Quantum expanders – Random constructions & Applications

Based on joint works with:
David Pérez-Garcìa (arXiv:1906.11682) and Pierre Youssef (arXiv:2302.07772)

Cécilia Lancien

Institut Fourier Grenoble & CNRS

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Plan

Classical expanders

G a d-regular graph on n vertices (d edges at each vertex).
A its (normalized) adjacency matrix, i.e. the $n \times n$ matrix s.t. $A_{kl} = e(k,l)/d$ for all $1 \leqslant k,l \leqslant n$.
number of edges between vertices k and l ↵

$\lambda_1(A),\ldots,\lambda_n(A)$ eigenvalues of A, ordered s.t. $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$.

G regular $\Longrightarrow \lambda_1(A) = 1$ with associated eigenvector the uniform probability $u = (1/n,\ldots,1/n)$.
The *spectral expansion parameter* of G is $\lambda(G) := |\lambda_2(A)|$.

Observation: $\lambda(G) = |\lambda_1(A-J)|$, where J is the adjacency matrix of the *complete graph* on n vertices, i.e. the matrix whose entries are all equal to $1/n$.
$\longrightarrow \lambda(G)$ is a distance measure between G and the complete graph.

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Definition [Classical expander]

A d-regular graph G on n vertices is an *expander* if it is sparse (i.e. $d \ll n$) and spectrally expanding (i.e. $\lambda(G) \ll 1$).

$\longrightarrow$ G is both 'economical' and 'resembling' the complete graph.
For instance, a random walk supported on G converges fast to equilibrium.
Indeed, for any probability p on $\{1, \ldots, n\}$, $\forall\, q \in \mathbf{N}$, $\|A^q p - u\|_1 \leqslant \sqrt{n}\|A^q p - u\|_2 \leqslant \sqrt{n}\lambda(G)^q$.
exponential convergence, at rate $|\log \lambda(G)|$ ↵

Quantum analogue of the transition matrix associated to a regular graph

Classical - Quantum correspondence:

- $p \in \mathbf{R}^n$ probability vector $\longleftrightarrow \rho \in \mathcal{M}_n(\mathbf{C})$ density operator (PSD and trace 1 operator).
- $A : \mathbf{R}^n \to \mathbf{R}^n$ transition matrix $\longleftrightarrow \Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$ quantum channel (CPTP map).
- G regular: A leaves u invariant $\longleftrightarrow \Phi$ unital: Φ leaves I/n invariant.

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- G regular: A leaves u invariant $\longleftrightarrow$ Φ unital: Φ leaves I/n invariant.

Question: What is the analogue of the degree?

Given a CP map Φ on $\mathcal{M}_n(\mathbf{C})$, its *Kraus representation* is:

$$\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^{d} K_i X K_i^* \in \mathcal{M}_n(\mathbf{C}), \text{ where } K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C}). \quad (\star)$$

↳ Kraus operators of Φ

The minimal d s.t. Φ can be written as $(\star)$ is the *Kraus rank* of Φ (it is always at most n^2).
[Note: Φ is TP iff $\sum_{i=1}^d K_i^* K_i = I$. Φ is unital iff $\sum_{i=1}^d K_i K_i^* = I$.]

Answer: The analogue of the degree is the Kraus rank.

- G a d-regular graph: If $|\mathrm{supp}(p)| = 1$, then $|\mathrm{supp}(Ap)| \leqslant d$.
- Φ a Kraus rank d unital quantum channel: If $\mathrm{rk}(\rho) = 1$, then $\mathrm{rk}(\Phi(\rho)) \leqslant d$.

$\longrightarrow$ Both quantify the 1-iteration spreading.

Quantum expanders

Φ a Kraus rank d unital quantum channel on $\mathcal{M}_n(\mathbf{C})$.
$\lambda_1(\Phi), \ldots, \lambda_{n^2}(\Phi)$ eigenvalues of Φ, ordered s.t. $|\lambda_1(\Phi)| \geqslant \cdots \geqslant |\lambda_{n^2}(\Phi)|$.

