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Two dimerized gapped ground state phases of $O(n)$ spin chains?

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Main questions regarding gapped ground state phases

1. Existence of a gap for specific Hamiltonians.
2. Stability of the gap under perturbations (existence of a 'phase').
3. Classification of equivalence classes of gapped phases, for example, those defined by gapped curves of Hamiltonians (Chen-Gu-Wen 2011).

Outline

1. AKLT chain
2. Ground state phase diagram
3. $O(n)$ spin chains
4. Answer the question of the title

Spin chains, Hamiltonians, ground states

Finite spin chain on $[a, b] \subset \mathbb{Z}$, Hilbert space $\mathcal{H}_{[a,b]} = \bigotimes_{x=a}^{b} \mathbb{C}^n$, $n \geq 2$, spins or qdits of dimension $n = 2J + 1$.

Translation-invariant nearest neighbor interaction is given by $h = h^* \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) = \mathcal{B}(\mathcal{H}_{[x,x+1]})$.

Hamiltonian: $H_{[a,b]} = \sum_{x=a}^{b-1} h_{x,x+1}$. Interested in ground states.

Heisenberg model: $h_{x,x+1} = \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} = S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + S_x^3 S_{x+1}^3$, where S_x^i, $i = 1, 2, 3$, $x \in [a, b]$, are n-dimensional spin matrices.

AKLT model, $n = 3$:
$h_{x,x+1} = \frac{1}{2} \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \frac{1}{6} (\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2 + \frac{1}{3} \mathbb{1} = P^{(2)}_{x,x+1}$.

Most general isotropic nearest neighbor interaction for $n = 3$:
$h_{x,x+1} = \cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi (\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

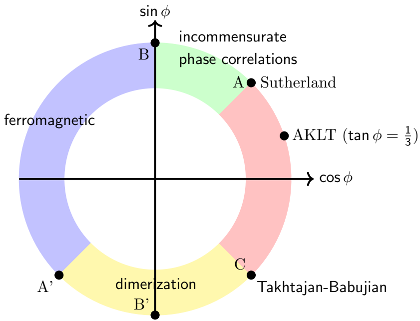
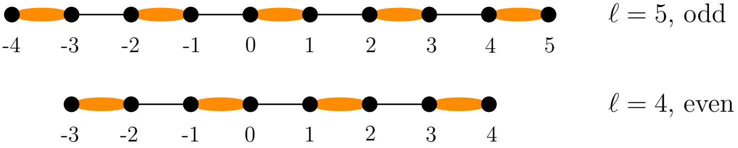


Figure: Ground state phase diagram for the $S = 1$ chain ($n = 3$) with nearest-neighbor interactions $\cos\phi \boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1} + \sin\phi (\boldsymbol{S}_x \cdot \boldsymbol{S}_{x+1})^2$.

- $\phi = 0$: Heisenberg antiferromagnet, Haldane phase (Haldane, 1983)
- $\tan\phi = 1/3$, AKLT point (Affleck-Kennedy-Lieb-Tasaki, 1987,1988), FF, MPS, gapped
- $\tan\phi = 1$, solvable, gapless, SU(3) invariant, (Sutherland, 1975)
- $\phi \in [\pi/2, 3\pi/2]$, ferromagnetic, FF, gapless
- $\phi = -\pi/2$, solvable, SU(3) invariant, Temperley-Lieb algebra, dimerized, gapped (Klümper; Affleck,1990)
- $\phi - = -\pi/4$ gapless, Bethe-ansatz, (Takhtajan; Babujian, 1982)

Dimerization

If a pair interaction favors a maximally entangled state (such as a spin singlet), monogamy of entanglement sets up a competition between pairings. In one dimension, this often leads to an instability and/or to spontaneous breaking of translation symmetry. In the family of $O(n)$ chains here, translation symmetry breaking occurs, called dimerization. For finite chains of 2ℓ spins the ground states can be viewed as chain of dimers:



The actual ground states need not consist of maximally entangled pairs. For the $O(n)$ chains maximally entangled pairs dominate for large n.

