Values of quantum non-local games

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Classical non-local game values

Notation: X a finite set $\rightsquigarrow$ M_X all $|X| \times |X|$ complex matrices; $\epsilon_{x,x'}$ matrix units.

$\mathcal{D}_X$ all diagonal $|X| \times |X|$ complex matrices.

Finite sets X, A, Y, B will be fixed throughout. Abbreviate $XY = X \times Y$.

Recall: $p = (p(a,b|x,y))$ no-signalling correlation if $\exists$ well-defined marginals:

Equivalently: Classical channels $\Gamma : \mathcal{D}_{XY} \to \mathcal{D}_{AB}$ with well-def.

$$p(a|x) = \sum_{b \in B} p(a,b|x,y'),\ p(b|y) = \sum_{a \in A} p(a,b|x',y) \qquad \Gamma_{X \to A} : \mathcal{D}_X \to \mathcal{D}_A,\ \Gamma_{Y \to B} : \mathcal{D}_Y \to \mathcal{D}_B$$

A correlation $p = (p(a,b|x,y))$ is called

- local if a convex combination of product correlations $p_1(a|x)p_2(b|y)$;
- quantum if
$$p(a,b|x,y) = \langle (E_{x,a} \otimes F_{y,b})\xi, \xi \rangle,$$
where $(E_{x,a})_{a \in A}$, $(F_{y,b})_{b \in B}$ fin. dim. POVM's. Notation: $\mathcal{C}_{\mathrm{q}}$.
- quantum commuting if $p(a,b|x,y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle$. Notation: $\mathcal{C}_{\mathrm{qc}}$.

Strict inclusions: $\mathcal{C}_{\mathrm{loc}} \subset \mathcal{C}_{\mathrm{q}} \subset \overline{\mathcal{C}_{\mathrm{q}}} \subset \mathcal{C}_{\mathrm{qc}} \subset \mathcal{C}_{\mathrm{ns}}$

If $\mathbb{G} = (XY, AB, \lambda, \mu)$ non-local game (μ a p.d. on XY, λ the rule function) $\rightsquigarrow$

$$\omega_{\mathrm{t}}(\mathbb{G}) = \sup_{p \in \mathcal{C}_{\mathrm{t}}} \sum_{x,y,a,b} \mu(x,y) p(a,b|x,y) \lambda(x,y,a,b).$$

Quantum versions of non-local games

Now: Pairs of inputs and outputs are quantum instead of classical.

- Rank 1 quantum games (Cooney-Junge-Palazuelos-Pérez-García 2015).
  $\xi \in \mathbb{C}^{XYR}$ unit vector, $P \in \mathcal{B}(\mathbb{C}^{ABR})$ rank one projection.
  The probability distribution is incorporated in ξ, the rules in P.
  Strategies (not necessarily no-signalling) transform XY to AB.
- XOR quantum games (Regev-Vidick 2015).
  $M = M^* \in M_{XX}$ representing the rules and the probability distribution.
  Strategies: POVM's $\{P, I - P\}$ on $\mathbb{C}^{XR}$, $\{Q, I - Q\}$ on $\mathbb{C}^{XR'}$, $\eta \in \mathbb{C}^{RR'}$ unit.
  $\rightsquigarrow$ quantum value $\sup_{P,Q,\eta} \mathrm{Tr}((P \otimes Q)(M \otimes \eta\eta^*))$.
- Quantum graph homom. (T-Turowska 2020, Brannan-Harris-T-Turowska 2023)
  Quantum graphs $\leftrightarrow$ proj. $P \in M_{XX}$, $Q \in M_{AA}$.
  Rules: $\mathrm{supp}(\omega_{\mathrm{quest}}) \leq P \Rightarrow \mathrm{supp}(\omega_{\mathrm{ans}}) \leq Q$.
  Strategies: same as above + quantum commuting no-signalling.
- More generally (T-Turowska 2020): rules $\varphi : \mathbb{P}_{XY} \to \mathcal{P}_{AB}$, p.d. μ over $\mathbb{P}_{XY}$.

($\mathbb{P}_X$ pure states on $\mathbb{C}^X$, $\mathcal{P}_A$ projections on $\mathbb{C}^A$) Leung-Toner-Watrous, 2013

Note: Finite implication games are finite rank quantum games:
$p_i \to \varphi(p_i)$, $i = 1, \ldots, k \rightsquigarrow p_i = \xi_i\xi_i^*$,

$$\rightsquigarrow \xi = \sum_{i=1}^k \sqrt{\mu(p_i)}\xi_i \otimes e_i \in \mathbb{C}^X \otimes \mathbb{C}^k \text{ and } P = \sum_{i=1}^k \varphi(p_i) \otimes \epsilon_{i,i}.$$

Probabilistic quantum hypergraphs

Note: Rules $\lambda \leftrightarrow$ subset $E \subseteq XY \times AB$

$\leftrightarrow$ hypergraph on AB with edges $E_{x,y} = \{(a,b) : \lambda(x,y,a,b) = 1\}$.

Simplify: $E \subseteq X \times A \leftrightarrow$ hypergraph on A with edges $E_x = \{a : (x,a) \in E\}$.

$\leftrightarrow$ map $\varphi : \epsilon_{x,x} \to \mathrm{span}\{\epsilon_{a,a} : a \in E_x\}$

Quantise: A probabilistic quantum hypergraph: (φ, μ); $\varphi : \mathbb{P}_X \to \mathcal{P}_A$ and μ p.d. on $\mathbb{P}_X$.

The finite rank viewpoint: (ξ, P), where $\xi \in \mathbb{C}^{XR}$ and $P \in \mathcal{P}_{AR}$.