Φ unital $\Longrightarrow \lambda_1(\Phi) = 1$ with associated eigenstate the maximally mixed state I/n.
The *spectral expansion parameter* of Φ is $\lambda(\Phi) := |\lambda_2(\Phi)|$.

Observation: $\lambda(\Phi) = |\lambda_1(\Phi - \Pi)|$, where Π is the *maximally mixing channel* on $\mathcal{M}_n(\mathbf{C})$, i.e. $\Pi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\, I/n \in \mathcal{M}_n(\mathbf{C})$.
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A Kraus rank d unital quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$ is an *expander* if it is little noisy (i.e. $d \ll n^2$) and spectrally expanding (i.e. $\lambda(\Phi) \ll 1$).

$\longrightarrow$ Φ is both ‘economical’ and ‘resembling’ the maximally mixing channel.
For instance, the dynamics associated to Φ converges fast to equilibrium.
Indeed, for any state ρ on $\mathbf{C}^n$, $\forall\, q \in \mathbf{N}$, $\|\Phi^q(\rho) - I/n\|_1 \leqslant \sqrt{n}\|\Phi^q(\rho) - I/n\|_2 \leqslant \sqrt{n}\lambda(\Phi)^q$.
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Constructions of optimal classical expanders

Fact: For any d-regular graph G on n vertices, $\lambda(G) \geqslant 2\sqrt{d-1}/d - o_n(1)$.
$\longrightarrow$ G is called a *Ramanujan graph* if it is an optimal expander, i.e. $\lambda(G) \leqslant 2\sqrt{d-1}/d$.

Question: Do Ramanujan graphs exist?

1. Explicit constructions of exactly Ramanujan graphs only for $d = p^m + 1$, p prime.
2. Random constructions of almost Ramanujan graphs for all d.
3. Existence of exactly Ramanujan graphs for all d.

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In fact, for large n, almost all regular graphs are almost Ramanujan:

Theorem [Uniform random regular graph (Friedman, Bordenave)]

Fix $d \in \mathbf{N}$. Let G be uniformly distributed on the set of d-regular graphs on n vertices.
Then, for all $\varepsilon > 0$, $\mathbf{P}\left(\lambda(G) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon\right) = 1 - o_n(1)$.

Remarks: ↱ permutation model
• First proven for a simpler model of random regular graphs: for d even, pick $\sigma_1, \ldots, \sigma_{d/2} \in \mathcal{S}_n$ independent uniformly distributed and let G have edges $\{(k, \sigma_i(k)), (k, \sigma_i^{-1}(k))\}_{1 \leqslant k \leqslant n, 1 \leqslant i \leqslant d/2}$.
• Remains true for d_n growing with n, up to a constant multiplicative factor (Cook/Goldstein/Johnson, Tikhomirov/Youssef): $\mathbf{P}\left(\lambda(G) \leqslant C/\sqrt{d_n} + \varepsilon\right) = 1 - o_n(1)$.

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Constructions of optimal quantum expanders

Fact: For any Kraus rank d unital quantum channel Φ on $\mathcal{M}_n(\mathbf{C})$, $\lambda(\Phi) \geqslant c/\sqrt{d}$.
$\longrightarrow$ Φ is an optimal expander if $\lambda(\Phi) \leqslant C/\sqrt{d}$.

Question: Do optimal quantum expanders exist?
First attempts at exhibiting explicit constructions (inspired by classical ones): not optimal.
$\longrightarrow$ What about random constructions?

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Question: How to sample a unital quantum channel randomly?

Idea: Pick random Kraus operators $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$, under the constraint $\begin{cases} \sum_{i=1}^d K_i^* K_i = I \\ \sum_{i=1}^d K_i K_i^* = I \end{cases}$.

Let $\Phi : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \sum_{i=1}^d K_i X K_i^* \in \mathcal{M}_n(\mathbf{C})$ be the associated random unital quantum channel.

