The Arithmetic and Modularity of Black Holes in String Theory

Part II

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Field Theory

A field is a set of numbers for which $+$ and $\times$ are defined.

$\mathbb{F}$ is an abelian group wrt $+$
and $\mathbb{F} \setminus \{0\}$ is an abelian group wrt $\times$.

$\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ are fields, but $\mathbb{Z}$ is not.

Also $\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$ is a field.

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2}.$$

$\mathbb{F}_p = \{0, 1, \dots, p-1\}$ is a field, for p prime.

The integers mod 6, say, are not a field since $2 \times 3 = 0$.

$\mathbb{F}_7$:

x	0	1	2	3	4	5	6
x^{-1}	*	1	4	5	2	3	6
x^2	0	1	4	2	2	4	1

0, 1, 2, 4 are $\square$ mod 7 but 3, 5, 6 are $\not\square$.

$$\mathbb{F}_{7^2} = \{a + b\sqrt{3} \mid a, b \in \mathbb{F}_7\}$$

There is a field $\mathbb{F}_{p^k}$ for each p and $k = 1, 2, \dots$

Fermat's little theorem

$$c^p = c \mod p$$

So in $\mathbb{F}_p$ we have $c^p = c$.

In $\mathbb{F}_{p^k}$ the relation is $c^{p^k} = c$.

If $\alpha, \beta \in \mathbb{F}_{p^k}$ then $(\alpha+\beta)^p = \alpha^p + \beta^p$.

Consider a polynomial

$$F(x) = \sum_{m} c_m x^m \quad ; \quad x^m = x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

and let the coefficients be in $\mathbb{F}_p$ but the x_r can be in $\mathbb{F}_{p^k}$, some k.

$$F(x) = 0$$

$$\Longrightarrow \quad F(x)^p = 0$$

$$\Longrightarrow \quad F(x^p) = 0$$

$$\sum_m (c_m x^m)^p$$

$$\sum_m c_m^p x^{mp}$$

Frob: $x \to x^p$ is the Frobenius map.

The Frobenius map is an automorphism that every $X/\mathbb{F}_p$ has.

The fixed points are the x's for which

$$x^p = x$$

and these lie in $\mathbb{F}_p \subset \mathbb{F}_{p^k}$.

$$N_k = \{ x \in \mathbb{F}_{p^k} \mid F(x) = 0 \}.$$

These are the fixed points of Frob^k.

The ζ-function

$$\zeta_X = \exp\left(\sum_{k=1}^{\infty} \frac{N_k T^k}{k}\right)$$

If X is a point, then $N_k = 1 \ \forall k$.

$$\zeta_{pt} = \exp\left(\sum_k \frac{T^k}{k}\right) = \exp\left(-\log(1-T)\right) = \frac{1}{1-T}$$

$$\prod_p \zeta_{pt}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}} = \sum \frac{1}{n^s} = \zeta_{\mathbb{R}}(s).$$

If X is an elliptic curve

$$\zeta_{\mathcal{E}}(T) = \frac{1-\alpha_p T + pT^2}{(1-T)(1-pT)}$$

Breuil, Conrad, Taylor and Wiles proved that every $\mathcal{E}$ is modular. As a consequence for each $\mathcal{E}$ there is modular group $\Gamma_0(N)$ and a weight two modular form $g_2(q)$ such that

$$g_2(q) = \sum \alpha_n q^n \; ; \quad q = e^{2\pi i \tau}$$

and the α_p are the p'th coefficients in this series.

Weil Conjectures

$$\zeta_X = \frac{R_1 R_3 \cdots R_{2n-1}}{R_0 R_2 \cdots R_{2n}} \quad ; \quad \begin{aligned} R_0 &= 1-T \\ R_{2n} &= 1-p^n T \end{aligned}$$

$$R_k(T) = \det(\mathbb{1} - T\,\mathrm{Frob}_k^{-1})$$

$$\deg R_k(T) = b^k$$

For a 1 parameter CY threefold

$$\zeta_X = \frac{\cancel{R_1}\, R_3\, \cancel{R_5}}{(1-T)(1-pT)^{h''}(1-p^2T)^{h''}(1-p^3T)}$$

