^1-invariant

$$\Phi_{sing} = \frac{\omega}{\epsilon} - \sum_{\mu=0}^{rk} \sum_{i=1}^{n_\mu} \frac{1}{2|x - p_\mu^i|} \alpha_\mu^\vee + \sum_{p \in NUT} \frac{1}{2|x-p|} \gamma_p$$

- near $p \in NUT$

$\epsilon^* \mathbb{A}_\epsilon \approx$ abelian instanton on $TN_{k=1}^{\epsilon=1}$ of charge γ_p

<u>Lemma</u>

Abelian instanton on TN is a rigid G instanton iff γ_p is cominuscule

- near p_μ^i, $\mu \neq 0$

$\epsilon^* \mathbb{A}_\epsilon \approx \rho_\mu (\mathbb{A}^+_{BPS})$

$\rho_\mu : SU(2) \hookrightarrow G$.
$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto \alpha_\mu^\vee$

$\mathbb{A}^+_{BPS} =$ charge 1 $SU(2)$ monopole (A_{BPS}, Φ_{BPS})
mass $\approx \frac{1}{2} \alpha_\mu^\vee(\omega)$

- near p_μ^i, $\mu = 0$

$\epsilon^* \mathbb{A}_\epsilon \approx \rho_0 (\mathbb{A}^-_{BPS})$

$\rho_0 : \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \mapsto -\alpha_0^\vee$

$\mathbb{A}^-_{BPS} = g^* \mathbb{A}^+_{BPS}$ mass $\approx \frac{1}{2}(1 + \alpha_0^\vee(\omega))$

$g(x,t) = \exp\left(\frac{1}{2} t \hat{\Phi}_{BPS}(x) \right)$

Theorem (F. – Ross)

$$\mathcal{M}_{\epsilon}^{TN_k, G}\left(\omega, \gamma_m, n_0, \{\gamma_p\}_{p\in NUT}\right)$$

- non-empty if (and only if) $n_\mu \geq 0 \quad \forall\, \mu = 0, \dots, rk$
- $\dim = 4 \sum_{\mu=0}^{rk} n_\mu$, smooth, hyperkähler, in general incomplete
- contains a full-dimensional family

  $A_\epsilon(\{p_\mu^i\})$ s.t. as $\epsilon \to 0$

  - $A_\epsilon \approx$ trivial $+ \omega\, h^{-1}\theta$
  - $\epsilon^* A_\epsilon \approx$ rigid instanton on TN $\quad T \xrightarrow{\gamma_p} G$
  - $\epsilon^* A_\epsilon \approx A^+_{BPS}$ near p_μ^i, $\mu \neq 0$
  - $\epsilon^* A_\epsilon \approx A^-_{BPS}$ near p_μ^i, $\mu = 0$

Rmks

- dimension formula via quantitative excision principle
- $n_0 = 0 \Rightarrow A_\epsilon$ S^1-invariant $\rightsquigarrow$ singular monopoles on $\mathbb{R}^3$

γ_p cominuscule $\Rightarrow$ completeness of L^2-metric (no monopole bubbling)

for $G = SO(3)$ can prove this via energy of S^1-invariant instantons on S^4

- converse compactness statement...
- instantons on $M^3 \times S^1_\epsilon$, M closed ?

§. Hypertoric manifolds

or cyclic QALF hK metrics

(Bielawski–Dancer 2000, Goto)

$T^n = \mathfrak{t}/\Lambda \curvearrowright M^{4n}$ triholo $\qquad \mu : M \longrightarrow \mathfrak{t}^* \otimes \mathbb{R}^3$

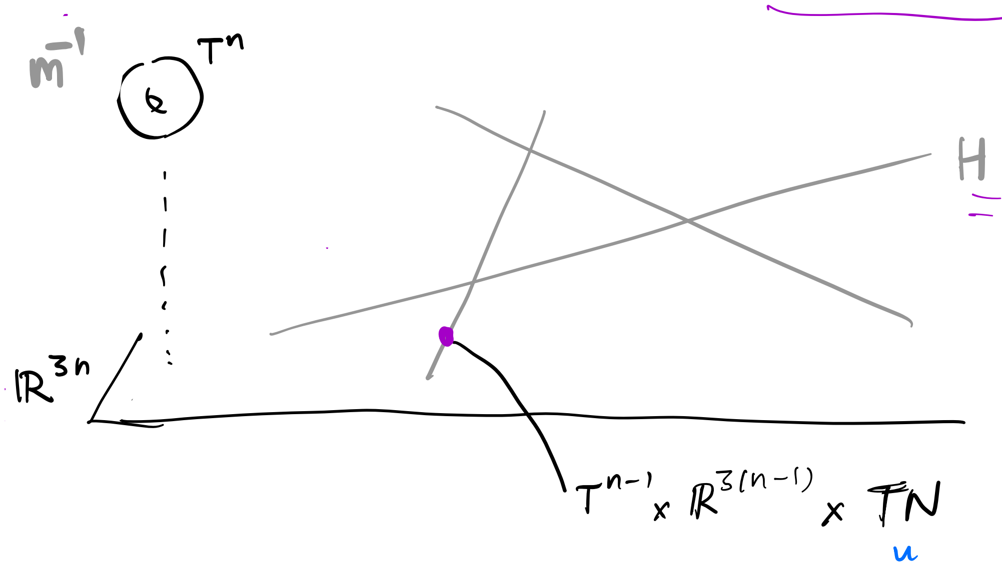
$m \in \mathrm{Sym}^2_{>0}(\mathfrak{t})$ Taub-NUT deformation parameters