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Theorem [Independent paired Haar unitaries as Kraus operators (Hastings, Pisier)]

Fix $d \in \mathbf{N}$ even. Pick $U_1, \ldots, U_{d/2} \in \mathcal{M}_n(\mathbf{C})$ independent Haar unitaries. Let $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d/2$. The random CP map Φ associated to the K_i's, K_i^*'s is TP and unital by construction.
Then, for all $\varepsilon > 0$, $\mathbf{P}\left(\lambda(\Phi) \leqslant \frac{2\sqrt{d-1}}{d} + \varepsilon\right) = 1 - o_n(1)$.

Remarks:

- Optimal for quantum channels Φ with d unitary Kraus operators, for which $\lambda(\Phi) \geqslant 2\sqrt{d-1}/d$.
- Same result, up to a constant multiplicative factor, for d independent unitary Kraus operators.

More random examples of optimal quantum expanders

Question: Can the previous result be extended to other random models? And to a regime where d is not fixed but grows with n?

Difficulty: Imposing that Φ is both TP and unital is very constraining.
However, the definition of expander can be extended to 'close to unital' quantum channels, whose fixed point ρ_* has a large entropy: $S(\rho_*) \geqslant \alpha S(I/n) = \alpha \log n$, for some $0 < \alpha < 1$.
[Note: We now have $\lambda(\Phi) = |\lambda_1(\Phi - \Pi_{\rho^*})|$, where $\Pi_{\rho_*} : X \in \mathcal{M}_n(\mathbf{C}) \mapsto \mathrm{Tr}(X)\,\rho_* \in \mathcal{M}_n(\mathbf{C})$.]

Classical analogy: Relaxation of the exact regularity condition, e.g. to look at Erdős-Rényi graphs.

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Theorem [Independent Gaussians as Kraus operators (Lancien/Pérez-García)**]**

Pick $G_1, \ldots, G_d \in \mathcal{M}_n(\mathbf{C})$ independent Gaussian matrices. Let $\tilde{K}_i = G_i/\sqrt{d}$, $1 \leqslant i \leqslant d$.
↳ i.i.d. Gaussian entries (mean 0 and variance $1/n$)

The random CP map $\tilde{\Phi}$ associated to the $\tilde{K}_i$'s is not TP but almost: $\mathbf{P}\left(\Sigma := \sum_{i=1}^d \tilde{K}_i^* \tilde{K}_i \simeq I\right) \simeq 1$.
With $K_i = \tilde{K}_i \Sigma^{-1/2}$, $1 \leqslant i \leqslant d$, the random CP map Φ associated to the K_i's is TP by construction.
Then, $\mathbf{P}\left(S(\rho_*) \geqslant \log n - \frac{C'}{\sqrt{d}} \text{ and } \lambda(\Phi) \leqslant \frac{C}{\sqrt{d}} \right) \geqslant 1 - e^{-cn}$, for $C, C', c > 0$ constants.

Remark: Other model that was proven to be a.s. an optimal expander as n grows (for d fixed): blocks of a Haar isometry $V : \mathbf{C}^n \hookrightarrow \mathbf{C}^n \otimes \mathbf{C}^d$ as Kraus operators (González-Guillén/Junge/Nechita).

How much can the previous examples be generalized?

Theorem [Independent general random matrices as Kraus operators (Lancien/Youssef)]