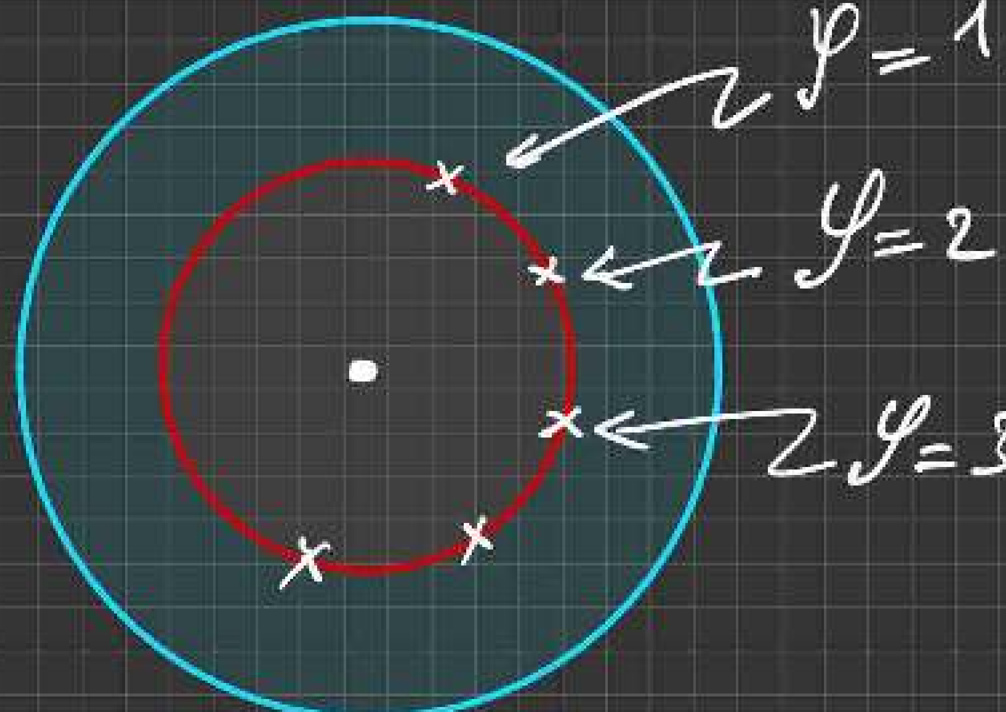
$$R_3 = \det\left(1 - T\,\mathrm{Frob}_3^{-1}\right) = \det\left(\mathbb{1} - T\,U(\varphi)\right)$$

$$R_3 = 1 + aT + bpT^2 + ap^3T^3 + p^6T^4$$

$$U(\varphi) = E^{-1}(\varphi^p)\,U(0)\,E(\varphi); \qquad E_j{}^k = \dot{\theta}^k \varpi_j$$

$$U(0) = \begin{bmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ p^3\gamma & & & p^3 \end{bmatrix}$$