The AKLT chain

The AKLT chain (Affleck-Kennedy-Lieb-Tasaki 1987-88) is the spin-1 chain with nearest neighbor interaction given by the projection onto the spin-2 states:

$$H_{[a,b]} = \sum_{x\in[a,b]} P^{(2)}_{x,x+1}, \quad P^{(2)}_{x,x+1} = \frac{1}{3}\mathbb{1} + \frac{1}{2}\mathbf{S}_x \cdot \mathbf{S}_{x+1} + \frac{1}{6}(\mathbf{S}_x \cdot \mathbf{S}_{x+1})^2.$$

$[a,b] \in \mathbb{Z}$, $H_{[a,b]}$ is the Hamiltonian, acts on $\bigotimes_{x\in[a,b]} \mathbb{C}^3$, self-adjoint.

Ground state space is 4-dimensional and given by $\ker H_{[a,b]}$, for all $b > a \in \mathbb{Z}$. AKLT proved that the infinite chain has a unqiue ground state with a spectral gap and exponential decay of correlations (Haldane's Conjecture).

- $\lim_n \langle \psi_n, A\psi_n \rangle = \omega(A)$, independent of the sequence of unit vectors $\psi_n \in \ker H_{[a_n,b_n]}$, $a_n \to -\infty$, $b_n \to \infty$.
- There exists $\gamma > 0$ such that $\mathrm{spec} \ker H_{[a,b]} \subset \{0\} \cup [\gamma, \infty)$, for all $b > a \in \mathbb{Z}$.
- $|\omega(A_x B_y)| \leq 4\|A_x\|\|B_y\| \frac{1}{3}^{|x-y|}$.

The exact ground state is a Matrix Product State (MPS) (Fannes-N-Werner 1989-1992).

AKLT settled Q1 (existence of the gap).

Q2 (stability) was first addressed by Yarotsky (2004), who proved that translation-invariant, finite-range perturbations of the AKLT chain do not close the gap for sufficiently small coupling constants.

$$H(s) = \sum_x P^{(2)}_{x,x+1} + s \sum_{X \subset \mathbb{Z}} \Phi(X).$$

$\Phi(X) = \Phi(X)^*$ acts non-trivially only on spins at $x \in X \subset \mathbb{Z}$. Finite range R: $\Phi(X) = 0$ if $\mathrm{diam}\, X > R$.

Other proofs and generalizations of stability for the AKLT chain by Michalakis-Zwolak 2013, Szehr-Wolf 2015, Moon-N 2018, Sims-N-Young 2021,

and for other models by Bravyi-Hastings-Michalakis 2010-11, Sims-N-Young 2018, De Roeck-Salmhofer 2019, Hastings 2019, Fröhlich-Pizzo 2018-2020, Del-Vecchio-Fröhlich-Pizzo-Rossi 2020-2022.

Q3 (classification of phases)

One can construct a C^1-curve of projections $P(s)$ such that $P(1) = P^{(2)}$ and the model with nn interaction $P(0)$ has a unique product ground state (for the infinite chain) and prove a uniform positive lower bound for the gap for $s \in [0, 1]$ (Bachmann-N 2014).

This implies that the AKLT chain belongs to the same phase as the model with a unique product ground state (the trivial phase).

In contrast, if we one restricts to interpolations $P(s)$ that respect spin rotation symmetry about 1 axis and an additional $\mathbb{Z}_2$ symmetry, an index argument shows that any curve connecting the AKLT model with a model in the trivial phase, must pass through a phase transition where the gap closes (Tasaki 2018, Ogata 2019-20).

This implies that the AKLT chain belongs to a SPT phase distinct from the trivial phase.