A collection $\mathcal{Q}$ of quantum channels $\Gamma : M_X \to M_A \rightsquigarrow$

- $\omega_{\mathcal{Q}}(\xi, P) = \sup_{\Gamma \in \mathcal{Q}} \mathrm{Tr}((\Gamma \otimes \mathrm{id}_R)(\xi\xi^*)P)$;
- $\omega_{\mathcal{Q}}(\varphi, \mu) = \sup_{\Gamma \in \mathcal{Q}} \int_{\mathbb{P}_X} \mathrm{Tr}(\Gamma(p)\varphi(p))d\mu(p)$.

Note: $\Gamma : M_X \to M_A$ channel $\Leftrightarrow \Gamma_* : M_A \to M_X$ u.c.p map

Add coefficients: $\Gamma_* : M_A \to M_X \otimes \mathcal{B}(H)$ u.c.p map

$\rightsquigarrow$ Choi matrix $E = (E_{a,a'})_{a,a'} = (E_{x,x'a,a'})_{x,x',a,a'}$ stochastic operator matrix,

i.e. $E \in M_{XA}(\mathcal{B}(H))^+$ with $\mathrm{Tr}_A(E) = I_X \otimes I_H$.

Observe: Stinespring's Theorem $\rightsquigarrow$ factorisation $E_{x,x',a,a'} = U_{a,x}^* U_{a',x'}$, where $U = (U_{a,x})_{a,x}$ block operator isometry.

Therefore: Collection $\mathcal{Q}$ of channels $\longleftrightarrow$ collection $\mathcal{R}$ of block operator isometries.

Resources and operator space structures

Ternary Ring of Operators $\mathcal{V}_{X,A} = [u_{a,x} : (u_{a,x})_{a,x}$ universal block operator isometry].

Meaning: Ternary mor. $\theta : \mathcal{V}_{X,A} \to \mathcal{B}(H,K) \longleftrightarrow$ isometries $(U_{a,x})_{a,x}$ via $\theta(u_{a,x}) = U_{a,x}$.

Recall: $\mathcal{U}$ a TRO $\rightsquigarrow \mathcal{L}_{\mathcal{U}} = [\mathcal{U}\mathcal{U}^*]$ and $\mathcal{R}_{\mathcal{U}} = [\mathcal{U}\mathcal{U}^*]$ ($\mathcal{U}$ Morita imprimitivity $\mathcal{L}_{\mathcal{U}}, \mathcal{R}_{\mathcal{U}}$-bimodule.)

$\rightsquigarrow$ $\mathcal{C}_{X,A} = \mathcal{R}_{\mathcal{V}_{X,A}}$, C*-algebra, universal for stochastic operator matrices.

$\rightsquigarrow$ $\mathcal{T}_{X,A} = \mathrm{span}\{e_{x,x',a,a'} := u_{a,x}^* u_{a',x'}\}$, operator system, universal for stochastic operator matrices.

Operator isometry $U = (U_{a,x})_{a,x}$, $U_{a,x} \in \mathcal{B}(H,K)$ and state σ on H

$\rightsquigarrow \Gamma_{U,\sigma}(\epsilon_{x,x'}) = \sum_{a,a'} \sigma(U_{a,x}^* U_{a',x'})\epsilon_{a,a'} \rightsquigarrow s_{U,\sigma}(e_{x,x',a,a'}) = \sigma(U_{a,x}^* U_{a',x'})$ state on $\mathcal{T}_{X,A}$.

Define a resource over (X,A): collection $\mathcal{R} = \{(U_{a,x})_{a,x} :$ isometry$\}$ s.t.

- $\mathcal{R}$ closed under direct sums;
- the family $\{s_{U,\sigma}\}$ is separating.

Note: Resource $\mathcal{R} \rightsquigarrow \mathrm{QC}(\mathcal{R}) = \{\Gamma_{U,\sigma} : U \in \mathcal{R},\ \sigma \text{ state}\}$, collection of channels.

Note: $\mathcal{O}_{X,A} := \mathrm{span}\{u_{a,x} : x \in X, a \in A\} \simeq_{\text{c.i.}} \mathcal{S}_1^{A,X} \equiv_{\text{c.i.}} \mathcal{B}(\mathbb{C}^X, \mathbb{C}^A)^*$.

Define: $u \in M_n(\mathcal{O}_{X,A}) \rightsquigarrow \|u\|_{\mathcal{R}}^{(n)} = \sup_{U\in\mathcal{R}} \|\theta_U^{(n)}(u)\|$.

Notice: $(\|\cdot\|_{\mathcal{R}}^{(n)})_{n\in\mathbb{N}}$ an operator space structure $\mathcal{O}_{X,A}^{\mathcal{R}}$ on $\mathcal{O}_{X,A}$, for which

$\mathrm{id} : \mathcal{O}_{X,A} \to \mathcal{O}_{X,A}^{\mathcal{R}}$ is completely contractive.

The $\mathcal{R}$-value of a probabilistic quantum hypergraph

Abberviate: For a resource $\mathcal{R}$ over (X, A), write

- $\omega_{\mathcal{R}}(\xi, P) = \sup_{\Gamma \in \mathrm{QC}(\mathcal{R})} \mathrm{Tr}((\Gamma \otimes \mathrm{id}_R)(\xi\xi^*)P)$
- $\omega_{\mathcal{R}}(\varphi, \mu) = \sup_{\Gamma \in \mathrm{QC}(\mathcal{R})} \int_{\mathbb{P}_X} \mathrm{Tr}(\Gamma(p)\varphi(p))d\mu(p)$.

