Super-exponential distinguishability of correlated quantum states

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MOMENTUM OF INNOVATION

Lendület program

Single-shot state discrimination

Communication problem: Alice wants to send one bit of information to Bob, encoded into a quantum system.

Alice		Bob		
$0 \mapsto \varrho$		POVM		
	$\longrightarrow$		$\longrightarrow$	decoded message $= 0$ or 1.
$1 \mapsto \sigma$		$T_0 = T, \quad T_1 = I - T$		

Error probabilities:

type I: $\alpha(T) := \mathrm{Prob}(\text{decoded} = 1|\text{message} = 0) = \mathrm{Tr}\,\varrho(I - T)$

type II: $\beta(T) := \mathrm{Prob}(\text{decoded} = 0|\text{message} = 1) = \mathrm{Tr}\,\sigma T.$

Type I can be made 0 by choosing $T := I$, but then type II $= 1$.

Vice versa, type II $= 0$ with $T = 0$, but then type I $= 1$.

Trade-off between the two error probabilities.

$$\exists T : \quad \alpha(T) = 0 = \beta(T) \qquad \Longleftrightarrow \qquad \mathrm{supp}\,\varrho \perp \mathrm{supp}\,\sigma$$

Asymptotic state discrimination

The error can be reduced by introducing redundancy.

Alice | Bob

$$0 \mapsto \underbrace{00\ldots0}_{n \text{ times}} \mapsto \varrho^{\otimes n}$$

$$1 \mapsto \underbrace{11\ldots1}_{n \text{ times}} \mapsto \sigma^{\otimes n}$$

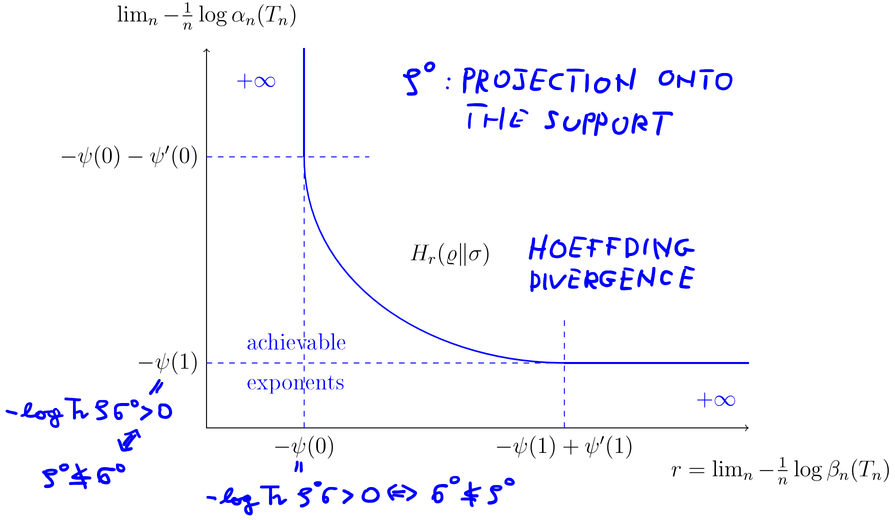
$\longrightarrow$ POVM $T_0 = T, \quad T_1 = I - T$ $\longrightarrow$ decoded message $= 0$ or 1.

Error probabilities:

$$\alpha_n(T) := \operatorname{Tr} \varrho^{\otimes n}(I - T), \qquad \beta_n(T) := \operatorname{Tr} \sigma^{\otimes n} T$$

What is the best asymptotics along arbitrary test sequences $0 \le T_n \le I_{\mathcal{H}^{\otimes n}}$, $n \in \mathbb{N}$?

Achievable exponents



$$\lim_n -\frac{1}{n}\log\alpha_n(T_n) = +\infty = \lim_n -\frac{1}{n}\log\alpha_n(T_n) \iff \varrho \perp \sigma$$

The exponential scale

- The exponential scale is fundamental in (quantum) information theory.
- Trade-off relations are studied on the level of exponents for state compression, channel coding, state conversion, etc.
- Measuring information in bits is a manifestation of this.
- Mathematical reason 1: The dimension of the tensor product grows exponentially in the number of copies.
- Mathematical reason 2: Large deviation theory; the probability of the mean deviating from the expectation value by a constant decays exponentially fast.
- Information theory reason: Most problems are closely related to state discrimination.
- Can we have non-trivial behaviour on a super-exponential scale?
  Yes, at least in binary state discrimination.