1. Let $A \in \mathcal{M}_n(\mathbf{R})$ be a doubly stochastic matrix s.t. $|\lambda_2(A)| \leqslant \frac{C}{\sqrt{d}}$, with $d \geqslant (\log n)^4$.
   E.g. A the adjacency matrix of a d-regular graph G on n vertices s.t. $\lambda(G) \leqslant \frac{C}{\sqrt{d}}$.
2. Let $W \in \mathcal{M}_n(\mathbf{C})$ be a random matrix with independent centered entries, s.t.
   $\forall\, 1 \leqslant k, l \leqslant n$, $\mathbf{E}|W_{kl}|^2 = A_{kl}$ and $\left(\mathbf{E}|W_{kl}|^{2p}\right)^{1/p} \leqslant C' p^{\beta} A_{kl},\ p \in \mathbf{N}$.
   [$\beta = 0$: bounded entries. $\beta = 1$: sub-Gaussian entries. $\beta = 2$: sub-exponential entries.]
3. Pick $W_1, \ldots, W_d \in \mathcal{M}_n(\mathbf{C})$ independent copies of W. Let $K_i = \frac{W_i}{\sqrt{d}}$, $1 \leqslant i \leqslant d$, and Φ be the random CP map with the K_i's as Kraus operators.

Then, Φ is on average TP and unital, and s.t. $\mathbf{E}\lambda(\Phi) \leqslant \dfrac{C''}{\sqrt{d}}$.

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Then, Φ is on average TP and unital, and s.t. $\mathbf{E}\,\lambda(\Phi) \leqslant \dfrac{C''}{\sqrt{d}}$.

Interest: Constructing a random optimal quantum expander from any optimal classical expander.
$\longrightarrow$ Optimal quantum expanders can be obtained from random Kraus operators which are sparse and whose entries have any distribution following the moments' growth assumption.

Proof idea to show that $\mathbf{E}\lambda(\Phi) \leqslant C/\sqrt{d}$

Goal: In all cases, we want to upper bound $\mathbf{E}|\lambda_2(\Phi)| = \mathbf{E}|\lambda_1(\Phi - \Pi_{\rho^*})|$.
First step: Upper bound $\mathbf{E}|\lambda_1(\Phi - \mathbf{E}(\Phi))|$ (and then show that $\mathbf{E}(\Phi)$ is close to Π_{ρ^*}).

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• Observation 1: $|\lambda_1(\Psi)| \leqslant s_1(\Psi) = \|\Psi\|_\infty$.
• Observation 2: $\|\Psi\|_\infty = \|M_\Psi\|_\infty$, where for $\Psi : X \mapsto \sum_{i=1}^d K_i X L_i^*$, $M_\Psi = \sum_{i=1}^d K_i \otimes \bar{L}_i$.
[Identification $\Psi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C}) \equiv M_\Psi : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$ preserves the operator norm.]
$\longrightarrow$ We want to upper bound $\mathbf{E}\| \underbrace{M_\Phi - \mathbf{E}(M_\Phi)}_{=:X} \|_\infty$, where $M_\Phi = \sum_{i=1}^d K_i \otimes \bar{K}_i$ with the K_i's random.

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↱ Haar unitaries, Gaussians, blocks of Haar isometry

- For concrete models, this can be done by a moments' method:

By Jensen's inequality, we have: $\forall\, p \in \mathbf{N},\ \mathbf{E}\|X\|_\infty \leqslant \mathbf{E}\|X\|_p \leqslant (\mathbf{E}\,\mathrm{Tr}|X|^p)^{1/p}$.
The term on the r.h.s. can be estimated and provides a good upper bound for $p \simeq n^\gamma$.
↳ by Weingarten or Wick calculus

- For the general case, we use recent results on estimating the operator norm of random matrices with dependencies and non-homogeneity (Bandeira/Boedihardjo/van Handel, Brailovskaya/van Handel):

Setting $X = \sum_{i=1}^d Z_i$, with $Z_i := K_i \otimes \bar{K}_i - \mathbf{E}(K_i \otimes \bar{K}_i)$, $1 \leqslant i \leqslant d$, we have for $p \simeq \log n$,

$$\mathbf{E}\|X\|_\infty \lesssim \|\mathbf{E}(XX^*)\|_\infty^{1/2} + \|\mathbf{E}(X^*X)\|_\infty^{1/2} + (\log n)^{3/2}\|\mathbf{Cov}(X)\|_\infty^{1/2} + (\log n)^2\left(\sum_{i=1}^d \mathbf{E}\,\mathrm{Tr}|Z_i|^p\right)^{1/p}.$$

Plan

Outline

Small subset of ‘physically relevant’ states of many-body quantum systems

Curse of dimensionality: Exponential growth of system’s dimension with number of subsystems. However, ‘physically relevant’ states of many-body quantum systems are often well approximated by so-called *tensor network states (TNS)*, which form a small subset of the global state space.