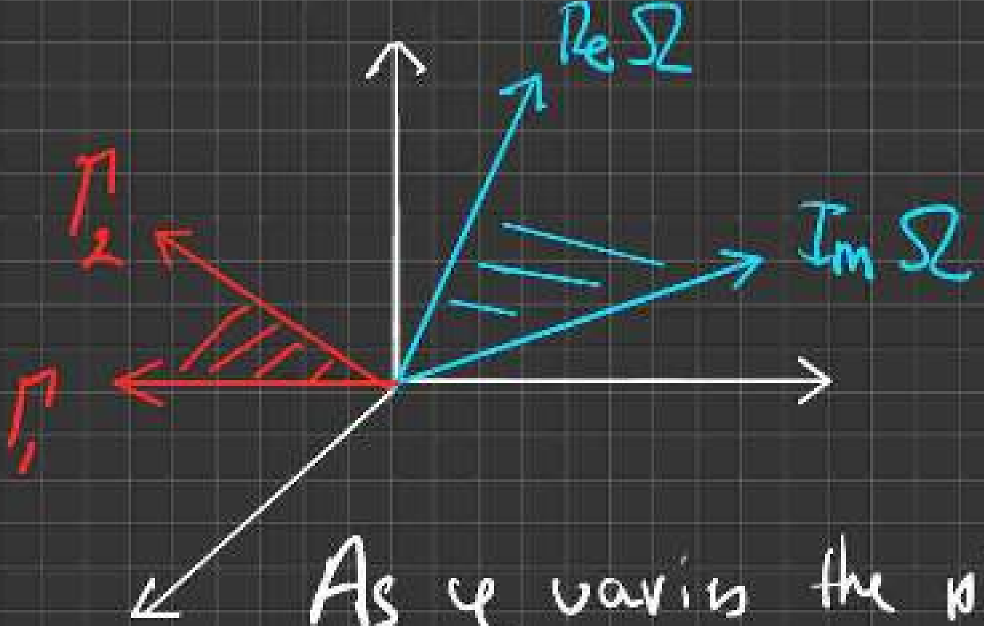
$$\frac{\zeta_p(3)\,\chi}{y}$$



$\boxed{\text{rank } 2}$ attractors

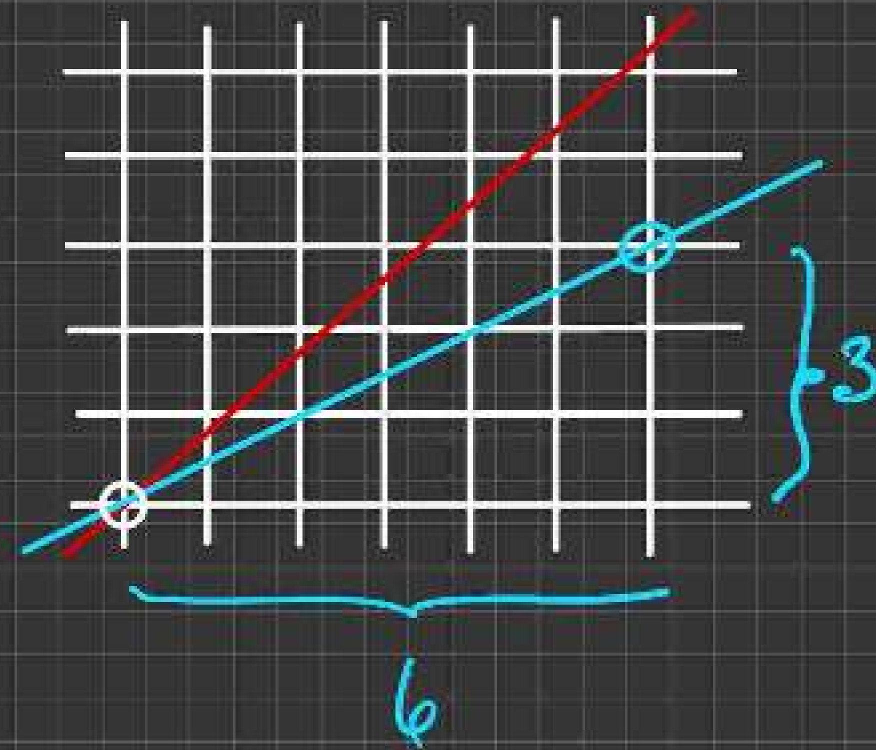
At an attractor point of rank 2 there are $\boxed{\text{two}}$ vectors in $H^3(X,\mathbb{Z})$

$$\Gamma_1, \Gamma_2 \in H^{(3,0)} \oplus H^{(0,3)}$$



As φ varies the plane $V_{\mathbb{R}}(\varphi)$ moves and at a rank 2 attractor point $\varphi = \varphi_*$ the plane $V_{\mathbb{R}}(\varphi)$ coincides with the plane generated by Γ_1 & Γ_2

A general plane or line will not pass through any lattice points

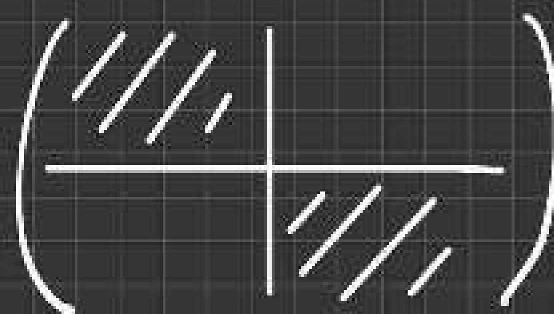


If it does pass through the origin and another lattice point then the slope is necessarily rational.

In the rank 2 case:

$$R = \det\left(\mathbb{1} - T\,\mathrm{Frob}_3^{-1}\right)$$

Frob_3^{-1} :



$$R(T) = \left(\underbrace{1 - p\alpha T + p^3 T^2}_{H^{1,2} \oplus H^{2,1}}\right)\left(\underbrace{1 - \beta T + p^3 T^2}_{H^{30} \oplus H^{03}}\right)$$