$(\mathbb{P}_X, \varphi, \mu)$ a probabilistic quantum hypergraph $\rightsquigarrow$

- $\mathbb{P}_X \ni p = \xi(p)\xi(p)^*$, where $\xi : \mathbb{P}_X \to \mathbb{C}^X$ is Borel;
- $\mathcal{P}_A \ni q = \sum_{a \in A} \eta_a(q)\eta_a(q)^*$, where $\eta_a : \mathcal{P}_X \to \mathbb{C}^X$ are Borel;
- $\xi(\bar{\eta} \circ \varphi)^* : (p, a) \to \bar{\xi}(p)(\bar{\eta}_a \circ \varphi(p))^*$; $\mathbb{P}_X \times A \to \mathcal{B}(\mathbb{C}^A, \mathbb{C}^X)$.

A "fuzzy" game: $\xi \in \mathbb{C}^{XR}$, $P \in \mathcal{B}(\mathbb{C}^{XR})^+$, $\|P\| \leq 1$. $\rightsquigarrow$ $P = \sum_{k=1}^{\infty} \lambda_k \gamma_k \gamma_k^*$, $0 < \lambda_k \leq 1$.

Theorem

- Let $M_k = \sqrt{\lambda_k}\mathrm{Tr}_R(\bar{\xi}\bar{\gamma}_k^*)$; then $\omega_{\mathcal{R}}(\xi, P) = \|[M_k]_{k=1}^{\infty}\|_{M_{\infty,1}(\mathcal{O}_{X,A}^{\mathcal{R}})}$;
- $\omega_{\mathcal{R}}(\varphi, \mu) = \|\xi(\bar{\eta} \circ \varphi)^*\|_{L^2(\mathbb{P}_X \times A, \tilde{\mu}) \otimes_{\mathrm{h}} \mathcal{O}_{X,A}^{\mathcal{R}}}$. ($\tilde{\mu}$ ampliation of μ by the counting measure on A.)

Fix $U \in \mathcal{R}$ acting from H, and let σ a state on H; write $\xi = \sum_{x \in X} e_x \otimes \xi_x$ and $\gamma_n = \sum_{a \in A} e_a \otimes \gamma_{n,a}$

$\rightsquigarrow \mathrm{Tr}((\Gamma_{U,\sigma} \otimes \mathrm{id}_R)(\xi\xi^*)P) =$

$$\sum_{n=1}^{\infty} \sum_{x,x',a,a'} \lambda_n \sigma(U_{a,x}^* U_{a',x'}) \langle \gamma_{n,a'}, \xi_{x'} \rangle \langle \xi_x, \gamma_{n,a} \rangle = \sum_{n=1}^{\infty} \sigma\left(\theta_U(M_n)^* \theta_U(M_n)\right).$$

QNS correlation types

Now: Back to having X, Y, A, B (bipartite systems).

Quantum no-signalling (QNS) correlations $\mathcal{Q}_{\rm ns}$: quantum channels $\Gamma : M_{XY} \to M_{AB}$ s.t.

- $\rho \in M_{XY}$, ${\rm Tr}_X \rho = 0 \implies {\rm Tr}_A \Gamma(\rho) = 0$;
- $\rho \in M_{XY}$, ${\rm Tr}_Y \rho = 0 \implies {\rm Tr}_B \Gamma(\rho) = 0$. (Duan-Winter, 2016)

Local QNS correlations $\mathcal{Q}_{\rm loc}$: $\Phi : M_X \to M_A$, $\Psi : M_Y \to M_B$

$\rightsquigarrow \Gamma = \Phi \otimes \Psi$ and their convex combinations (i.e. LOSR).

Def: $E = (E_{x,x',a,a'}) \in M_{XA} \otimes \mathcal{B}(H)$ stochastic operator matrix if ${\rm Tr}_A E = I_X \otimes I_H$

(i.e. $\sum_{a \in A} E_{x,x',a,a'} = \delta_{x,x'} I_H, \quad x, x' \in X$)

Note: $\{(E_{x,a})_{a \in A} : x \in X\}$ POVM's $\rightsquigarrow E = \sum_{x,a} \epsilon_{x,x} \otimes \epsilon_{a,a} \otimes E_{x,a}$.

A QNS correlation $\Gamma : M_{XY} \to M_{AB}$ is called

- quantum ($\mathcal{Q}_{\rm q}$) if $\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}) = \sum_{a,a',b,b'} \langle (E_{x,x',a,a'} \otimes F_{y,y',b,b'})\xi, \xi \rangle \epsilon_{a,a'} \otimes \epsilon_{b,b'}$, with E and F are fin. dim. acting stochastic operator matrices;
- quantum commuting ($\mathcal{Q}_{\rm qc}$) when $E_{x,x',a,a'} \otimes F_{y,y',b,b'}$ is replaced by $E_{x,x',a,a'} F_{y,y',b,b'}$, acting on the same Hilb. space, mutually commuting.

(T-Turowska, 2020)

Strict inclusions: $\mathcal{Q}_{\rm loc} \subset \mathcal{Q}_{\rm q} \subset \overline{\mathcal{Q}_{\rm q}} \subset \mathcal{Q}_{\rm qc} \subset \mathcal{Q}_{\rm ns}$.

The quantum resource

Next: Apply the hypergraph setup to the bipartite context $\rightsquigarrow$ values of quantum games
(ξ, P), where $\xi \in \mathbb{C}^{XYR}$ and P a projection in $\mathcal{B}(\mathbb{C}^{ABR})$,
or (φ, μ), where $\varphi : \mathbb{P}_{XY} \to \mathcal{P}_{AB}$ and μ prob. measure on $\mathbb{P}_{XY}$.

Abberviate: $\omega_{\rm t} := \omega_{\mathcal{Q}_{\rm t}} \rightsquigarrow$ loc, q, qs and ns value of a game.

$\rightsquigarrow$ Four type of resources $\mathcal{R}$: local, quantum, quantum commuting and no-signalling.