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Example: A *matrix product state (MPS)* on $(\mathbf{C}^d)^{\otimes M}$ is a pure state $\chi \in (\mathbf{C}^d)^{\otimes M}$ of the form

$$\chi = \sum_{i_1,\ldots,i_M=1}^{d} \mathrm{Tr}\left(K_{i_1}^{(1)} \cdots K_{i_M}^{(M)} \right) |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_M}\rangle, \text{ where } K_1^{(\ell)},\ldots,K_d^{(\ell)} \in \mathcal{M}_n(\mathbf{C}),\ 1 \leqslant \ell \leqslant M.$$

$\longrightarrow$ Such state is described by Mdn^2 parameters, which is linear rather than exponential in M.

[Vocabulary: d is the *physical dimension*. n is the *bond dimension*.]

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Fact: On a 1D system (M subsystems disposed on a line), the *ground state of a gapped local Hamiltonian* is well approximated by an MPS (Hastings, Landau/Vazirani/Vidick...)

spectral gap lower bounded by a constant independent of M ←

composed of terms which act non-trivially only on nearby sites ←

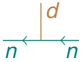
$\longrightarrow$ In condensed-matter physics, MPS are used as Ansatz in ground energy computations: optimization over a manageable number of parameters, even for large M.

Outline

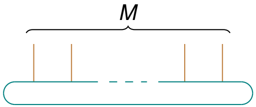
Model of random translation-invariant MPS

Idea: Pick $K_1,\dots,K_d \in \mathcal{M}_n(\mathbf{C})$ at random.
Let $\chi \in (\mathbf{C}^d)^{\otimes M}$ be the corresponding *random translation-invariant MPS*, i.e.

$$\chi = \sum_{i_1,\dots,i_M=1}^{d} \mathrm{Tr}\left(K_{i_1}\cdots K_{i_M}\right) |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_M}\rangle .$$



$$K = \sum_{i=1}^{d} K_i \otimes |e_i\rangle \in \mathcal{M}_n(\mathbf{C}) \otimes \mathbf{C}^d$$



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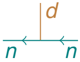
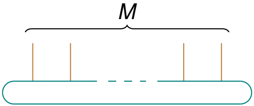
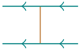




$$K = \sum_{i=1}^{d} K_i \otimes |e_i\rangle \in \mathcal{M}_n(\mathbf{C}) \otimes \mathbf{C}^d \qquad \chi = \sum_{i_1,\dots,i_M=1}^{d} \mathrm{Tr}\left(K_{i_1} \cdots K_{i_M}\right) |e_{i_1}\rangle \otimes \cdots \otimes |e_{i_M}\rangle \in (\mathbf{C}^d)^{\otimes M}$$

Associated *transfer operator*: $T = \sum_{i=1}^{d} K_i \otimes \bar{K}_i \in \mathcal{M}_n(\mathbf{C}) \otimes \mathcal{M}_n(\mathbf{C})$,



obtained by contracting the d-dimensional indices of K and $\bar{K}$.

Observation: $T = \sum_{i=1}^{d} K_i \otimes \bar{K}_i$ is the matrix version of the CP map $\Phi_T : X \mapsto \sum_{i=1}^{d} K_i X K_i^*$.
[Identification $T : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n \equiv \Phi_T : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$.]
In particular, both have the same spectrum.

Correlations in an MPS

Let A, B be 1-site observables, i.e. observables on $\mathbf{C}^d$.

$A : \mathbf{C}^d \to \mathbf{C}^d$

$B : \mathbf{C}^d \to \mathbf{C}^d$

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Compute the value on the MPS χ of the observable $A_1 \otimes I_q \otimes B_1 \otimes I_{M-q-2}$, i.e.