Gapped ground state phases

Gapped phase
Def: two interactions, Φ_0 and Φ_1, with a (unique) gapped ground state belong to the same gapped phase if there exists a (piecewise) differentiable interpolation $[0,1] \ni s \mapsto \Phi_s$, that is uniformly gapped (Chen-Gu-Wen 2011).

Symmetry Protected / Enhanced gapped phase
Def: Given a symmetry G, defined as 'gapped phase' above, but with G-symmetric Φ_s, for all $s \in [0,1]$ (Pollman-Turner-Berg-Oshikawa, 2010).

There are other definitions in the literature which, under suitable conditions, are equivalent to those above.
See *From Lieb-Robinson bounds to automorphic equivalence*, in Rupert L. Frank, Ari Laptev, Mathieu Lewin, and Robert Seiringer (eds), The Physics and Mathematics of Elliott Lieb, vol. 2, pp. 79–92, European Mathematical Society Press, 2022, arXiv:2205.10460.

$O(n)$ chains and generalizations of the AKLT model

There is a local unitary change of basis in which the AKLT interaction is given by

$$P^{(2)} = \frac{1}{2}(T - 2Q + \mathbb{1}),$$

where T is the swap operator and Q is the projection onto $\frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_0 \otimes e_0 + e_{-1} \otimes e_{-1})$.

This generalizes to n-dimensional spins and arbitrary coupling constants as follows

$$uT + vQ, \quad u, v \in \mathbb{R}$$

where Q is the projection to

$$\psi = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle.$$

Both T and Q commute with the natural action of $O(n)$ on the spins in this basis. It is the general $O(n)$ invariant nearest neighbor interaction for $n \geq 2$, which was studied by Tu & Zhang, 2008.

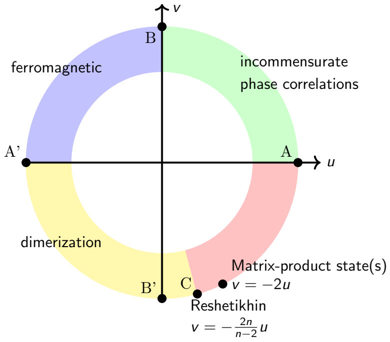


Figure: Ground state phase diagram for the chain with nearest-neighbor interactions $uT + vQ$ for $n \geq 3$.

- $v = -2nu/(n-2)$, $n \geq 3$, Bethe ansatz point (Reshetikhin, 1983)
- $v = -2u$: frustration free point, equivalent to $\perp$ projection onto symmetric vectors $\ominus$ one. Unique g.s. if n odd; two 2-periodic g.s. for even n; spectral gap in all cases and stable phase (N-Sims-Young, 2022).
- $u = 0, v = -1$. Equivalent to the $SU(n) - P^{(0)}$ models aka Temperley-Lieb chain; Affleck, 1990, Nepomechie-Pimenta 2016). Dimerized for all $n \geq 3$ (Aizenman, Duminil-Copin, Warzel, 2020). Proof of some stability for large n (Björnberg-Mühlbacher-N-Ueltschi, 2021).

Two distinct gapped phases for all $n \geq 3$

- MPS/FF point
  - odd n: unique gapped ground state
  - even n: two 2-periodic gapped ground states
- South Pole: two dimerized gapped ground states for all $n \geq 3$.

All these MPS/FF gapped phase are fully stable (N-Sims-Young 2022). The dimerized phase (South Pole) is also expected to be fully stable under translation-invariant short-range perturbations, but only specific stability has been proved (Björnberg-Mühlbacher-N-Ueltschi 2021).

New results (N-Ragone, in prep)

- the MPS/FF point and the South Pole always belong to distinct phases.
- the two ground states for even n at the MPS/FF point have identical entanglement properties

Phase structure of the MPS states (N-Ragone, in prep)

An equivalent parent Hamiltonian is given by

$$h = \frac{1}{2}(T + \mathbb{1}) - Q,$$

which is the projection on the symmetric states in ker Q.