For each one, need to specify a family of isometries $W : \mathbb{C}^{XY} \otimes H \to \mathbb{C}^{AB} \otimes K$.

Recall: $(\xi, P) \rightsquigarrow P = \sum_{k=1}^{\infty} \gamma_k \gamma_k^* \rightsquigarrow M_k = \mathrm{Tr}_R(\bar{\xi}\bar{\gamma}_k^*) \rightsquigarrow$ column operator $M = [M_k]_{k=1}^{\infty}$.

$\Rightarrow$ For each chosen resource $\mathcal{R}$ we have $\omega_{\mathcal{R}}(\xi, P) = \|M\|_{\mathcal{O}^{\mathcal{R}}_{XY,AB}}$.

Thus: For each resource $\mathcal{R}$, we need to

- link $\mathcal{R}$ to one of the QNS correlation types $\mathcal{Q}_{\rm t}$, and
- identify the operator space structure $\mathcal{O}^{\mathcal{R}}_{XY,AB}$.

The quantum resource (revisited! – recall Cooney-Junge-Palazuelos-Pérez-García)

$$\mathcal{R}_{\rm q} = \{U \otimes V \ : \ U : \mathbb{C}^X \otimes H_X \to \mathbb{C}^A \otimes K_A, V : \mathbb{C}^Y \otimes H_Y \to \mathbb{C}^B \otimes K_B\}$$

(H_X and H_Y finite dimensional).

Straightforward: $\mathrm{QC}(\mathcal{R}_{\rm q}) = \mathcal{Q}_{\rm q}$

However: $\mathcal{S}_1^{A,X} \otimes_{\min} \mathcal{S}_1^{B,Y} \subseteq \mathcal{V}_{X,A} \otimes_{\min} \mathcal{V}_{Y,B} \quad \Longrightarrow \quad \mathcal{O}^{\mathcal{R}_{\rm q}}_{XY,AB} = \mathcal{S}_1^{A,X} \otimes_{\min} \mathcal{S}_1^{B,Y}$.

$\Longrightarrow$ the formulas of Cooney-Junge-Palazuelos-Pérez-García.

The quantum commuting resource

Say that: two families $\mathcal{E} \subseteq \mathcal{B}(K,L)$ and $\mathcal{F} \subseteq \mathcal{B}(H,K)$ semi-commute if

$$\mathcal{E}^*\mathcal{E} \text{ and } \mathcal{F}\mathcal{F}^* \text{ commute in } \mathcal{B}(K) \qquad (H \xrightarrow{\mathcal{F}} K \xrightarrow{\mathcal{E}} L)$$

Further say: the block operator matrices $U = (U_{a,x})_{a,x}$ and $V = (V_{b,y})_{b,y}$ semi-commute if the families $\{U_{a,x}\}_{a,x}$ and $\{V_{b,y}\}_{b,y}$ semi-commute.

Define: $\mathcal{R}_{\rm qc} = \{(U_{a,x}V_{b,y})_{ab,xy} : U = (U_{a,x})_{a,x}, V = (U_{b,y})_{b,y}$ semi-commuting isom.$\}$.

Note: $\mathcal{R}_{\rm qc}$ a resource over $(X \times Y, A \times B)$ as $(U_{a,x}V_{b,y})_{ab,xy} = U_{1,3}V_{2,3}$

Theorem

We have that $\mathrm{QC}(\mathcal{R}_{\rm qc}) = \mathcal{Q}_{\rm qc}$.

What is needed? $\Gamma \in \mathcal{Q}_{\rm qc} \leftrightarrow$ Choi matrices $(\langle E_{x,x',a,a'}F_{y,y',b,b'}\xi,\xi\rangle)$, where $(E_{x,x',a,a'})$ and $(F_{y,y',b,b'})$ mutually comm. stochastic op. matrices.

But: $E_{x,x',a,a'} = U_{a,x}^*U_{a',x'}$ and $F_{y,y',b,b'} = V_{b,y}^*V_{b',y'}$, with $(U_{a,x})_{a,x}$, $(V_{b,y})_{b,y}$ isom..

Hence for $\mathcal{Q}_{\rm qc} \subseteq \mathrm{QC}(\mathcal{R}_{\rm qc})$ wish:

$$\langle U_{a,x}^*U_{a',x'}V_{b,y}^*V_{b',y'}\xi,\xi\rangle = \langle \tilde{U}_{a',x'}\tilde{V}_{b',y'}\xi, \tilde{U}_{a,x}\tilde{V}_{b,y}\xi\rangle = \langle \tilde{V}_{b,y}^*\tilde{U}_{a,x}^*\tilde{U}_{a',x'}\tilde{V}_{b',y'}\xi,\xi\rangle.$$

For $\mathrm{QC}(\mathcal{R}_{\rm qc}) \subseteq \mathcal{Q}_{\rm qc}$ need: To reverse the last line.

TRO's and semi-commutation

Notation: $\mathcal{U}$ TRO $\rightsquigarrow$ $\mathcal{L}_{\mathcal{U}} = [\mathcal{U}\mathcal{U}^*]$ (left) and $\mathcal{R}_{\mathcal{U}} = [\mathcal{U}^*\mathcal{U}]$ (right) C*-algebra.

Note: $\mathcal{R}_{\mathcal{U}} = \mathcal{U}^* \otimes_{\mathrm{h}}^{\mathcal{L}_{\mathcal{U}}} \mathcal{U}$ and $\mathcal{L}_{\mathcal{U}} = \mathcal{U} \otimes_{\mathrm{h}}^{\mathcal{R}_{\mathcal{U}}} \mathcal{U}^*$.

