

The Quantum Wasserstein Distance of Order 1

Giacomo De Palma

GdP, Milad Marvian, Dario Trevisan, Seth Lloyd

[IEEE Transactions on Information Theory 67\(10\), 6627-6643 \(2021\)](#)

GdP, Milad Marvian, Cambyse Rouzé, Daniel Stilck França

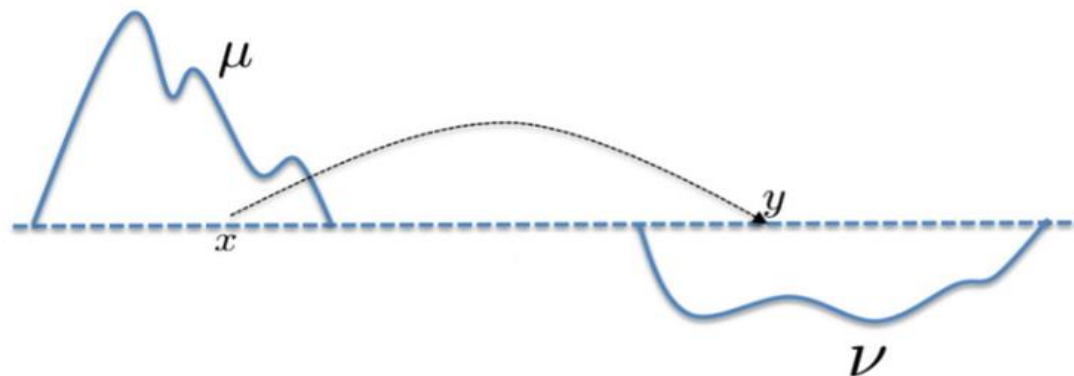
[PRX Quantum 4, 010309 \(2023\)](#)

GdP, Cambyse Rouzé

[Annales Henri Poincaré 23, 3391 \(2022\)](#)

The classical Wasserstein distance

- Probability distributions as distributions of unit amount of mass
- Moving unit mass from x to y has cost $d(x,y)$
- $W_1(p,q)$: minimum cost to transport p onto q
- Recovers d for Dirac deltas
- Induced by a norm
- Countless applications in geometric analysis, probability, information theory, machine learning
- For bit strings, $d =$ Hamming distance



Quantum W_1 distance: Why?

- Hamming distance ubiquitous in classical probability, information theory, machine learning
- Yet no quantum version for qudits!!
- Bit flip small change wrt Hamming distance, but can generate orthogonal state
- Orthogonal states maximally far for any unitarily invariant distance
- Desired properties:
 - Recovery of Hamming distance for canonical basis states
 - Robust wrt few-qudit operations
 - Intensive quantities (e.g., entropy per qudit) continuous

Quantum W_1 norm

- Neighboring states: coincide after discarding one qudit
- Require: neighboring states have distance at most one, *i.e.*, differences between neighboring states belong to unit ball
- Quantum W_1 norm: maximum norm that assigns distance at most one to any couple of neighboring states
- Unit ball: convex hull of differences between neighboring states
- Semidefinite program!

Properties

- Recovers classical W_1 distance for states diagonal in computational basis
- Recovers Hamming distance for canonical basis states
- Extensivity

$$\|\rho_{AB} - \sigma_{AB}\|_{W_1} \geq \|\rho_A - \sigma_A\|_{W_1} + \|\rho_B - \sigma_B\|_{W_1}$$

- Relation with trace distance

$$\frac{1}{2} \|\rho - \sigma\|_1 \leq \|\rho - \sigma\|_{W_1} \leq \frac{n}{2} \|\rho - \sigma\|_1$$

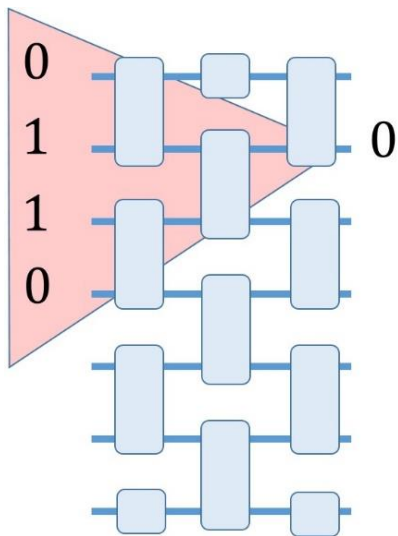
- Robust wrt local operations

$$\text{Tr}_A \rho = \text{Tr}_A \sigma \quad \implies \quad \|\rho - \sigma\|_{W_1} \leq 2|A|$$