Case of odd n: up to a local unitary basis transformation this is the following $SU(2)$-invariant interaction for a spin-J chain with $2J + 1 = n$:

$$h = P^{(2)} + P^{(4)} + \cdots + P^{(2J)}.$$

These spin chains have a unique gapped ground state.
The case $n = 3$ is the well-known Haldane SPT phase with string order (den Nijs-Rommelse 1989), a hidden $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry (Kennedy-Tasaki 1992) and characterized by a non-trivial index Pollmann-Berg-Turner-Oshikawa 2012, Tasaki 2018, Ogata 2020.

Tu-Zhang 2008 found this model has a unique matrix product state ground state with several string order parameters and a hidden $(\mathbb{Z}_2 \times \mathbb{Z}_2)^J$ symmetry.

Clifford MPS (Tu-Zhang 2008, Fannes-N-Werner, 2010 unpub notes)

Case of odd n:

The unique MPS ground states is given by

$$\psi^{(\ell)}(B) = \sum_{i_1,\ldots,i_\ell=1}^{n} (\mathrm{Tr} B\gamma_{i_\ell}\cdots\gamma_{i_1})\,|i_1\ldots i_\ell\rangle.$$

where the γ_i are an irrep of the Clifford algebra:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

Up to unitary equivalence, there is a unique irrep given by n traceless Hermitian $2^{(n-1)/2} \times 2^{(n-1)/2}$ matrices.

Case of even n:

The MPS ground states are also given by

$$\psi^{(\ell)}(B) = \sum_{i_1,\ldots,i_\ell=1}^{n} (\mathrm{Tr} B\gamma_{i_\ell}\cdots\gamma_{i_1})\,|i_1\ldots i_\ell\rangle.$$

where the γ_i are an irrep of the Clifford algebra:

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}.$$

Again, there is a unique irrep, now given by $2^{n/2} \times 2^{n/2}$ matrices.

In contrast to the odd n case, the transfer matrix

$$\mathbb{E}(B) := \sum_{i=1}^{n} \gamma_i B \gamma_i$$

has an eigenvalue -1, and two projections $P_\pm \in M_{2^{n/2}}$ such that $P_+ + P_- = \mathbb{1}$, and

$$\mathbb{E}(P_\pm) = P_\mp.$$

This implies the existence of two 2-periodic gapped ground states.

Entanglement structures

For the infinite chain we have two pure ground states ω_+ and ω_-, in FCS form given by

$$\omega_\pm(A_1 \otimes \cdots \otimes A_\ell) = \frac{2}{n} \mathrm{Tr} \mathbb{E}_{A_1} \circ \mathbb{E}_{A_2} \circ \mathbb{E}_{A_3} \circ \cdots \circ \mathbb{E}_{A_{\ell-1}} \circ \mathbb{E}_{A_\ell}(P_\pm).$$

where, for $A \in M_n$, $\mathbb{E}_A(B) = \sum_{ij} A_{ij} \gamma_i B \gamma_j$. ω_+ and ω_- are selected by different b.c.. Define automorphisms α on $M_{2^{n/2}}$, and σ on M_n by

$$\alpha(B) = \gamma_1 B \gamma_1, \quad \sigma(A) = RAR,$$

with

$$R = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Then

$$\alpha(\mathbb{E}_A(B)) = \mathbb{E}_{\sigma(A)}(\alpha(B)), \quad \alpha(P_+) = P_-.$$

One finds two MPS states with primitive transfer matrices by defining $\mathbb{F}_A^{(i)} : P_+ M_{2^{n/2}} P_+ \to P_+ M_{2^{n/2}} P_+$, $i = 1, 2$, as follows:

$$\mathbb{F}_A^{(1)} = \alpha \circ \mathbb{E}_{\sigma(A)}, \quad \mathbb{F}_A^{(2)} = \alpha \circ \mathbb{E}_A.$$