Correlated states

finite chains:

$$\begin{array}{ccc} \mathcal{C}_{[k,l]} := \otimes_{k\le i\le l}\mathcal{B}(\mathbb{C}^2) & \hookrightarrow & \mathcal{C}_{[k-1,l+1]} \\ \Cup & & \Cup \\ A & \sim & I\otimes A\otimes I \end{array}$$

infinite spin chain:

$$\mathcal{C} := \text{completion of } \left(\bigcup_{k,l\in\mathbb{Z}} \mathcal{C}_{[k,l]}\right)\Big/_{\sim}$$

generated by

$$\sigma_j^{(k)} = \ldots \otimes I \otimes \underbrace{\sigma_j}_{\text{site } k} \otimes I \otimes \ldots, \quad k\in\mathbb{Z}, \qquad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

translation

$$\tau : \sigma_j^{(k)} \mapsto \sigma_j^{(k+1)}, \qquad j=1,2, \quad k\in\mathbb{Z}$$

translation-invariant state:

$\omega_{[k,l]} \in \mathcal{C}_{[k,l]}$ density operators, $\quad \mathrm{Tr}_{k-1}\,\omega_{[k-1,l]} = \mathrm{Tr}_{l+1}\,\omega_{[k,l+1]} = \omega_{[k,l]} = \omega_{k-l}$

$\iff$ ω' positive normalized functional on $\mathcal{C}$, $\quad \omega'\circ\tau = \omega'$

XX-model

Hamilton operators:

$$H_{[k,l]} := \frac{1}{2} \sum_{i=k}^{l-1} \left(\sigma_1^{(i)} \sigma_1^{(i+1)} + \sigma_2^{(i)} \sigma_2^{(i+1)} \right) + h \sum_{i=k}^{l} \sigma_3^{(i)}$$

ground states:

$$\gamma_{[k,l]} := \lim_{\beta \to +\infty} \frac{e^{-\beta H_{[k,l]}}}{\mathrm{Tr}\, e^{-\beta H_{[k,l]}}}$$

thermodynamic limit:

$$\lim_{n \to +\infty} \mathrm{Tr}\, \gamma_{[-n,n]} A = \mathrm{Tr}\, \omega_{[k,l]} A, \qquad A \in \mathcal{C}_{[k,l]}$$

symmetries:

$$\pi : \quad \sigma_k^{(i)} \mapsto -\sigma_k^{(i)}, \quad k = 1, 2, \quad i \in \mathbb{Z}, \qquad \text{parity automorphism}$$

$$\omega_{[k,l]} \in \mathcal{C}_+ := \{ x \in \mathcal{C} : \pi(x) = x \} \qquad \text{even part}$$

$$\tau(\omega_{[k,l]}) = \omega_{[k+1,l+1]} \qquad \text{translation-invariant}$$

Super-exponential distinguishability

Theorem: The TDL ground states of the XX-model corresponding to different $h_0, h_1 \in (-1, 1)$ can be super-exponentially distinguished, i.e.,

$$\lim_{n\to+\infty} -\frac{1}{n}\log \operatorname{Tr}\omega^{(0)}_{[1,n]}(I - T_n) = +\infty = \lim_{n\to+\infty} -\frac{1}{n}\log \operatorname{Tr}\omega^{(1)}_{[1,n]}T_n$$

for some test sequence $T_n \in \mathcal{C}_{[1,n]}$, $n \in \mathbb{N}$.