Shallow quantum circuits

- Expand W_1 distance by at most twice the size of the largest light-cone of a qudit

$$\|U \rho U^\dagger - U \sigma U^\dagger\|_{W_1} \leq 2B(U) \|\rho - \sigma\|_{W_1}$$



Continuity of the von Neumann entropy

- von Neumann entropy

$$S(\rho) = -\text{Tr} [\rho \log \rho]$$

- Quantifies uncertainty
- Continuity bound wrt trace distance void for orthogonal states, but flipping one qudit changes entropy by at most $2 \log d$
- Continuity bound wrt quantum W_1 distance

$$\frac{1}{n} |S(\rho) - S(\sigma)| \leq h_2 \left(\frac{1}{n} \|\rho - \sigma\|_{W_1} \right) + \frac{1}{n} \|\rho - \sigma\|_{W_1} \log (d^2 - 1)$$

$$h_2(p) = -p \log p - (1 - p) \log (1 - p) \quad [\text{GdP, Trevisan, } \text{arXiv:2210.11446}]$$

- Only intensive quantities!

Transportation-Cost Inequality (TCI)

- Quantum relative entropy

$$S(\rho\|\omega) = \text{Tr} [\rho (\ln \rho - \ln \omega)]$$

- Pinsker's inequality

$$\frac{1}{2} \|\rho - \omega\|_1 \leq \sqrt{\frac{1}{2} S(\rho\|\omega)}$$

- Quantum TCI: for any product state ω

$$\frac{1}{n} \|\rho - \omega\|_{W_1} \leq \sqrt{\frac{1}{2n} S(\rho\|\omega)}$$

The quantum Lipschitz constant

- Lipschitz constant $\|f\|_L = \max_{x, y} \frac{|f(x) - f(y)|}{d(x, y)}$

- Quantum generalization

$$\|H\|_L = 2 \max_{i \in [n]} \min_{H_{i^c}} \|H - \mathbb{I}_i \otimes H_{i^c}\|_\infty$$

- Recovers classical Lipschitz constant for observables diagonal in computational basis
- Provides dual SDP for quantum W_1 distance

$$\|\rho - \sigma\|_{W_1} = \max_{\|H\|_L \leq 1} \text{Tr}[(\rho - \sigma)H]$$

Gaussian concentration for maximally mixed state

- In high dimension, smooth functions are essentially constant
- Upper bound on partition function

$$\frac{1}{n} \ln \operatorname{Tr} e^H \leq \ln d + \frac{1}{8} \|H\|_L^2 \quad \operatorname{Tr} H = 0$$

- Spectrum of H lies in interval with size $O(\sqrt{n} \|H\|_L)$

$$\frac{1}{d^n} \dim(H \geq n\delta) \leq e^{-\frac{2n\delta^2}{\|H\|_L^2}}$$

Quadratic concentration for product states

- ω product state

$$\text{Var}_\omega H \leq n \|H\|_L^2$$

- ρ output of quantum circuit with blow-up B

$$\text{Var}_\rho H \leq 4n B^2 \|H\|_L^2$$

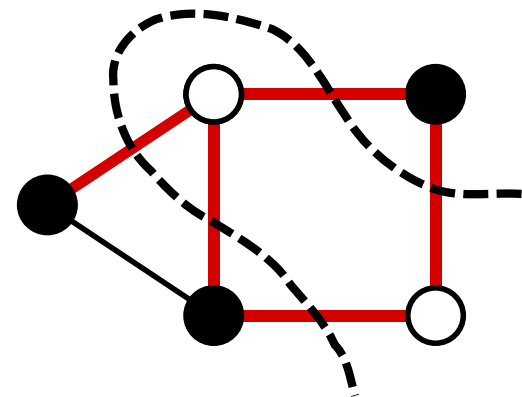
- See [Anshu, Metger, [arXiv:2209.02715](https://arxiv.org/abs/2209.02715)] for Gaussian concentration of observables diagonal in computational basis