In terms of these, we have

$$\begin{aligned} \omega_+(A_1 \otimes \cdots \otimes A_\ell) &= \tfrac{2}{n} \mathrm{Tr} \mathbb{F}_{A_1}^{(2)} \circ \mathbb{F}_{A_2}^{(1)} \circ \mathbb{F}_{A_3}^{(2)} \circ \cdots \circ \mathbb{F}_{A_{\ell-1}}^{(2)} \circ \mathbb{F}_{A_\ell}^{(1)}(P_+) \\ \omega_-(A_1 \otimes \cdots \otimes A_\ell) &= \tfrac{2}{n} \mathrm{Tr} \mathbb{F}_{A_1}^{(1)} \circ \mathbb{F}_{A_2}^{(2)} \circ \mathbb{F}_{A_3}^{(1)} \circ \cdots \circ \mathbb{F}_{A_{\ell-1}}^{(1)} \circ \mathbb{F}_{A_\ell}^{(2)}(P_+). \end{aligned}$$

This shows ω_+ and ω_- are translates of each other. But also

$$\omega_+(A_1 \otimes \cdots \otimes A_\ell) = \omega_-(\sigma(A_1) \otimes \cdots \otimes \sigma(A_\ell)).$$

Hence, ω_+ and ω_- are related by a local unitary transformation and have the same entanglement.

The two-fold degeneracy of the ground state turns out to be breaking of this local symmetry of the Hamiltonian, $R \in O(n)$. Since the $O(n)$ symmetry is fully preserved in the dimerized ground state of the South Pole model, this suffices to show that the two 2-periodic phases are distinct (N-Sims-Young 2022).

Properties of the South Pole phase

Chain of n-dimensional spins with $O(n)$-invariant nearest neighbor interaction $h = uT + vQ$, $u, v \in \mathbb{R}$, T is the swap operator and Q projects onto $\psi = n^{-1/2} \sum_{\alpha=1}^{n} |\alpha, \alpha\rangle$. South Pole means $u = 0, v = -1$.

Finite chains of 2ℓ spins, with Hamiltonian: $H_\ell = \sum_{x=-\ell+1}^{\ell-1} h_{x,x+1}$.
Consider ground states as limits of Gibbs states:

$$\langle A \rangle_{\ell,\beta,u} = \frac{\mathrm{Tr} A e^{-\beta H_\ell}}{\mathrm{Tr} e^{-\beta H_\ell}}.$$

Basic observables: generators of $O(n)$:

$$L^{\alpha,\alpha'} = |\alpha\rangle\langle\alpha'| - |\alpha'\rangle\langle\alpha|, 1 \le \alpha < \alpha' \le n.$$

Theorem (Dimerization, Björnberg-Mühlbacher-N-Ueltschi 2021)

There exist constants $n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$, $v = -1$, and $|u| < u_0$, we have that for all $1 \le \alpha < \alpha' \le n$,

$$\lim_{\beta\to\infty} \left[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \right] \quad > \quad c \quad \text{for } \ell \text{ odd;}$$

$$\lim_{\beta\to\infty} \left[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \right] \quad < \quad -c \quad \text{for } \ell \text{ even.}$$

Theorem (Exponential decay of correlations, Björnberg-Mühlbacher-N-Ueltschi 2021)

There exist constants $n_0, u_0, c_1, c_2, C > 0$ (independent of ℓ) such that for $n > n_0$, $v = -1$, and $|u| < u_0$, we have

$$\lim_{\beta\to\infty} \left| \langle L_x^{\alpha,\alpha'} e^{tH_\ell} L_y^{\alpha,\alpha'} e^{-tH_\ell} \rangle_{\ell,\beta,u} \right| \leq C e^{-c_1|x-y|-c_2|t|}$$

for all $\ell \in \mathbb{N}$, all $x, y \in \{-\ell+1, \ldots, \ell\}$, all $1 \leq \alpha < \alpha' \leq n$, and all $t \in \mathbb{R}$.