Fermionic chain

antisymmetric Fock space:

$\phi_1, \ldots, \phi_k \in \mathcal{H} := \ell^2(\mathbb{Z}) = \{(\ldots, x_{-1}, x_0, x_1, \ldots) : \sum_k |x_k|^2 < +\infty\}$

$$\phi_1 \wedge \ldots \wedge \phi_k := \frac{1}{\sqrt{k!}} \sum_{\pi \in \mathfrak{S}_k} \varepsilon(\sigma) \phi_{\sigma(1)} \otimes \ldots \otimes \phi_{\sigma(k)}$$

$$\Gamma(\mathcal{H}) := \bigoplus_{k \in \mathbb{N}} \underbrace{\overline{\text{span}\{\phi_1 \wedge \ldots \wedge \phi_k : \phi_i \in \mathcal{H}\}}}_{=\mathcal{H}^{\wedge k}}$$

Creation and annihilation:

$c(\phi) : \quad \phi_1 \wedge \ldots \wedge \phi_k \mapsto \phi \wedge \phi_1 \wedge \ldots \wedge \phi_k$ creation operator

$a(\phi) := c(\phi)^*$ annihilation operator

Canonical Anti-commutation Relations (CAR):

$$a(\phi)a(\psi) + a(\psi)a(\phi) = 0, \qquad a(\phi)a^*(\psi) + a^*(\psi)a(\phi) = \langle \phi, \psi \rangle I.$$

CAR algebra:

$$\mathcal{A} := \text{CAR}_{\mathcal{H}} := \overline{\text{span}\{I, a^*(\phi_1) \ldots a^*(\phi_n) a(\psi_m) \ldots a(\psi_1) : \phi_i, \psi_j \in \mathcal{H}, n, m \in \mathbb{N}\}}$$

Fermionic chain

$\mathbf{1}_{\{k\}}$, $k \in \mathbb{Z}$, canonical ONB of $\ell^2(\mathbb{Z})$, $a_i := a(\mathbf{1}_{\{i\}})$

$$\mathcal{A} = \overline{\mathrm{span}\{I, a^*_{i_1} \dots a^*_{i_n} a_{j_m} \dots a_{j_1} \,:\, i_1 < \dots < i_n, j_1 < \dots < j_m\}}$$

locality:

$$\mathcal{A}_{[k,l]} := \mathrm{span}\{I, a^*_{i_1} \dots a^*_{i_n} a_{j_m} \dots a_{j_1} \,:\, k \leq i_t, j_s \leq l\} = \mathrm{CAR}_{\ell^2([k,l])}$$

translation:

$$\hat{\tau} : a_i \mapsto a_{i+1}$$

parity automorphism:

$$\hat{\pi} : a_i \mapsto -a_i$$

even part:

$$\mathcal{A}_+ := \{x \in \mathcal{A} : \hat{\pi}(x) = x\} = \overline{\mathrm{span}\{I, a^*_{i_1} \dots a^*_{i_n} a_{j_m} \dots a_{j_1} \,:\, n + m \text{ even}\}}$$

Jordan-Wigner isomorphism

Jordan-Wigner for finite chain:

$e_1, \dots, e_n$ ONB in $\mathbb{C}^{\langle n \rangle} = \ell^2(\{0, \dots, n-1\})$; $|0\rangle, |1\rangle$ ONB in $\mathbb{C}^2$

$$U_e : e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto \otimes_{j=1}^n |x_j\rangle, \qquad x_j := \begin{cases} 1, & j \in \{i_1, \dots, i_k\}, \\ 0, & \text{otherwise}, \end{cases}$$

unitary from $\Gamma(\mathbb{C}^{\langle n \rangle})$ to $(\mathbb{C}^2)^{\otimes n}$, and

$$\alpha : a(e_j)^* \mapsto U_e\, a(e_j)^*\, U_e^* = \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \underbrace{I \otimes \dots \otimes I}_{n-j \text{ times}}$$

isomorphism from $\mathcal{A}_{[0,n-1]}$ to $\mathcal{C}_{[0,n-1]}$; extends to $\mathcal{A}_{[0,+\infty)} \cong \mathcal{C}_{[0,+\infty)}$

Not compatible with the translations of the half-infinite chains:

$$\alpha\left(\hat{\tau}(a(e_j)^*)\right) = \alpha\left(a(e_{j+1}^*)\right) = \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{j \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes I \otimes I \otimes \dots$$

$$\tau(\alpha\left(a(e_j)^*\right) = I \otimes \underbrace{\sigma_3 \otimes \dots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes I \otimes I \otimes \dots$$

Jordan-Wigner isomorphism

Jordan-Wigner for finite chain:

$e_1, \ldots, e_n$ ONB in $\mathbb{C}^{\langle n \rangle} = \ell^2(\{0, \ldots, n-1\})$; $|0\rangle, |1\rangle$ ONB in $\mathbb{C}^2$

$$U_e : e_{i_1} \wedge \ldots \wedge e_{i_k} \mapsto \otimes_{j=1}^n |x_j\rangle, \quad x_j := \begin{cases} 1, & j \in \{i_1, \ldots, i_k\}, \\ 0, & \text{otherwise}, \end{cases}$$

unitary from $\Gamma(\mathbb{C}^{\langle n \rangle})$ to $(\mathbb{C}^2)^{\otimes n}$, and

$$\alpha : a(e_j)^* \mapsto U_e\, a(e_j)^*\, U_e^* = \underbrace{\sigma_3 \otimes \ldots \otimes \sigma_3}_{j-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \otimes \underbrace{I \otimes \ldots \otimes I}_{n-j \text{ times}}$$

isomorphism from $\mathcal{A}_{[0,n-1]}$ to $\mathcal{C}_{[0,n-1]}$

Jordan-Wigner for doubly infinite chain:

$$a_j \mapsto \left(\otimes_{k=-\infty}^{j-1} \sigma_3\right) \otimes \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{\text{site } j} \otimes I \otimes \ldots \quad \notin \mathcal{C}$$

but gives an isomorphism of $(\mathcal{A}_{\mathbb{Z}})_+$ and $(\mathcal{C}_{\mathbb{Z}})+$, compatible with translations.

Bijection between even translation-invariant states on fermions and spins.

Quasi-free states

Symbol:

$$\hat{a} : [0, 2\pi) \to [0, 1] \text{ measurable}$$

$$A_{k,j} := \langle \mathbf{1}_{\{k\}}, A\mathbf{1}_{\{j\}} \rangle = \frac{1}{2\pi} \int_{[0,2\pi)} e^{-i(k-j)x} \hat{a}(x) \, \mathrm{d}x, \qquad k, j \in \mathbb{Z}$$

Toeplitz operator on $\ell^2(\mathbb{Z})$: constant on diagonals

Quasi-free state on $\mathrm{CAR}_{\ell^2(\mathbb{Z})}$:

$$\omega'_A \left(a^*_{i_1} \dots a^*_{i_n} a_{j_m} \dots a_{j_1} \right) = \delta_{m,n} \det \{A_{i_k, j_l}\}_{k,l=1}^{n}$$

Translation-invariant, even $\Longrightarrow$ state on the spin chain

Example: TDL ground state of the XX-model with h

$$\hat{a}_h = \mathbf{1}_{[\arccos f(h), 2\pi - \arccos f(h))}, \qquad f(h) := \max\{-1, \min\{h, 1\}\}$$

Super-exponential distinguishability

Theorem: $\{\hat{q}_i\}_{i\in\mathcal{I}}$, $\{\hat{r}_j\}_{j\in\mathcal{J}}$ symbols

$\exists\, [\mu,\nu] \subseteq [0,2\pi)$ of positive length such that

$$\hat{q}_i\big|_{[\mu,\nu]} \equiv 0, \qquad \hat{r}_i\big|_{[\mu,\nu]} \equiv 1, \qquad i\in\mathcal{I},\ j\in\mathcal{J}$$

There exists a sequence $T_n \in (\mathcal{A}_{[0,n-1]})_+$, $n\in\mathbb{N}$,

$$\sup_{i\in\mathcal{I}} \omega'_{Q^{(i)}}(I-T_n) \le e^{-cn\log n} \qquad\qquad \sup_{j\in\mathcal{J}} \omega'_{R^{(j)}}(T_n) \le e^{-cn\log n}.$$

If $\hat{q}_i \not\equiv 0$, $\hat{r}_j \not\equiv 1$ then the local densities are invertible.

Example: Hypotheses $h \le h_0$ and $h_1 \le h$ can be superexponentially distinguished when $h_0 < h_1$.

$$\mu := \arccos f(h_1), \qquad \nu := \arccos f(h_0)$$

Fourier transform

- All Toeplitz operators on $\ell^2(\mathbb{Z})$ commute

  simultaneously diagonalized by the Fourier transform.