$\phi : \mathcal{U} \to \mathcal{B}(H, K)$ ternary morphism $\rightsquigarrow$ $\pi_\phi^R : \mathcal{R}_{\mathcal{U}} \to \mathcal{B}(H)$ and $\pi_\phi^L : \mathcal{L}_{\mathcal{U}} \to \mathcal{B}(K)$, with

$$\pi_\phi^R(u_1^* \otimes u_2) = \phi(u_1)^*\phi(u_2) \quad \text{and} \quad \pi_\phi^L(u_1 \otimes u_2^*) = \phi(u_1)\phi(u_2)^*$$

$\mathcal{D}_{\mathcal{U}} = \begin{pmatrix} \mathcal{L}_{\mathcal{U}} & \mathcal{U} \\ \mathcal{U}^* & \mathcal{R}_{\mathcal{U}} \end{pmatrix}$ linking alg. $\rightsquigarrow$ $\tilde{\pi}_\phi : \mathcal{D}_{\mathcal{U}} \to \mathcal{B}(H \oplus K)$ s.t. $\tilde{\pi}_\phi\left(\begin{pmatrix} l & u \\ v^* & r \end{pmatrix}\right) = \begin{pmatrix} \pi_\phi^L(l) & \phi(u) \\ \phi(v)^* & \pi_\phi^R(r) \end{pmatrix}$

Terminology: $\mathcal{U}$, $\mathcal{V}$ TRO's $\rightsquigarrow$ $\phi : \mathcal{U} \to \mathcal{B}(K, L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H, K)$ semi-commuting

if $\phi(\mathcal{U})$ and $\psi(\mathcal{V})$ semi-commuting. $\quad H \overset{\psi(\mathcal{V})}{\longrightarrow} K \overset{\phi(\mathcal{U})}{\longrightarrow} L$

Lemma

$\phi : \mathcal{U} \to \mathcal{B}(K, L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H, K)$ semi-commuting ternary morphisms, ψ left non-degenerate. Then $\exists$ *-homomorphism $\rho : \mathcal{R}_{\phi(\mathcal{U})} \to \mathcal{R}'_{\psi(\mathcal{V})}$ s.t.

$$\rho(b)\psi(v)^* = \psi(v)^* b, \quad b \in \mathcal{R}_{\phi(\mathcal{U})}, \ v \in \mathcal{V}.$$

$b \in \mathcal{R}_{\phi(\mathcal{U})} \rightsquigarrow \tilde{b} : \sum_{k=1}^n \psi(v_k)^*\xi_k \mapsto \sum_{k=1}^n \psi(v_k)^* b\xi_k$, $v_k \in \mathcal{V}$, $\xi_k \in K$, $k \in [n]$.

$[\pi_\psi^L(v_l v_k^*)]_{k,l=1}^n \in M_n(\mathcal{B}(K))^+$ and semi-comm. $\Longrightarrow \tilde{b} \in \mathcal{B}(H) \rightsquigarrow$ set $\rho(b) = \tilde{b}$.

Semi-commutation $\Longrightarrow$ ρ *-preserving; the rest follow from the def..

A TRO tensor product

$\mathcal{U}$ and $\mathcal{V}$ TRO's $\rightsquigarrow$ ternary product on $\mathcal{U} \otimes \mathcal{V}$,

$$(u_1 \otimes v_1)(u_2 \otimes v_2)^*(u_3 \otimes v_3) := u_1 u_2^* u_3 \otimes v_1 v_2^* v_3.$$

$\rightsquigarrow$ $\|w\|_{\max} := \sup\{\|\theta(w)\| \ : \ \theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H,K) \text{ ternary morphism}\}$ $\rightsquigarrow$ $\mathcal{U} \otimes_{\max} \mathcal{V}$.

Theorem $\rightsquigarrow$ an identification of $\|\cdot\|_{\mathcal{R}_{qc}}$.

$\theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H,L)$ is a ternary morphism iff $\exists$ a Hilbert space K and semi-commuting ternary morphisms $\phi : \mathcal{U} \to \mathcal{B}(K,L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H,K)$, s.t. $\theta = \phi \cdot \psi$.

$((\phi \cdot \psi)(u \otimes v) = \phi(u)\psi(v))$

$\phi : \mathcal{U} \to \mathcal{B}(K,L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H,K)$ be semi-commuting $\Rightarrow$

$$\begin{aligned}\phi(u_1 u_2^* u_3)\psi(v_1 v_2^* v_3) &= \phi(u_1)\pi_\phi^R(u_2^* u_3)\pi_\psi^L(v_1 v_2^*)\psi(v_3) = \phi(u_1)\pi_\psi^L(v_1 v_2^*)\pi_\phi^R(u_2^* u_3)\psi(v_3) \\ &= (\phi \cdot \psi)(u_1 \otimes v_1)(\phi \cdot \psi)(u_2 \otimes v_2)^*(\phi \cdot \psi)(u_3 \otimes v_3).\end{aligned}$$