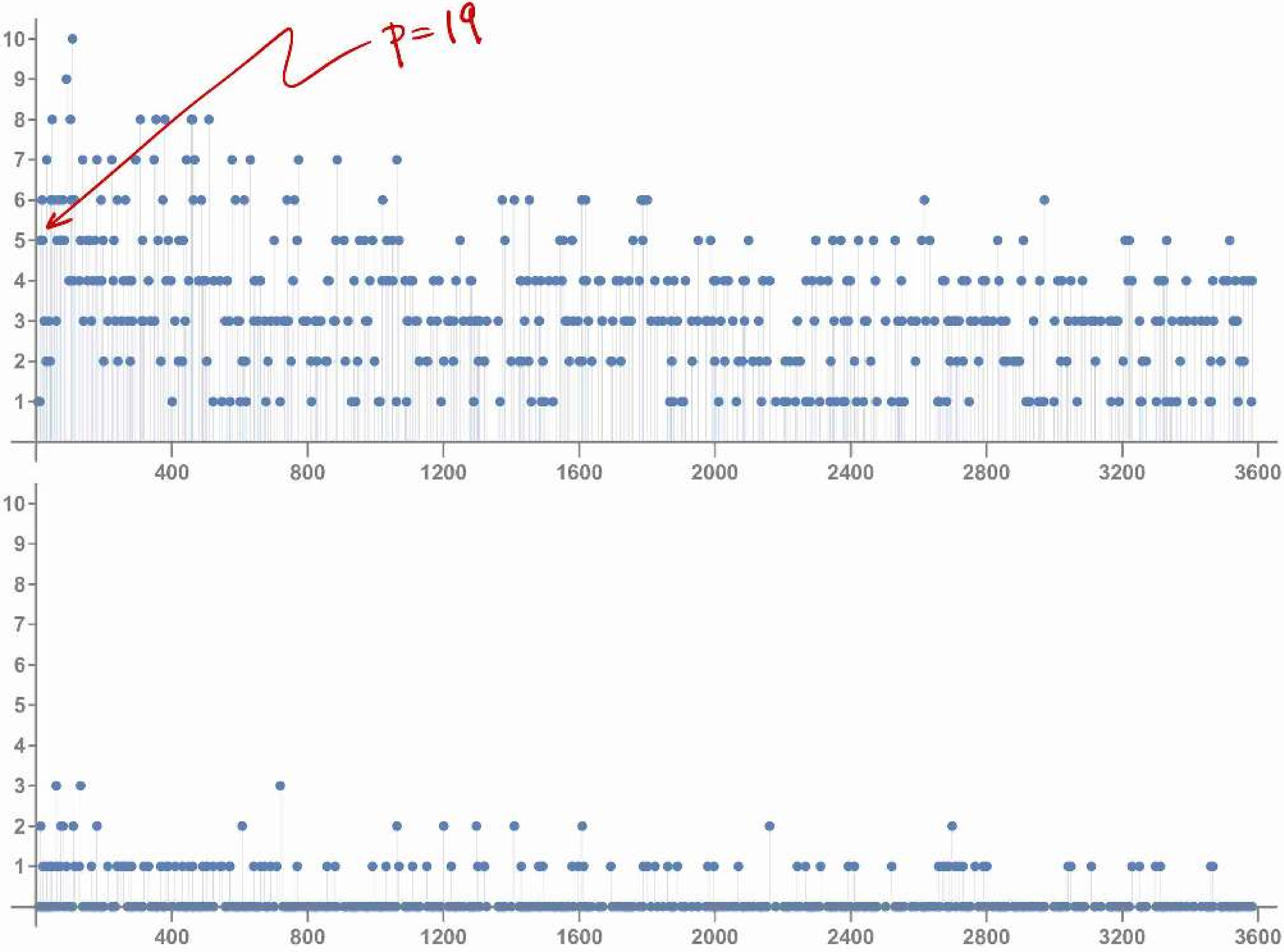
Strategy: Make tables of $R(\varphi, T)$ for many p and φ and look for <u>persistent</u> <u>factorisations</u>

That is factorisations that occur when φ_* is the root of a polynomial $G(\varphi)$ with rational (so integer) coefficients.