- The n-site restrictions of ω'_Q, ω'_R are quasi-free with symbol operators

$$Q_n = (Q_{k,l})_{k,l=0}^{n-1}, \qquad R_n = (R_{k,l})_{k,l=0}^{n-1}.$$

  Q_n, R_n need not commute.

- Discrete Fouier transform:

$$\mathcal{F}_n : \mathbb{C}^{\langle n \rangle} \to \mathbb{C}^{\langle n \rangle}, \qquad \mathcal{F}_n \mathbf{1}_{\{k\}} := \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} e^{i \frac{2\pi}{n} kj} \mathbf{1}_{\{j\}}, \qquad k \in \langle n \rangle.$$

  Simultaneously diagonalizes all rotation-invariant (circular) operators.

- Idea: Try to diagonalize Q_n, R_n with $\mathcal{F}_n$.

Fourier transform

$$0 \le (\mathcal{F}_n\, A_n\, \mathcal{F}_n^*)_{k,k} = \sum_{j,l=0}^{n-1} (\mathcal{F}_n)_{k,j} (A_n)_{j,l} (\mathcal{F}_n^*)_{l,k}$$

assume: $0 \le \hat{a} \le 1$

$$= \frac{1}{n} \sum_{j,l=0}^{n-1} e^{i\frac{2\pi}{n}kj} e^{-i\frac{2\pi}{n}kl} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-i(j-l)x} \hat{a}(x)\,\mathrm{d}x$$

$m := j - l$

$$= \sum_{m=-n+1}^{n-1} \frac{n-|m|}{n} e^{i\frac{2\pi}{n}km} \frac{1}{2\pi} \int_{[0,2\pi)} e^{-imx} \hat{a}(x)\,\mathrm{d}x$$

$$= \int_{[0,2\pi)} \hat{a}(x) \underbrace{\frac{1}{2\pi} \sum_{m=-n+1}^{n-1} \frac{n-|m|}{n} e^{im\left(\frac{2\pi k}{n}-x\right)}}_{=F_n\left(\frac{2\pi k}{n}-x\right)=\frac{1}{2\pi n}\frac{\sin^2\left(n\left(\frac{2\pi k}{n}-x\right)/2\right)}{\sin^2\left(\left(\frac{2\pi k}{n}-x\right)/2\right)}} \mathrm{d}x$$

$$= \int_{[-\pi,\pi)} \hat{a}\left(\frac{2\pi k}{n} - x\right) F_n(x)\,\mathrm{d}x$$

Fejér kernel

$$F_n(x) = \frac{1}{2\pi n}\frac{\sin^2(nx/2)}{\sin^2(x/2)}$$

assume: $\hat{a}\big|_{[\mu,\nu]} \equiv 0, \qquad \frac{2\pi k}{n} \in [\mu+\delta, \nu-\delta]$

$$\begin{aligned}
(\mathcal{F}_n\, A_n\, \mathcal{F}_n^*)_{k,k} &= \int_{[-\pi,\pi)} \hat{a}\left(\frac{2\pi k}{n} - x\right) F_n(x)\,\mathrm{d}x \\
&= \int\limits_{|x|<\delta} F_n(x)\,\underbrace{\hat{a}\left(\frac{2\pi k}{n} - x\right)}_{=0}\mathrm{d}x + \int\limits_{\delta\le|x|\le\pi} F_n(x)\,\underbrace{\hat{a}\left(\frac{2\pi k}{n} - x\right)}_{\le 1}\mathrm{d}x \\
&\le \frac{1}{2\pi n}\int\limits_{\delta\le|x|\le\pi} \frac{\sin^2\frac{nx}{2}}{\sin^2\frac{x}{2}}\,\mathrm{d}x \le \frac{1}{2\pi n}\int\limits_{\delta\le|x|\le\pi} \frac{1}{\sin^2\frac{\delta}{2}}\,\mathrm{d}x \\
&= \frac{\pi-\delta}{\pi n\sin^2\frac{\delta}{2}} \le \frac{1}{n\sin^2\frac{\delta}{2}} =: \frac{\gamma_\delta}{n}
\end{aligned}$$