Conversely: Let $\theta : \mathcal{U} \otimes \mathcal{V} \to \mathcal{B}(H,L)$ ternary morphism $\rightsquigarrow \pi_\theta^R : \mathcal{R}_{\mathcal{U}} \otimes \mathcal{R}_{\mathcal{V}} \to \mathcal{B}(H)$,
$\pi_\theta^R(u^* u' \otimes v^* v') = \theta(u \otimes v)^* \theta(u' \otimes v') \rightsquigarrow \pi_\theta^R = \pi_{\mathcal{U}}^R \times \pi_{\mathcal{V}}^R$ for commuting rep. of $\mathcal{R}_{\mathcal{U}}$ and $\mathcal{R}_{\mathcal{V}}$.
Equip $\mathcal{V} \otimes H$ with $\langle v_1 \otimes \xi_1, v_2 \otimes \xi_2\rangle = \langle \pi_{\mathcal{V}}^R(v_2^* v_1)\xi_1, \xi_2\rangle \rightsquigarrow$ Hilbert space $\mathcal{V} \otimes_{\mathcal{V}} H$.
Define $\psi : \mathcal{V} \to \mathcal{B}(H, \mathcal{V} \otimes_{\mathcal{V}} H)$ by $\psi(v)\xi = v \otimes_{\mathcal{V}} \xi \Rightarrow \pi_\psi^R = \pi_{\mathcal{V}}^R$ and $\pi_\psi^L(a)(v \otimes_{\mathcal{V}} \xi) = av \otimes_{\mathcal{V}} \xi$, $a \in \mathcal{L}_{\mathcal{V}}$.
For $b \in R_{\mathcal{U}}$, define $\tilde\pi_{\mathcal{U}}^R(b) \in \mathcal{B}(\mathcal{V} \otimes_{\mathcal{V}} H)$ by $\tilde\pi_{\mathcal{U}}^R(b)(v \otimes \xi) = v \otimes \pi_{\mathcal{U}}^R(b)\xi \Rightarrow \tilde\pi_{\mathcal{U}}^R : \mathcal{R}_{\mathcal{U}} \to \mathcal{B}(\mathcal{V} \otimes_{\mathcal{V}} H)$ *-hom.
Define $\phi : \mathcal{U} \to \mathcal{B}(\mathcal{V} \otimes_{\mathcal{V}} H, \mathcal{U} \otimes_{\mathcal{U}} (\mathcal{V} \otimes_{\mathcal{V}} H))$ by

$$\phi(u)(v \otimes_{\mathcal{V}} \xi) = u \otimes_{\mathcal{U}} (v \otimes_{\mathcal{V}} \xi), \ u \in \mathcal{U}, v \in \mathcal{V}, \xi \in H \ \rightsquigarrow \ \pi_\phi^R = \tilde\pi_{\mathcal{U}}^R.$$

Then $\pi_\phi^R(\mathcal{R}_{\mathcal{U}})$ commutes with $\pi_\psi^L(\mathcal{L}_{\mathcal{V}})$ in $\mathcal{B}(\mathcal{V} \otimes_{\mathcal{V}} H)$, i.e. ϕ and ψ semi-commute.

Further, $\langle \phi(u_1)\psi(v_1)\xi_1, \phi(u_2)\psi(v_2)\xi_2\rangle = \langle \theta(u_1 \otimes v_1)\xi_1, \theta(u_2 \otimes v_2)\xi_2\rangle$
$\Rightarrow W : \mathcal{U} \otimes_{\mathcal{U}} (\mathcal{V} \otimes_{\mathcal{V}} H) \ni \sum_{i=1}^k u_i \otimes_{\mathcal{U}} v_i \otimes_{\mathcal{V}} \xi_i \mapsto \sum_{i=1}^k \theta(u_i \otimes v_i)\xi_i \in L$ isometry and $\theta = (W \circ \phi) \cdot \psi$.

The tmax-tensor product

Terminology: Ternary morphisms $\phi : \mathcal{U} \to \mathcal{B}(H)$ and $\psi : \mathcal{V} \to \mathcal{B}(H)$ are commuting if

$$\phi(u)\psi(v) = \psi(v)\phi(u) \text{ and } \phi(u)\psi(v)^* = \psi(v)^*\phi(u), \quad u \in \mathcal{U}, v \in \mathcal{V}.$$

Set: $\|w\|_{\rm tmax} := \sup\{\|(\phi\cdot\psi)(w)\| : \phi,\psi \text{ commuting pair}\} \rightsquigarrow \mathcal{U}\otimes_{\rm tmax}\mathcal{V}$ Kaur-Ruan, 2002

Note: $\mathcal{U}\otimes_{\rm tmax}\mathcal{V} \subseteq_{\rm c.i.} \mathcal{D}_{\mathcal{U}}\otimes_{\max}\mathcal{D}_{\mathcal{V}}$, and $\|\cdot\|_{\rm tmax} \leq \|\cdot\|_{\max}$.

Theorem

$\|\cdot\|_{\max} = \|\cdot\|_{\rm tmax}$ on $\mathcal{U}\otimes\mathcal{V}$, i.e. $\mathcal{U}\otimes_{\rm tmax}\mathcal{V} = \mathcal{U}\otimes_{\max}\mathcal{V}$.

First note $\mathcal{R}_{\mathcal{U}}\otimes_{\max}\mathcal{R}_{\mathcal{V}} \subseteq \mathcal{D}_{\mathcal{U}}\otimes_{\max}\mathcal{D}_{\mathcal{V}}$, because of u.c.p. maps $\mathcal{R}_{\mathcal{U}} \to \mathcal{D}_{\mathcal{U}} \to \mathcal{R}_{\mathcal{U}}$.
Let $\phi : \mathcal{U} \to \mathcal{B}(K, L)$ and $\psi : \mathcal{V} \to \mathcal{B}(H, K)$ semi-commuting ternary morphisms
$\Rightarrow \exists$ *-homom. $\rho : \mathcal{R}_{\phi(\mathcal{U})} \to \mathcal{R}'_{\psi(\mathcal{V})} \subseteq \mathcal{B}(H)$ s. t. $\rho(b)\psi(v)^* = \psi(v)^* b$, for $b \in \mathcal{R}_{\phi(\mathcal{U})}$, $v \in \mathcal{V}$.