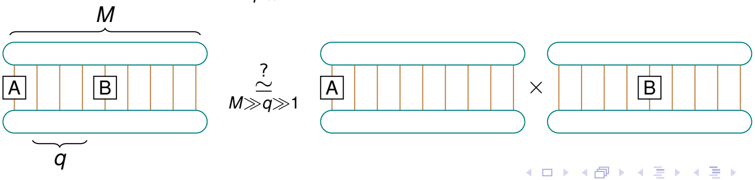
$$v_\chi(A,B,q) := \frac{\langle\chi|\, A_1 \otimes I_q \otimes B_1 \otimes I_{M-q-2}\, |\chi\rangle}{\langle\chi|\chi\rangle}.$$

Compare it to the product of the values on χ of $A_1 \otimes I_{M-1}$ and $I_{q+1} \otimes B_1 \otimes I_{M-q-2}$, i.e.

$$v_\chi(A)v_\chi(B) := \frac{\langle\chi|\, A_1 \otimes I_{M-1}\, |\chi\rangle \langle\chi|\, I_{q+1} \otimes B_1 \otimes I_{M-q-2}\, |\chi\rangle}{\langle\chi|\chi\rangle^2}.$$

Correlations in the MPS χ: $\gamma_\chi(A,B,q) := \left| v_\chi(A,B,q) - v_\chi(A)v_\chi(B) \right|$.

Question: Do we have $\gamma_\chi(A,B,q) \underset{q \ll M \to \infty}{\longrightarrow} 0$? And if so, at which speed?

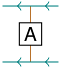


Correlation length in an MPS and spectrum of its transfer operator

$\chi \in (\mathbf{C}^d)^{\otimes M}$ an MPS.
T its associated transfer operator on $\mathbf{C}^n \otimes \mathbf{C}^n$.
Its correlation function can be re-written as:

$$\gamma_\chi(A,B,q) = \left| \frac{\mathrm{Tr}\left(\hat{A}T^q\hat{B}T^{M-q-2}\right)}{\mathrm{Tr}(T^M)} - \frac{\mathrm{Tr}\left(\hat{A}T^{M-1}\right)\mathrm{Tr}\left(\hat{B}T^{M-1}\right)}{(\mathrm{Tr}(T^M))^2} \right| .$$

$A : \mathbf{C}^d \to \mathbf{C}^d$



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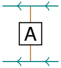

$\hat{A} : \mathbf{C}^n \otimes \mathbf{C}^n \to \mathbf{C}^n \otimes \mathbf{C}^n$

Important consequence:
$\lambda_1(T), \lambda_2(T), \ldots$ eigenvalues of T (with multiplicities), ordered s.t. $|\lambda_1(T)| > |\lambda_2(T)| > \cdots$.
Set $\Delta(T) := |\lambda_1(T)| - |\lambda_2(T)|$ and $\varepsilon(T) = |\lambda_2(T)|/|\lambda_1(T)|$.

$$\gamma_\chi(A,B,q) \leqslant C \left(\frac{\mathrm{Tr}(T)}{\Delta(T)} \right)^2 \varepsilon(T)^q \|A\|_\infty \|B\|_\infty .$$

$\longrightarrow$ Correlations between two 1-site observables decay exponentially with the distance separating the two sites, at a rate $\tau(\chi) = |\log \varepsilon(T)|$.
Correlation length in the MPS χ: $\xi(\chi) := 1/\tau(\chi) = 1/|\log \varepsilon(T)|$.

Conclusion: Estimating $\xi(\chi)$ boils down to estimating $|\lambda_1(\Phi_T)|$ and $|\lambda_2(\Phi_T)|$.