$p = 19$			
φ	smooth/sing.	singularity	$R(T)$
1	singular	1	$(1 - pT)(1 - 20T + p^3T^2)$
2	smooth		$1 + 4pT + 2pT^2 + 4p^4T^3 + p^6T^4$
3	smooth		$1 - 8T + 242pT^2 - 8p^3T^3 + p^6T^4$
4	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
5	smooth		$(1 + 4pT + p^3T^2)(1 - 60T + p^3T^2)$
6	smooth		$1 + 8T - 318pT^2 + 8p^3T^3 + p^6T^4$
7	smooth		$1 - 44T - 238pT^2 - 44p^3T^3 + p^6T^4$
8	smooth		$(1 - 2pT + p^3T^2)(1 - 80T + p^3T^2)$
9	smooth		$(1 + 4pT + p^3T^2)(1 - 160T + p^3T^2)$
10	smooth		$1 + 12T + 562pT^2 + 12p^3T^3 + p^6T^4$
11	smooth		$(1 + 4pT + p^3T^2)(1 - 140T + p^3T^2)$
12	smooth		$1 + 12T + 82pT^2 + 12p^3T^3 + p^6T^4$
13	smooth		$1 + 178T + 1082pT^2 + 178p^3T^3 + p^6T^4$
14	smooth		$1 + 12T - 158pT^2 + 12p^3T^3 + p^6T^4$
15	smooth		$1 + 42T - 2p^2T^2 + 42p^3T^3 + p^6T^4$
16	singular	$\frac{1}{25}$	$(1 - pT)(1 + 76T + p^3T^2)$
17	singular	$\frac{1}{9}$	$(1 - pT)(1 - 20T + p^3T^2)$
18	smooth		$1 - 54T + 322pT^2 - 54p^3T^3 + p^6T^4$

the same

Table 1: *The R-factors for $\varphi \in \mathbb{F}_{19}$. Note the factorisations into two quadrics for the five values $\varphi = 4, 5, 8, 9, 11$.*