Fourier transform

$$(\mathcal{F}_n\, A_n\, \mathcal{F}_n^*)_{k,k} \le \frac{\gamma_\delta}{n}$$

Corollary: $$E_{n,\delta} := \sum_{k:\ \frac{2\pi k}{n} \in [\mu+\delta, \nu-\delta]} \big|\mathcal{F}_n^*\, \mathbf{1}_{\{k\}}\big\rangle\big\langle \mathcal{F}_n^*\, \mathbf{1}_{\{k\}}\big|,$$

$$\operatorname{Tr} E_{n,\delta} \ge \left\lfloor \frac{\nu-\mu-2\delta}{2\pi} n \right\rfloor \ge cn, \qquad \operatorname{Tr} E_{n,\delta} A_n \le (\operatorname{Tr} E_{n,\delta}) \frac{\gamma_\delta}{n}, \qquad n \in \mathbb{N}.$$

Corollary: $\hat{q}^{(i)}\big|_{[\mu,\nu]} \equiv 0, \quad \hat{r}^{(j)}\big|_{[\mu,\nu]} \equiv 1, \qquad E_{n,\delta}$ as above

$$\operatorname{Tr} E_{n,\delta} \ge cn, \qquad \frac{\operatorname{Tr} E_{n,\delta} Q_n^{(i)}}{\operatorname{Tr} E_{n,\delta}} \le \frac{\gamma_\delta}{n}, \quad \frac{\operatorname{Tr} E_{n,\delta}(I - R_n^{(j)})}{\operatorname{Tr} E_{n,\delta}} \le \frac{\gamma_\delta}{n}, \qquad n \in \mathbb{N}.$$

Universal test sequence

particle number operator: $e_1, \ldots, e_{d_n}$ ONB in $\mathcal{H} := \operatorname{ran} E_{n,\delta} \subseteq \ell^2(\{0, \ldots, n-1\})$

$$N_{\mathcal{H}} := \bigoplus_{k=0}^{d_n} k I_{\mathcal{H}^{\wedge k}} = \sum_{i=1}^{d_n} a(e_i)^* a(e_i) = U_e^* \left(\sum_{i=1}^{d_n} \underbrace{I \otimes \ldots \otimes I}_{i-1 \text{ times}} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \otimes \underbrace{I \otimes \ldots \otimes I}_{d_n - i \text{ times}} \right) U_e.$$

The eigenvalues of $N_{\mathcal{H}}$ are $0, \ldots, d_n$, with spectral projections

$$P_k^{N_{\mathcal{H}}} = \underbrace{0 \oplus \ldots \oplus 0}_{k \text{ times}} \oplus I_{\mathcal{H}^{\wedge k}} \oplus \underbrace{0 \oplus \ldots \oplus 0}_{d_n - k \text{ times}}$$

$$= U_e^* \left(\sum_{\Lambda \subseteq [d_n],\, |\Lambda| = k} \left(\bigotimes_{i \in \Lambda} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \bigotimes \left(\bigotimes_{i \in [d_n] \setminus \Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) U_e$$

$$T_{n,\delta} := \sum_{k=0}^{\lfloor d_n/2 \rfloor} P_k^{N_{\mathcal{H}}}$$

Universal test sequence

$$T_{n,\delta} = \sum_{k=0}^{\lfloor d_n/2 \rfloor} U_e^* \left(\sum_{\Lambda \subseteq [d_n],\, |\Lambda|=k} \left(\bigotimes_{j\in\Lambda} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \otimes \left(\bigotimes_{j\in[d]\setminus\Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) \right) U_e .$$

$$\tilde{A}_n := E_{n,\delta} A E_{n,\delta} = \sum_{i=1}^{d_n} a_i \, |e_i\rangle\langle e_i|$$

$$\omega_{\tilde{A}_n} = U_e^* \left(\bigotimes_{j=1}^{d_n} \begin{bmatrix} 1-a_j & 0 \\ 0 & a_j \end{bmatrix} \right) U_e \le U_e^* \left(\bigotimes_{j=1}^{d_n} \begin{bmatrix} 1 & 0 \\ 0 & a_j \end{bmatrix} \right) U_e ,$$