$u \in \Lambda \subset \mathfrak{t}, \lambda \in \mathbb{R}^3 \rightsquigarrow H_{u,\lambda} = \{ \langle u, \cdot \rangle = \lambda \} \subset \mathbb{R}^3 \otimes \mathfrak{t}^*$

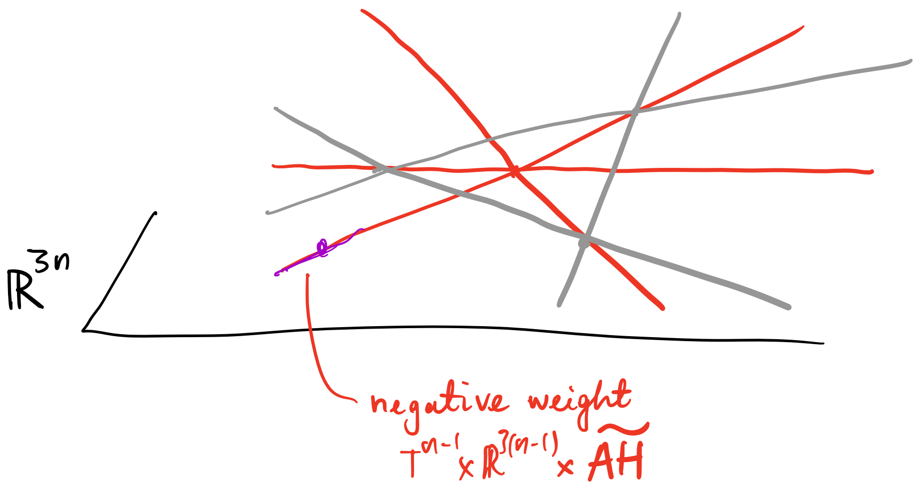
flat

(Pedersen–Poon 1988) $\quad \Phi : \mathbb{R}^3 \otimes \mathfrak{t}^* \longrightarrow \mathrm{Sym}^2(\mathfrak{t})$

$$\Phi(x) = m + \sum_{H_{u,\lambda}} \frac{\langle u, \cdot \rangle \otimes \langle u, \cdot \rangle}{2 \left| \langle u, x \rangle - \lambda \right|}$$



Q: "Dihedral" QALF hk metrics? (cf. Bielawski's talk)



$/W$

Weyl gp

§. Asymptotic geometry of instantons moduli spaces

gluing construction $\rightsquigarrow$

$$g_{L^2}^{\mathcal{M}_\epsilon} \approx \text{hypertoric}/W + O(\epsilon^2)$$

T = maximal torus of $\prod_{\mu=0}^{rk} U(n_\mu)$ up to finite quotients

$W = \prod_{\mu=0}^{rk} \Sigma_{n_\mu}$

$\frac{\partial}{\partial \varphi_\mu^i}$ generate T-action

"phase" of p_μ^i

Four types of flats:

- $p_\mu^i - p_\mu^{i'} = 0$ mult -2
- $C_{\mu'\mu} p_\mu^i - C_{\mu\mu'} p_{\mu'}^j = 0$ mult $= |\langle \alpha_\mu^\vee, \alpha_{\mu'}^\vee \rangle|$

  $\alpha_\mu(\alpha_{\mu'}^\vee)$

- $p_\mu^i = p$ mult 1 if $\gamma_p = \varpi_\mu^\vee$ for $\mu \in \{1, .., rk\}$

  $\{0,1\} \ni \alpha_0(\gamma_p) + 1$ if $\mu = 0$

Example: $\mathbb{R}^3 \times S^1$ $\quad G = SU(2)$ $\quad n_1 = 2$ $\quad n_0 = 1$ $\quad r_m = 1$