Combinatorial optimization

- Goal: find bit string that maximizes cost function C
- Local cost: sum of functions each depending on $O(1)$ bits
- Efficient classical algorithms usually achieve

$$C = a C_{\max} \quad 0 < a \leq 1$$

- **Example:** maximum cut problem, i.e., find the bipartition of a graph that maximizes the # of edges connecting the two parts
- Associate one bit to each vertex, set to 1 bits in second half of bipartition
- NP complete!



Variational quantum algorithms

- Associate one qubit to each bit, quantum Hamiltonian to cost function

$$H = \sum_{x \in \{0,1\}^n} C(x) |x\rangle\langle x|$$

- Train parametric quantum circuit to generate high-energy states
- **Example:** Quantum Approximate Optimization Algorithm (QAOA)
- Alternate time evolution with H and mixing Hamiltonian

$$\left(\prod_{k=1}^P e^{-i\gamma_k \sum_{i=1}^n X_i} e^{-i\beta_k H} \right) |+\rangle^{\otimes n}$$

Limitations of QAOA for MaxCut

- Toy model: D -regular bipartite graph (maxcut = $n D / 2$)
- Technical assumption:

$$C(x) \geq \left(\frac{D}{2} - \sqrt{D-1} \right) \min \{ |x|, n - |x| \} \quad \forall x \in \{0, 1\}^n$$

- Satisfied by Ramanujan expander graphs with $D \geq 3$ and for large n by random D -regular graphs with high probability
- Observation [Bravyi *et al.*, [PRL 125, 260505 \(2020\)](#)]: QAOA circuit commutes with $X^{\otimes n}$
- Probability distribution of output measurement symmetric wrt flipping all bits and cannot be concentrated on single string

Limitations of QAOA for MaxCut

- Result: if

$$\text{Tr} [\rho H] \geq C_{\max} \left(\frac{5}{6} + \frac{\sqrt{D-1}}{3D} \right)$$

then the quadratic concentration inequality implies

$$P \geq \frac{1}{2 \log(D+1)} \log \frac{n}{576} = \Omega(\log n)$$

- Holds for any circuit and initial state commuting with $X^{\otimes n}$
- Improves Bravyi *et al.*

$$P \geq \frac{1}{3(D+1)} \log_2 \frac{n}{4096}$$

Limitations of noisy quantum circuits

- Goal: generate high-energy state of traceless local Hamiltonian H
- Model: L layers of 2-qubit gates with depolarizing noise on each qubit after each layer
- Result: output energy exponentially concentrated about 0

$$\mathbb{P}(|H| \geq n \delta) \leq \exp \left(-\frac{n}{2} \left(\frac{\delta^2}{\|H\|_L^2} - (1-p)^{2L} \right) \right)$$

- Quantum advantage exponentially unlikely for

$$L > \left\lceil \frac{\log \frac{\delta}{\|H\|_L}}{\log(1-p)} \right\rceil = O \left(\frac{1}{p} \right)$$

Proof idea

- 2-Rényi divergence wrt maximally mixed state ω decreases exponentially with L

$$D_2(\rho\|\omega) = \log \text{Tr} \left(\omega^{-\frac{1}{4}} \rho \omega^{-\frac{1}{4}} \right)^2 \leq n (1 - p)^{2L}$$

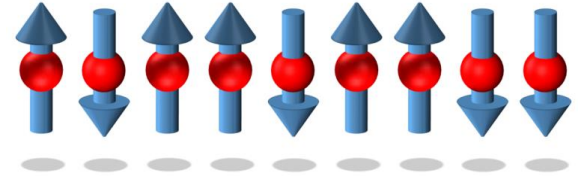
- Gaussian concentration for ω implies Gaussian concentration for ρ

$$\mathbb{P}_\rho (|H| \geq n \delta) \leq \sqrt{2} \exp \left(\frac{1}{2} D_2(\rho\|\omega) - \frac{n \delta^2}{\|H\|_L^2} \right)$$