Decay of correlations in random translation-invariant MPS

Examples of distribution for $K_1,\dots,K_d \in \mathcal{M}_n(\mathbf{C})$:

1. $K_i = W_i/\sqrt{d}$, $1 \leqslant i \leqslant d$, where the W_i's are i.i.d. matrices, with independent centered entries having variance profile a doubly stochastic matrix A s.t. $|\lambda_2(A)| \leqslant C/\sqrt{d}$. E.g. Gaussians with mean 0 and variance $1/n$.
2. $K_i = U_i/\sqrt{d}$, $1 \leqslant i \leqslant d$, where the U_i's are i.i.d. Haar unitaries.
3. $K_i = V_i$, $1 \leqslant i \leqslant d$, where $V = \sum_{i=1}^{d} V_i \otimes |e_i\rangle$ is a Haar isometry.

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Theorem [Correlation length of a random translation-invariant MPS (Lancien/Pérez-García)]

Let $\chi \in (\mathbf{C}^d)^{\otimes M}$ be a random translation-invariant MPS, with associated $K_1, \ldots, K_d \in \mathcal{M}_n(\mathbf{C})$ sampled according to one of the models above.
For large n, its correlation length is typically upper bounded by $2/\log d$.

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Question: What about more complicated models, where the random MPS has a local symmetry?
For instance: Let G be a compact group with unitary representation $U:\mathrm{G}\to\mathcal{M}_d(\mathbf{C})$.
Assume that $\chi\in(\mathbf{C}^d)^{\otimes M}$ is s.t. $U_g^{\otimes M}\chi=\chi$ for all $g\in\mathrm{G}$.
This means that $K_1,\dots,K_d\in\mathcal{M}_n(\mathbf{C})$ are s.t. $P_gK_iP_g^*=K_i$ for all $g\in\mathrm{G}$, where $P:\mathrm{G}\to\mathcal{M}_n(\mathbf{C})$ is a projective representation (Schuch/Pérez-García/Cirac).
↳ $P_gP_h=e^{i\omega(g,h)}P_{gh}$
$\longrightarrow$ Given $K_1,\dots,K_d\in\mathcal{M}_n(\mathbf{C})$ random, set $\hat{K}_i:=\Phi_{\mathrm{G}}(K_i)$, where $\Phi_{\mathrm{G}}(X)=\mathbf{E}_{g\in\mathrm{G}}[P_gXP_g^*]$.

One-slide summary of “Implications for random matrix product states”

Matrix product states (MPS) form a subset of *many-body quantum states*.

They are particularly useful because:

- They admit an *efficient description* (number of parameters that scales linearly rather than exponentially with the number of subsytems).
- They are *good approximations of several ‘physically relevant’ states* (such as ground states of gapped local Hamiltonians on 1D systems).

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with the distance separating the sites

between observables measured on distinct sites

Main result: Random MPS typically have correlations that decay exponentially fast, with a *small correlation length*.

Proof strategy: Observe that the correlation length is given by $1/|\log \lambda(\Phi)|$ for Φ a random quantum channel associated to the random MPS (its so-called *transfer operator*).

Some perspectives

- What about *explicit constructions* of optimal quantum expanders? Would be important in practice (quantum cryptography, error correction, condensed matter physics, etc.)

  Previously known constructions required a large amount of randomness. First step towards *derandomization*: taking Kraus operators as sparse Rademacher matrices works as well.

  Other direction to explore: unitary Kraus operators sampled according to a ‘simple’ measure that still ‘resembles’ the uniform one. E.g. an approximate t-design measure or the uniform measure on a subgroup of the unitary group (Bordenave/Collins).

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- What about identifying the *full spectral distribution* of random quantum channels?

  Known: For a random quantum channel $\Phi : \mathcal{M}_n(\mathbf{C}) \to \mathcal{M}_n(\mathbf{C})$ sampled according to various models, the eigenvalues of $\Phi - \Pi_{\rho_*}$ are typically inside a disc of radius $C/\sqrt{d}$ for large n.
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- What about looking at other, related, notions of expansions, such as *geometric* ones (Bannink/Briët/Labib/Maassen) or *linear-algebraic* ones (Li/Qiao/Wigderson/Wigderson/Zhang)?

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