In fact, the decay of correlations between any two local observables is bounded by an exponential with a fixed rate.

Let $E_0^{(\ell)} < E_1^{(\ell)} < \dots$ be the eigenvalues of $H_{[-\ell+1,\ell]}$, and define the ground state gap $\Delta^{(\ell)}$ by

$$\Delta^{(\ell)} = E_1^{(\ell)} - E_0^{(\ell)}.$$

The gap is obviously positive but is there is a positive lower bound independent of ℓ?

Theorem (Spectral gap, Björnberg-Mühlbacher-N-Ueltschi 2021)

There exist constants $n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$, $v = -1$, and $|u| < u_0$, we have

(a) $E_0^{(\ell)}$ *is non-degenerate.*

(b) $\Delta^{(\ell)} \geq c$ *for all* ℓ.

These results are proved using a ‘random’ loop representation of the partition function and the Gibbs states (Toth 1993, Aizenman-N 1994, Ueltschi 2013).

Some corollaries of the previous results and the random loop representation:
Consider the case $(u, v) = (0, -1)$, and intervals of the form $[-\ell + 1, \ell]$ (2ℓ spins), and denote the Hamiltonian by H_ℓ, and let ψ_ℓ be a normalized eigenvector of its smallest eigenvalue, which turn out to be simple. Then

$$|\psi_\ell\rangle\langle\psi_\ell| = \lim_{\beta\to\infty} \frac{e^{-2\beta H_\ell}}{\mathrm{Tr} e^{-2\beta H_\ell}},$$

and therefore, with $A = L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'}$, or $Q_{x,x+1}$, or any other observable,

$$\langle\psi_\ell, A\psi_\ell\rangle = \mathrm{Tr}[|\psi_\ell\rangle\langle\psi_\ell| A] = \lim_{\beta\to\infty} \frac{\mathrm{Tr} e^{-\beta H_\ell} A e^{-\beta H_\ell}}{\mathrm{Tr} e^{-2\beta H_\ell}}.$$

One can define weak limits over even and odd (sub-)sequences of ℓ:

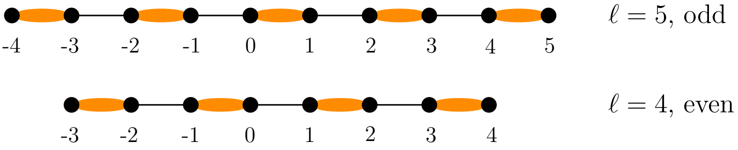
$$\omega_e(A) = \lim_{\ell,\mathrm{even}} \langle\psi_\ell, A\psi_\ell\rangle, \quad \omega_o(A) = \lim_{\ell,\mathrm{odd}} \langle\psi_\ell, A\psi_\ell\rangle.$$

ω_o and ω_e are the pure ground states with dimerization. For example:

$$\omega_o(Q_{0,1}) > \omega_e(Q_{0,1})$$

and for any observable A:

$$\omega_o(A) = \omega_e(\tau(A)) = \omega_o(\tau^2(A)), \tau \text{ is translation by one site.}$$



Both states are $O(n)$ invariant.

Conclusion

At and near the South Pole we have:

- ω_e and ω_o have full $O(n)$ invariance.
- ω_e and ω_o are 2-period, distinct, and translates of each other.
- ω_e and ω_o are dimerized: 2-periodic nearest neighbor entanglement.

In contrast, at and near the MPS/FF point, for even n, we have a 2-periodic phase with

- In ω_+ and ω_- the $O(n)$ symmetry is broken down to $SO(n)$.
- ω_e and ω_o are 2-periodic, distinct, and are translates of each other.
- ω_e and ω_o have translation invariant nearest neighbor entanglement; not dimerized.

Of the two 2-periodic gapped phases only the South Pole phase is dimerized. Of course, but there may well be other dimerized phases...