Entanglement in operator-algebraic quantum field theory

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FAU

ICMAT week on Functional Analysis and Quantum Information

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Setting for this talk: "algebraic quantum field theory" (AQFT):

- **maths:** operator-algebraic approach (von Neumann algebras, (normal) states)
- **physics:** relativistic quantum systems describing elementary particles

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Plan/purpose of this talk:

- sketch the setup of AQFT
- entanglement properties of the vacuum state
- quantifying entanglement in QFT
- examples

AQFT I

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3. Local observable algebras: For every open region $\mathcal{O} \subset \mathbb{R}^d$, have a von Neumann algebra

$$\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}).$$

Idea: $A \in \mathcal{A}(\mathcal{O})$ is an observable measurable in $\mathcal{O}$, e.g. in

$$\mathcal{O} = \text{today} \times \text{Madrid}.$$

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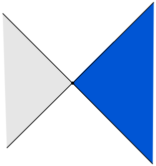

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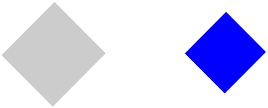

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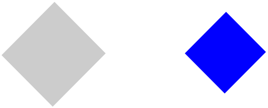

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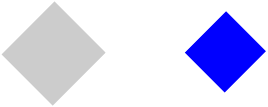

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State of prime interest: vacuum state $\omega = \langle \Omega, \cdot \Omega \rangle$ on bipartite systems of the form $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ with $\mathcal{O}_1$ spacelike to $\mathcal{O}_2$.

Elementary properties of the vacuum state ω

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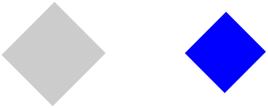

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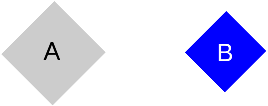

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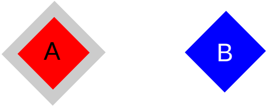

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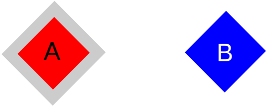

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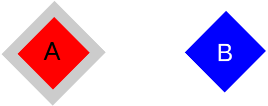

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$$\Rightarrow U(-x) = |\Omega\rangle\langle\Omega|.$$

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Under further physically natural assumptions (that are valid in models) one can show:

For every (causally convex) bounded open region $\mathcal{O}$, the von Neumann algebra $\mathcal{A}(\mathcal{O})$ is isomorphic to the unique hyperfinite type III_1 factor. [Buchholz/D'Antoni/Fredenhagen '87]

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- What about entanglement properties?

Entanglement for touching regions

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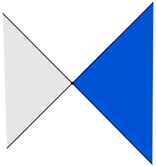

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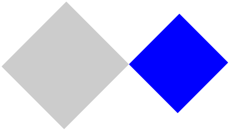

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- Consider two regions $\mathcal{O}_1$ and $\mathcal{O}_2$ that "touch". Then $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ does not have any normal separable state [Hollands/Sanders 18]

Entanglement at finite separation

For separated regions, the **cluster property of the vacuum** enters.

Assume:

- $\mathcal{O}_1, \mathcal{O}_2$ are spacelike separated regions.
- The spectrum of the Hamiltonian satisfies $\sigma(P_0) \subset \{0\} \cup [m, \infty)$, $m > 0$ ("massive theory").

Then for any $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$ [Fredenhagen 85]

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Nonetheless, it is still entangled!

If $\mathcal{N}, \mathcal{M}$ are commuting nonabelian von Neumann algebras on a Hilbert space $\mathcal{H}$ and $\Omega \in \mathcal{H}$ a unit vector cyclic for $\mathcal{N}$, then $\omega = \langle \Omega, \cdot\, \Omega \rangle$ is entangled on $\mathcal{N} \vee \mathcal{M}$. [Halvorson/Clifton 00]

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- This implies $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2) \cong \mathcal{A}(\mathcal{O}_1) \otimes \mathcal{A}(\mathcal{O}_2)$ for spacelike separated (with finite distance) bounded regions $\mathcal{O}_1, \mathcal{O}_2$.

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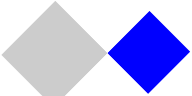

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At finite separation, some of the familiar structure of bipartite systems of QI reappears.

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Araki found a generalisation of relative entropy to arbitrary von Neumann algebras with arbitrary normal states ω, ω',

$$H(\omega, \omega') = \langle \Omega, \log \Delta_{\omega,\omega'} \Omega \rangle.$$

Relative entanglement entropy

Araki's relative entropy has many good properties, including

$$H(\omega,\omega') = 0 \iff \omega = \omega'.$$

Define relative entanglement entropy of ω on $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ as

$$E(\omega,\mathcal{O}_1,\mathcal{O}_2) = \inf\{H(\omega,\sigma) : \sigma \text{ normal and separable}\} \in [0,\infty].$$

[Hollands/Sanders 18].

This is a good entanglement measure that works in QFT. In particular, ω is entangled on $\mathcal{A}(\mathcal{O}_1) \vee \mathcal{A}(\mathcal{O}_2)$ if and only if $E(\omega,\mathcal{O}_1,\mathcal{O}_2) > 0$.

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$E(\omega,\mathcal{O}_1,\mathcal{O}_2)$ is very difficult to compute in concrete situations.

One usually has to estimate it (from above/below).

The modular partition function

An upper bound on the entanglement entropy is given by modular theory.

Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{B}(\mathcal{H})$ be an inclusion of factors with joint cyclic and separating vector Ω on Hilbert space $\mathcal{H}$. Consider the (linear, bnd) map

$$\Xi : \mathcal{N} \to \mathcal{H}, \qquad \Xi(A) := \Delta_{\mathcal{M}}^{1/4} A\Omega,$$

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This is a useful tool for estimates from above because for special regions (wedges), Δ has a simple form.

Examples

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Open questions:

- How do the entanglement properties of the vacuum depend on the model / interaction?
- Do entanglement properties between various regions determine a model? [Casini]
- Can we make contact with (non-rigorous) approaches from theoretical physics literature?

Ongoing work

- There exists a family of QFTs on $\mathbb{R}^2$ parametrized by pairs (U, T),
    - U irreducible positive energy rep of Poincaré group on a one-particle space $\mathcal{H}_1$
    - T a selfadjoint operator on $\mathcal{H}_1 \otimes \mathcal{H}_1$

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- The Ising model is included [calculations with Ian Koot yesterday]:

$$E(\omega, W + x, W') \leq c \, \frac{e^{-mx}}{\sqrt{mx}} \left(1 + \frac{1}{2mx}\right)$$

  already close to predictions of theoretical physics.

Outlook

- Entanglement is ubiquitous in QFT, in particular in the vacuum state across spacelike separated regions.
- Good but abstract entanglement measures exist that work in this setting (type III algebras)
- We still need better **lower** bounds on entanglement entropies.
- Investigation of entanglement entropies in interacting models in progress – does this characterize the interaction / the QFT?