p=19

For the HV manifold there are always 4, 5, 8, 9, 11 factorisations when

$$7\varphi + 1 = 0$$

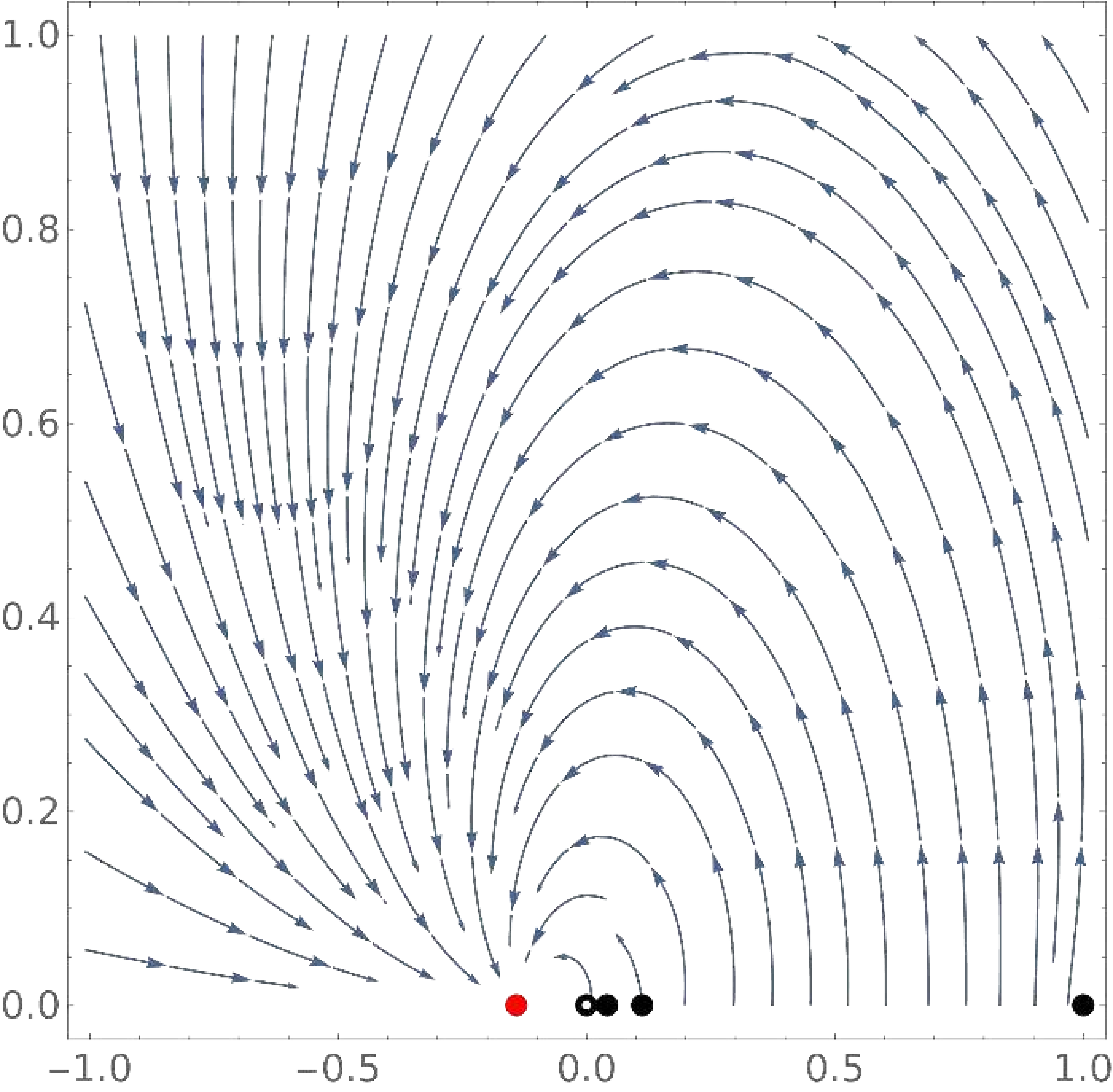
and also when

$$\varphi^2 - 66\varphi + 1 = 0 ; \quad \varphi_\pm = 33 \pm 8\sqrt{17}$$

For $p = 19$,

$$7 \times 8 = 56 = 3 \times 19 - 1 = -1 \quad \text{so} \quad -\frac{1}{7} = 8$$

and $17 = 6^2 - 19 = 6^2$ so $\varphi_\pm = 4, 5$.



1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0

Modularity

When $\psi = -1/7$ we have

$$R = (1 - p\alpha_p T + p^3 T^2)(1 - \beta_p T + p^3 T^2)$$

and there are modular forms f_2 and f_4 for $\Gamma_0(14)$, such that

$$f_2 = \sum_n \alpha_n q^n \quad \text{and} \quad f_4 = \sum_n \beta_n q^n .$$

When $\psi^2 - 66\psi + 1 = 0$, the corresponding group is $\Gamma_1(34)$.

$\Gamma_0(N) \subset SL_2(\mathbb{Z})$ is the subgroup

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } N \mid c$$

$\Gamma_1(N) \subset \Gamma_0(N)$ is the subgroup of $SL(2,\mathbb{Z})$

$$\text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N.$$

$$g\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k g(\tau)$$

Associated with these modular forms are L-functions

$$L_j(s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dy\, y^{s-1} f_j(iy) \ ; \qquad q = e^{-2\pi y}.$$

For $L_4(1)$ and $L_4(2)$ these are closely related to the products

$$\prod_p \left(1 - b_p T + p^3 T^2\right)^{-1} ; \quad T = p^{-s}$$

At φ_* the periods of Ω are given by simple rational multiples of $L_4(1)$ and $L_4(2)$.

Periods at the attractor point

There is a natural basis of periods $\xi_j(\varphi)$, $j = 0, 1, 2, 3$.

When $\varphi = -1/7$ we have:

$$\xi_0 = 7\frac{L_4(2)}{\pi^2} \quad , \quad \frac{\xi_1}{\pi} = -\frac{5}{2}\frac{L_4(1)}{\pi}$$

$$\frac{\xi_2}{\pi^2} = -\frac{7}{3}\frac{L_4(2)}{\pi^2} \quad , \quad \frac{\xi_3}{\pi^3} = \frac{11}{2}\frac{L_4(1)}{\pi}$$

So there are two linear relations between the periods. If we write these for $\tilde{\xi}_j = \xi_j / \pi^j$, then these are defined over $\mathbb{Q}$.

$$\tilde{\xi}_0 + 3\tilde{\xi}_2 = 0 \quad ; \quad 11\tilde{\xi}_1 + 5\tilde{\xi}_3 = 0$$

In an integral basis we have the following expression for the period vector.

$$\Pi(-1/7) = i\,\frac{L_4(1)}{4\pi}\begin{pmatrix} 8\kappa \\ -30\kappa \\ 0 \\ 5 \end{pmatrix} + \frac{7}{2}\,\frac{L_4(2)}{\pi^2}\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

The two integral vectors define a lattice plane, and so a two dimensional family of charge vectors.

For an attractor of rank 2 there is a two parameter family of black holes.