$\Rightarrow$ if $b \in \mathcal{R}_{\phi(\mathcal{U})}$ and $v_1, v_2 \in \mathcal{V}$ then $\rho(b)\pi^R_\psi(v_1^* v_2) = \rho(b)\psi(v_1)^*\psi(v_2) = \pi^R_\psi(v_1^* v_2)\rho(b)$

i.e. $\rho\circ\pi^R_\phi : \mathcal{R}_{\mathcal{U}} \to \mathcal{B}(H)$ and $\pi^R_\psi : \mathcal{R}_{\mathcal{V}} \to \mathcal{B}(H)$ commuting. Now for $w = \sum_{i=1}^n u_i\otimes v_i \in \mathcal{U}\otimes\mathcal{V}$:

$\|(\phi\cdot\psi)(w)\| = \|\sum_{i,j=1}^n \psi(v_j)^*\phi(u_j)^*\phi(u_i)\psi(v_i)\|^{1/2} = \|\sum_{i,j=1}^n \rho(\phi(u_j)^*\phi(u_i))\psi(v_j)^*\psi(v_i)\|^{1/2}$

$\leq \|\sum_{i,j=1}^n u_j^* u_i\otimes v_j^* v_i\|^{1/2}_{\mathcal{R}_{\mathcal{U}}\otimes_{\max}\mathcal{R}_{\mathcal{V}}} = \|\sum_{i,j=1}^n u_j^* u_i\otimes v_j^* v_i\|^{1/2}_{\mathcal{D}_{\mathcal{U}}\otimes_{\max}\mathcal{D}_{\mathcal{V}}} = \|\sum_{i=1}^n u_i\otimes v_i\|_{\mathcal{D}_{\mathcal{U}}\otimes_{\max}\mathcal{D}_{\mathcal{V}}}$.

$\Rightarrow \|w\|_{\max} \leq \|w\|_{\rm tmax}$.

Back to qc-values: Define $\mathcal{S}_1^{A,X}\otimes_{\max}\mathcal{S}_1^{B,Y} \subseteq \mathcal{V}_{X,A}\otimes_{\rm tmax}\mathcal{V}_{Y,B}$

$$\Rightarrow \mathcal{S}_1^{A,X}\otimes_{\max}\mathcal{S}_1^{B,Y} = \mathcal{O}_{XY,AB}^{\mathcal{R}_{\rm qc}} \Rightarrow \text{formula for } \omega_{\rm qc}(\xi, P) \text{ or } \omega_{\rm qc}(\varphi,\mu).$$

The local resource

Define: $\mathcal{R}_{\rm loc} = \langle\{U \otimes V : U \in \mathcal{B}(\mathbb{C}^X,\mathbb{C}^{AS}), V \in \mathcal{B}(\mathbb{C}^Y,\mathbb{C}^{BT})$ isom., S, T finite sets$\}\rangle$

Observe: $\mathrm{QC}(\mathcal{R}_{\rm loc}) = \mathcal{Q}_{\rm loc}$.

Elements of $\mathcal{Q}_{\rm loc}$: conv. comb. of $\Phi \otimes \Psi$, where Φ, Ψ quantum channels.
Write $\Phi(\omega) = \sum_{i\in S} V_i\omega V_i^*$ and $\Psi(\omega) = \sum_{j\in T} W_j\omega W_j^*$ in Kraus dec. $\rightsquigarrow$ $V = [V_i]_{i\in S}$ and $W = [W_j]_{j\in T}$.

Junge-Kubicki-Palazuelos-Pérez-García, 2021: $\mathcal{X}$ and $\mathcal{Y}$ op. sp., $\phi : \mathcal{X} \to \mathcal{Y}$ $\rightsquigarrow$

$$\|\phi\|_{w,\rm cb} = \sup\{\|\beta \circ \phi \circ \alpha\|_{\mathcal{S}_2(\ell_2)} : \alpha : R_\infty \to \mathcal{X}, \beta : \mathcal{Y} \to C_\infty \text{ completely contr.}\};$$

$\rightsquigarrow$ $\mathcal{S}_2^{w,\rm cb}(\mathcal{X},\mathcal{Y})$ weak-cb Hilbert-Schmidt op. $\rightsquigarrow$ op. space by amplifying $(\beta^{(n)} \circ (\phi_{i,j}) \circ \alpha)$.

In tensor terms: finite dimensional $\mathcal{X}$ and $\mathcal{Y}$ $\rightsquigarrow$ $\mathcal{X}^* \otimes_{w,\rm cb} \mathcal{Y} := \mathcal{S}_2^{w,\rm cb}(\mathcal{X},\mathcal{Y})$

Theorem

We have $\mathcal{O}_{XY,AB}^{\mathcal{R}_{\rm loc}} = \mathcal{S}_1^{A,X} \otimes_{w,\rm cb} \mathcal{S}_1^{B,Y}$.

$U \in \mathcal{B}(\mathbb{C}^X,\mathbb{C}^{AS})$, $V \in \mathcal{B}(\mathbb{C}^Y,\mathbb{C}^{BT})$ $\rightsquigarrow$ $(\theta_V \otimes \theta_W)(\omega) = \sum_{s\in S}\sum_{t\in T}\langle V_s \otimes W_t, \omega\rangle e_s \otimes e_t$
Set $\alpha(\xi) = \sum_{s\in S}\langle \xi, e_s\rangle V_s$ and $\beta(\rho) = \sum_{t\in T}\langle \rho, W_t\rangle e_t$ to see $\|(\theta_V \otimes \theta_W)(\omega)\| = \|\beta \circ \omega \circ \alpha\|_2$.
For the reverse, a c.c. $\alpha : R_\infty \to \mathcal{B}(\mathbb{C}^X, \mathbb{C}^A)$ gives only $[V_i]_{i=0}^\infty$: $\sum_{i=0}^\infty V_i^* V_i \leq I_X$ $\rightsquigarrow$ add some extra terms.