$$\omega'_A(I - T_{n,\delta}) = \mathrm{Tr}\, \omega_{\tilde{A}_n}(I - T_{n,\delta})$$

$$\le \mathrm{Tr} \sum_{k=\lfloor d_n/2\rfloor+1}^{d_n} \sum_{\Lambda \subseteq [d_n],\, |\Lambda|=k} \left(\bigotimes_{j\in\Lambda} \begin{bmatrix} 0 & 0 \\ 0 & a_j \end{bmatrix} \right) \otimes \left(\bigotimes_{j\in[d_n]\setminus\Lambda} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$= \sum_{k=\lfloor d_n/2\rfloor+1}^{d_n} \sum_{\Lambda \subseteq [d_n],\, |\Lambda|=k} \prod_{j\in\Lambda} a_j \le \sum_{\lfloor d_n/2\rfloor+1}^{d_n} \underbrace{\sum_{\Lambda \subseteq [d_n],\, |\Lambda|=k}}_{=\binom{d_n}{k}} \underbrace{\left(\frac{\sum_{j\in\Lambda} a_j}{k} \right)^k}_{\le \left(\frac{\mathrm{Tr}\, \tilde{A}_n}{d_n/2} \right)^{d_n/2}} \le 2^{d_n} \left(\frac{\mathrm{Tr}\, \tilde{A}_n}{d_n/2} \right)^{\frac{d_n}{2}}$$

Super-exponential discrimination

$$\omega'_A(I - T_{n,\delta}) \le 2^{d_n} \left(\frac{\operatorname{Tr} \tilde{A}_n}{d_n/2} \right)^{\frac{d_n}{2}}$$

$$\operatorname{Tr} E_{n,\delta} \ge cn, \qquad \frac{\operatorname{Tr} E_{n,\delta} Q_n^{(i)}}{\operatorname{Tr} E_{n,\delta}} \le \frac{\gamma_\delta}{n}, \quad \frac{\operatorname{Tr} E_{n,\delta}(I - R_n^{(j)})}{\operatorname{Tr} E_{n,\delta}} \le \frac{\gamma_\delta}{n}, \qquad n \in \mathbb{N}.$$

$$\omega_{Q^{(i)}}(I - T_{n,\delta}) \le \left(\frac{8 \operatorname{Tr} A E_{n,\delta}}{\operatorname{Tr} E_{n,\delta}} \right)^{\frac{\operatorname{Tr} E_{n,\delta}}{2}} \le e^{-nc \log n},$$

$$\omega_{R^{(j)}}(T_{n,\delta}) \le \left(\frac{8 \operatorname{Tr}(I - A) E_{n,\delta}}{\operatorname{Tr} E_{n,\delta}} \right)^{\frac{\operatorname{Tr} E_{n,\delta}}{2}} \le e^{-nc \log n}$$

Note: $T_{n,\delta}$ only depends on $[\mu, \nu]$, not on the specific states.

Orthogonality of states

Orthogonality and perfect distinguishability:

$$\min_{T\in\mathbb{T}(\mathcal{H})}\{\underbrace{\operatorname{Tr}\varrho(I-T)}_{=:\alpha(T)}+\underbrace{\operatorname{Tr}\sigma T}_{=:\beta(T)}\}=1-\frac{1}{2}\left\|\varrho-\sigma\right\|_1,$$

where $\|X\|_1 := \operatorname{Tr}|X|$, $X \in \mathcal{B}(\mathcal{H})$, is the trace-norm. In particular, we have

$$\exists\, T\in\mathbb{T}(\mathcal{H}):\ \alpha(T)=0=\beta(T) \iff \chi(\varrho\|\sigma):=-\log\left(1-\frac{1}{2}\left\|\varrho-\sigma\right\|_1\right)=+\infty$$

$$\iff \varrho\perp\sigma$$

Rényi (α, z)-divergence of ϱ and σ:

$$D_{\alpha,z}(\varrho\|\sigma):=\frac{1}{\alpha-1}\log\operatorname{Tr}\left(\varrho^{\frac{\alpha}{z}}\sigma^{\frac{1-\alpha}{z}}\varrho^{\frac{\alpha}{z}}\right)^z$$

$$\varrho\perp\sigma \iff D_{\alpha,z}(\varrho\|\sigma)=+\infty \text{ for some/all } \alpha\in(0,1),\ z\in(0,+\infty).$$