For $\varphi = -1/7$

$$Q_{k\ell} = k\,(4\kappa, -15\kappa, -5, 0) + \ell\,(0, 0, 2, 1)$$

Let $$v_* = \frac{7 L_4(2)}{\pi L_4(1)}$$

$$\boxed{A(-1/7) = 14\pi \left\{ k^2 v_* + \left(\ell - \frac{5k}{2}\right)^2 \frac{1}{v_*} \right\}}$$

For the attractor points at

$$\varphi_{\pm} = 33 \pm 8\sqrt{17}$$

we set

$$v_* = \frac{17}{2}\left(\frac{9-\sqrt{17}}{2}\right)\frac{\lambda_4(2)}{\pi\,\lambda_4(1)}$$

then

$$\boxed{A(\varphi_{\pm}) = 34\pi\left(\frac{k^2}{v_*} + \ell^2 v_*\right)}$$

Flux Vacua

Attractor points (of rank 2) associated with L_4

Flux vacua " " L_2

These also are revealed by ζ_X

Arithmetic reveals the elliptic curves of F-theory uplifts.

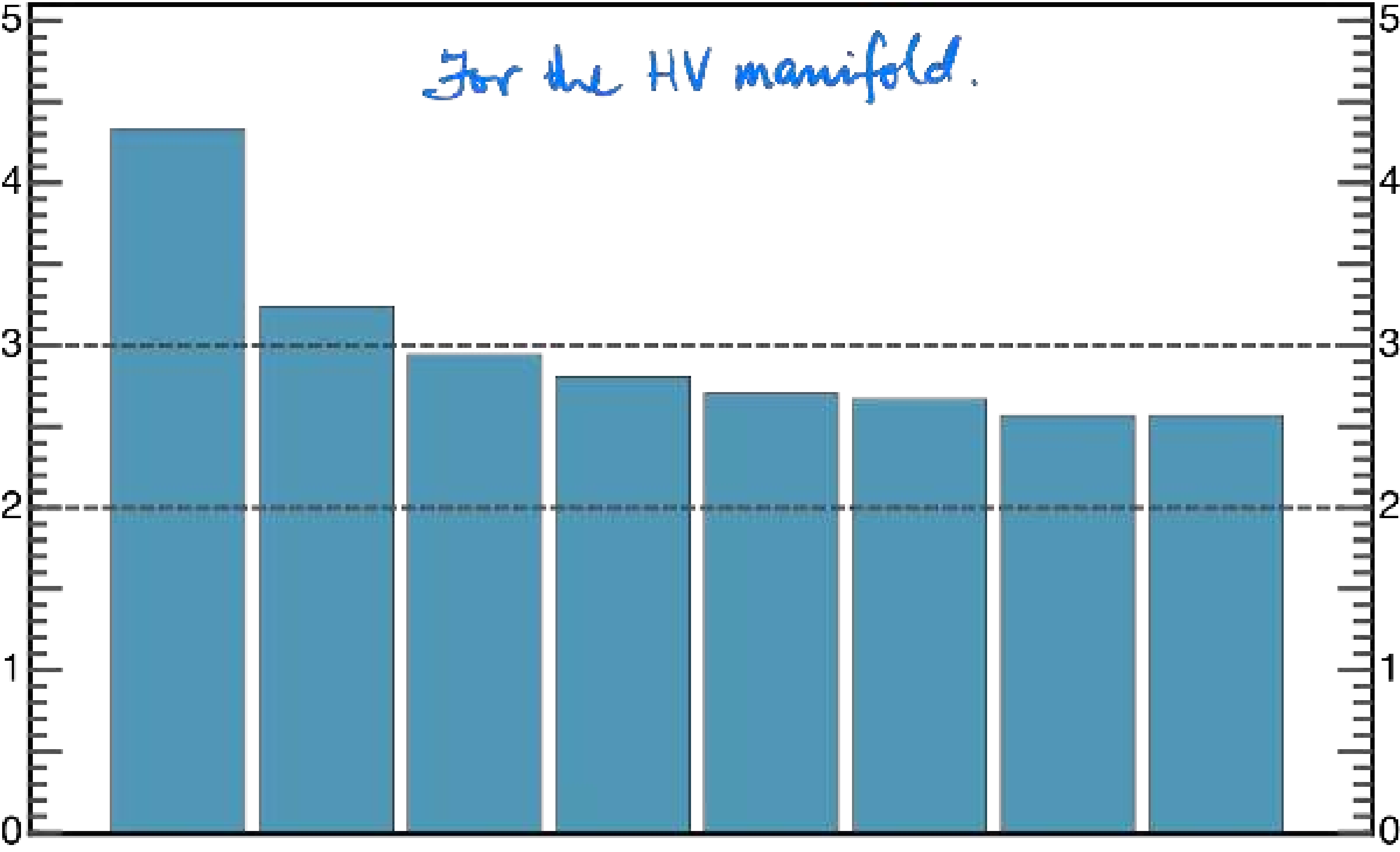
Kachru, Nally and Yang 2010.07285 v2

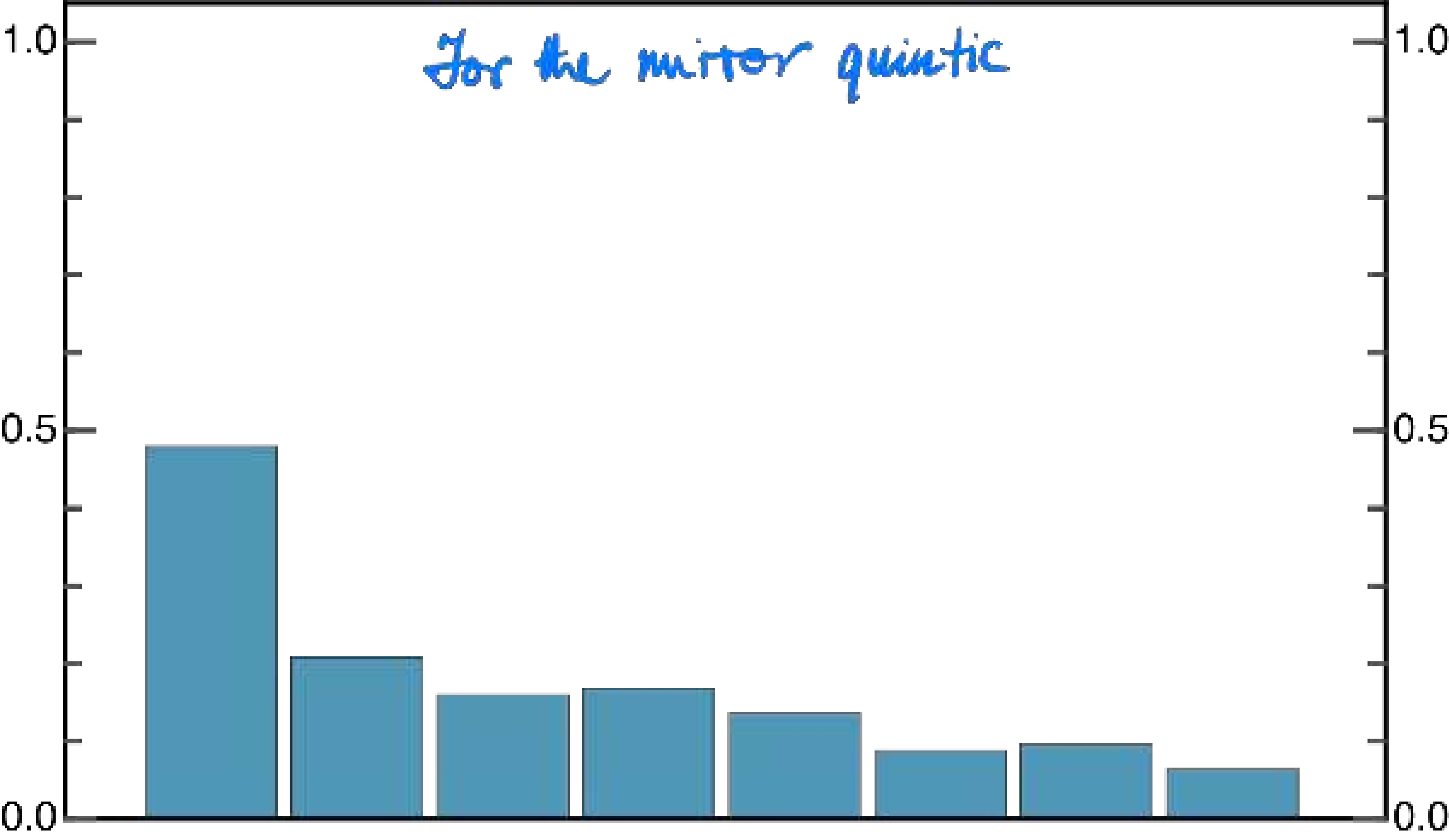
PC, XD, P. Kuusela and J. McGovern 2302.03047

Corollary to the Chebotarëv Theorem

Over $\mathbb{F}_p$, an irreducible polynomial $f(y)$, with integer coefficients has on average (over p) <u>one root</u>.

So if $G(y)$ has k irreducible factors, G has, on average k roots.


For the HV manifold.
5
4
3
2
1
0
5
4
3
2
1
0


For the mirror quintic
1.0
0.5
0.0
1.0
0.5
0.0