Corollary

$\xi\xi^*$ on $\mathbb{C}^{XY}$ and $\gamma\gamma^*$ on $\mathbb{C}^{AB}$ $\rightsquigarrow$ $\exists$ LOSR Γ such that $\Gamma(\xi\xi^*) = \gamma\gamma^*$ $\Leftrightarrow$ $\|\bar{\xi}\bar{\gamma}^*\|_{w,\rm cb} = 1$.

The no-signalling resource and inequalities

K a Hilbert space, $\mathcal{E} = \{\eta_{x,y,a,b}\}_{x,y,a,b} \subseteq K \;\rightsquigarrow\; \zeta_{x,y} = \sum_{a\in A}\sum_{b\in B} e_a \otimes e_b \otimes \eta_{x,y,a,b}$.

$\mathcal{E}$ no-signalling if $\exists$ PVM's $\{P_x\}_{x\in X}$ on $\mathbb{C}^A \otimes K$, and $\{Q_y\}_{y\in Y}$ on $\mathbb{C}^B \otimes K$ s.t.

$$\zeta_{x,y} \in \mathrm{ran}(P_x \otimes I_B) \cap \mathrm{ran}(Q_y \otimes I_A) \subseteq \mathbb{C}^{AB} \otimes K, \quad x \in X, y \in Y.$$

$\mathcal{E}$ no-signalling family $\rightsquigarrow$ $U_{\mathcal{E}} : \mathbb{C}^{XY} \to \mathbb{C}^{AB} \otimes K$, $U_{\mathcal{E}}(e_x \otimes e_y) = \zeta_{x,y}$, $x \in X$, $y \in Y$.

Define: $\mathcal{R}_{\mathrm{ns}} = \langle \{U_{\mathcal{E}} : \mathcal{E} \text{ no-signalling family}\}\rangle$.

Proposition

We have that $\mathrm{QC}(\mathcal{R}_{\mathrm{ns}}) = \mathcal{Q}_{\mathrm{ns}}$.

Proposition

We have $\omega_{\mathrm{qc}}(\xi, P) \leq \omega_{\mathrm{h}}(\xi, P)$.

One-way communication value, Cooney-Junge–Palazuelos-Pérez-García.

For $\mathcal{U} \otimes_{\mathrm{h}} \mathcal{V}$, one takes all c.c. ternary morphisms, while for $\mathcal{U} \otimes_{\mathrm{tmax}} \mathcal{V}$ only the commuting ones.

Classical-to-quantum values

Classical-to-quantum correlation types $\mathcal{CQ}_{\mathrm{t}}$ — T-Turowska, 2020

$$\mathcal{E} : \mathcal{D}_{XY} \to M_{AB} \quad \rightsquigarrow \quad \text{states } \sigma_{x,y} := \mathcal{E}(\epsilon_{x,x} \otimes \epsilon_{y,y}).$$

No-signalling: $\mathrm{Tr}_A \sigma_{x,y} = \mathrm{Tr}_A \sigma_{x',y}$ and $\mathrm{Tr}_B \sigma_{x,y} = \mathrm{Tr}_A \sigma_{x,y'}$

Other types: e.g. quantum commuting via $E_{x,a,a'} F_{y,b,b'} = F_{y,b,b'} E_{x,a,a'}$.

Universal objects: C*-alg. $\mathcal{B}_{X,A} = M_A *_1 \cdots *_1 M_A$ ($|X|$ times) and op.sys. $\mathcal{R}_{X,A} \subseteq \mathcal{B}_{X,A}$.

$\rightsquigarrow$ similar descriptions in terms of states on $\mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}$ etc.

Classical	Classical-to-quantum	Quantum
$\mathcal{N} : \mathcal{D}_{XY} \to \mathcal{D}_{AB}$	$\mathcal{E} : \mathcal{D}_{XY} \to M_{AB}$	$\Gamma : M_{XY} \to M_{AB}$
$\mathcal{A}_{X,A} = C^*(e_{x,a})$	$\mathcal{B}_{X,A} = C^*(e_{x,a,a'})$	$\mathcal{C}_{X,A} = C^*(e_{x,x',a,a'})$
POVM's $(E_{x,a})_{a \in A}$	$E_x \in M_A^+$ with $\mathrm{Tr}_A E_x = I$	$E = (E_{x,x',a,a'})$ stochastic

Classical-to-quantum game: $\varphi : \epsilon_{x,x} \otimes \epsilon_{y,y} \to \varphi(\epsilon_{x,x} \otimes \epsilon_{y,y})$, with p. d. π on $X \times Y$

$\rightsquigarrow$ $\omega_{\mathrm{t}}(\varphi, \pi) = \sup_{\Gamma \in \mathcal{CQ}_{\mathrm{t}}} \sum_{x \in X} \sum_{y \in Y} \pi(x,y) \mathrm{Tr}(\Gamma(\epsilon_{x,x} \otimes \epsilon_{y,y}) \varphi(\epsilon_{x,x} \otimes \epsilon_{y,y}))$

Set: $\tilde{G} = \sum_{x \in X} \sum_{y \in Y} \pi(x,y) \iota_{x,y}(\varphi(\epsilon_{x,x} \otimes \epsilon_{y,y}))$.

Theorem

- $\omega_{\mathrm{q}}(\varphi, \pi) = \|\widehat{G}\|_{\ell^1_X(M_A) \otimes_{\min} \ell^1_Y(M_B)} = \|\tilde{G}\|_{\mathcal{R}_{X,A} \otimes_{\min} \mathcal{R}_{Y,B}}$;
- $\omega_{\mathrm{qc}}(\varphi, \pi) = \|\tilde{G}\|_{\mathcal{R}_{X,A} \otimes_{\mathrm{c}} \mathcal{R}_{Y,B}} = \|\tilde{G}\|_{\mathcal{B}_{X,A} \otimes_{\max} \mathcal{B}_{Y,B}}$.

THANK YOU VERY MUCH