Orthogonality of states

$T \in \mathbb{T}(\mathcal{H})$ test:

$$\mathcal{T}(X) := (\operatorname{Tr} XT) \, |0\rangle\langle 0| + (\operatorname{Tr} X(I-T)) \, |1\rangle\langle 1| \, ,$$

Test-measured Rényi α-divergence

$$D_\alpha^{\mathrm{test}}(\varrho\|\sigma) := \max_{T \in \mathbb{T}(\mathcal{H})} D_\alpha(\mathcal{T}(\varrho)\|\mathcal{T}(\sigma))$$

$$\varrho \perp \sigma \iff D_\alpha^{\mathrm{test}}(\varrho\|\sigma) = +\infty \quad \text{for some/all} \quad \alpha \in (0,1),$$

Divergences for infinite-size states

$\omega^{(k)}_{[1,n]} = \omega_k^{\otimes n}$

D^q_α strictly positive, monotone, and additive quantum Rényi α-divergence

$$D^q_\alpha\left(\omega^{(0)}\|\omega^{(1)}\right) \geq D^q_\alpha\left(\omega_0^{\otimes n}\|\omega_1^{\otimes n}\right) = nD_\alpha\left(\omega_0\|\omega_1\right) \xrightarrow[n\to+\infty]{} +\infty$$

Fuchs-van de Graaf inequality $\chi(\varrho\|\sigma) \geq \frac{1}{2}D_{1/2,1/2}(\varrho\|\sigma)$

$$\chi(\omega^{(0)}\|\omega^{(1)}) \geq \chi(\omega_0^{\otimes n}\|\omega_1^{\otimes n}) \geq \frac{1}{2}D_{1/2,1/2}(\omega_0^{\otimes n}\|\omega_1^{\otimes n}) \xrightarrow[n\to+\infty]{} +\infty$$
$$\iff \quad \left\|\omega^{(0)} - \omega^{(1)}\right\| = 2$$

This gives one possible notion of orthogonality for states on a spin chain.

Any two different i.i.d. states are orthogonal
$\implies$ not well suited for state discrimination

Divergences and super-exponential distinguishability

$\Delta = \chi, D_{\alpha,z}, D_\alpha^{\text{test}}$

$$\overline{\Delta}(\omega^{(0)}\|\omega^{(1)}) := \liminf_{n\to+\infty} \frac{1}{n} \Delta\left(\omega^{(0)}_{[1,n]}\middle\|\omega^{(1)}_{[1,n]}\right),$$

Theorem: $\omega^{(0)}$ and $\omega^{(1)}$ translation-invariant states on the infinite spin-chain algebra $\mathcal{B}(\mathcal{H})_{\mathbb{Z}}$. T.f.a.e.:

1. $\omega^{(0)}$ and $\omega^{(1)}$ can be super-exponentially distinguished.
2. $\overline{\chi}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$.
3. $\overline{D}_\alpha^{\text{test}}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ for every $\alpha \in (0,1)$.
4. $\overline{D}_\alpha^{\text{test}}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ for some $\alpha \in (0,1)$.
5. $\overline{D}_{\alpha,z}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ for every $\alpha \in (0,1)$ and every $z \geq \max\{\alpha, 1-\alpha\}$.
6. $\overline{D}_{\alpha,z}\left(\omega^{(0)}\middle\|\omega^{(1)}\right) = +\infty$ for some $\alpha \in (0,1)$ and some $z \geq \max\{\alpha, 1-\alpha\}$.

Open questions

- What is the optimal asymptotics of error probabilities in the quasi-free example?
- What is the optimal asymptotics of error probabilities achievable with arbitrary correlated states on $\otimes_{k\in\mathbb{Z}}\mathcal{B}(\mathbb{C}^d)$?

  Does it depend on the local local dimension d?
- Trade-off relations on a super-exponential scale?

  Suitable regularization of Rényi divergences?
- What properties guarantee super-exponential asymptotics?

  Quantitative relation to correlations?