- Improves over [Stilck França, García-Patrón, [Nat Phys 17, 1221 \(2021\)](#)], which considers expectation value

1D Markovian states

- Markovian state ω of 1D qudit chain



$$I(X_1 \dots X_{i-1}; X_{i+1} \dots X_n | X_i) = 0$$

- Quantum conditional mutual information

$$I(X; Y | Z) = S(X|Z) + S(Y|Z) - S(XY|Z) \geq 0$$

$$S(X|Y) = S(XY) - S(Y)$$

- Recovery map Φ_i^ω : quantum counterpart of stochastic map associated to $\omega(X_{i+1}|X_i)$

TCl for 1D Markovian states

- Quantum Dobrushin uniqueness condition: each Φ_i^ω contracts trace distance between any quantum states ρ, σ with $\text{Tr}_i \rho = \text{Tr}_i \sigma$

$$\|\Phi_i^\omega(\rho) - \Phi_i^\omega(\sigma)\|_1 \leq \eta_\omega \|\rho - \sigma\|_1 \quad 0 \leq \eta_\omega < 1$$

- For any quantum state ρ of the chain

$$\frac{1}{n} \|\rho - \omega\|_{W_1} \leq \frac{2 - \eta_\omega}{1 - \eta_\omega} \sqrt{\frac{2}{n} S(\rho||\omega)}$$

Gibbs states of commuting local Hamiltonians

- Qudits arranged in hypergraph with edge set \mathcal{E}
- Two qudits interact iff connected by an edge
- Mutually commuting Hamiltonian terms $H_e \otimes \mathbb{I}_{e^c}$
- Commuting local Hamiltonian

$$H = \sum_{e \in \mathcal{E}} H_e \otimes \mathbb{I}_{e^c}$$

- Gibbs state $\omega = \frac{e^{-\beta H}}{\text{Tr} e^{-\beta H}}$

TCl from coarse Ricci curvature

- Ψ_ω : quantum counterpart of stochastic map that selects a random site i and resets it according to $\omega(X_i|X_i^c)$
- Coarse Ricci curvature κ_ω : contraction coefficient of Ψ_ω wrt W_1 distance

$$\|\Psi_\omega(\rho) - \Psi_\omega(\sigma)\|_{W_1} \leq \left(1 - \frac{\kappa_\omega}{n}\right) \|\rho - \sigma\|_{W_1}$$

- Assume that each qudit interacts with at most q qudits. Then, for any quantum state ρ

$$\frac{1}{n} \|\rho - \omega\|_{W_1} \leq \frac{q}{1 - e^{-\kappa_\omega}} \sqrt{\frac{2}{n} S(\rho||\omega)}$$

Equivalence of ensembles

- Canonical ensemble: Gibbs states
- Microcanonical ensemble: uniform convex combination of all states in energy shell
- Assume that Gibbs state satisfies TCI

$$\frac{1}{n} \|\rho - \omega\|_{W_1} \leq \sqrt{\frac{c}{n} S(\rho||\omega)}$$

- Then, any ρ with same average energy as ω and approximately same entropy as ω is close to ω

$$\text{Tr} [\rho H] = \text{Tr} [\omega H] \quad \Longrightarrow \quad \frac{1}{n} \|\rho - \omega\|_{W_1} \leq \sqrt{\frac{c}{n} (S(\omega) - S(\rho))}$$

- Ok if fraction of states in shell is

$$e^{-o(n)}$$

Perspectives

- Quantum Wasserstein Generative Adversarial Networks
[Kiani, GdP, Marvian, Liu, Lloyd, [Quantum Sci Technol 7, 045002 \(2022\)](#)]
- Design of quantum error correcting codes
[Zoratti, GdP, Giovannetti, [arXiv:2301.05273](#)]
- Quantum spin systems on infinite lattices
[GdP, Trevisan, [arXiv:2210.11446](#)]
- Efficient learning of ground & thermal states
[Rouzé, França, [arXiv:2107.03333](#)]
[Onorati, Rouzé, França, Watson, [arXiv:2301.12946](#)]
- Quantum rate-distortion theory