Classification of topologically ordered phases of matter in 2D

Pieter Naaijkens

Cardiff University

Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

Gapped quantum phases

$$H \geq 0, \quad H\Omega = 0, \quad \mathrm{spec}(H) \cap (0, \gamma) = \emptyset$$

Two states in the same phase if they are connected by a continuous path of gapped Hamiltonians

- What are interesting phases?
- Can we find invariants?

Topological order

Wen, *Int. J. Mod. Phys. B.* **4**,1990

Topological order

Quantum phase outside of Landau theory

Wen, *Int. J. Mod. Phys. B.* **4**,1990

Topological order

Quantum phase outside of Landau theory

> No good definition known

Wen, *Int. J. Mod. Phys. B.* **4**,1990

Topological order

Quantum phase outside of Landau theory

- No good definition known
- ground space degeneracy

Wen, *Int. J. Mod. Phys. B.* **4**,1990

Topological order

Quantum phase outside of Landau theory

- No good definition known
- ground space degeneracy
- long range entanglement

Wen, *Int. J. Mod. Phys. B.* **4**,1990

Topological order

Quantum phase outside of Landau theory

- No good definition known
- ground space degeneracy
- long range entanglement
- anyonic excitations

Wen, *Int. J. Mod. Phys. B.* **4**,1990

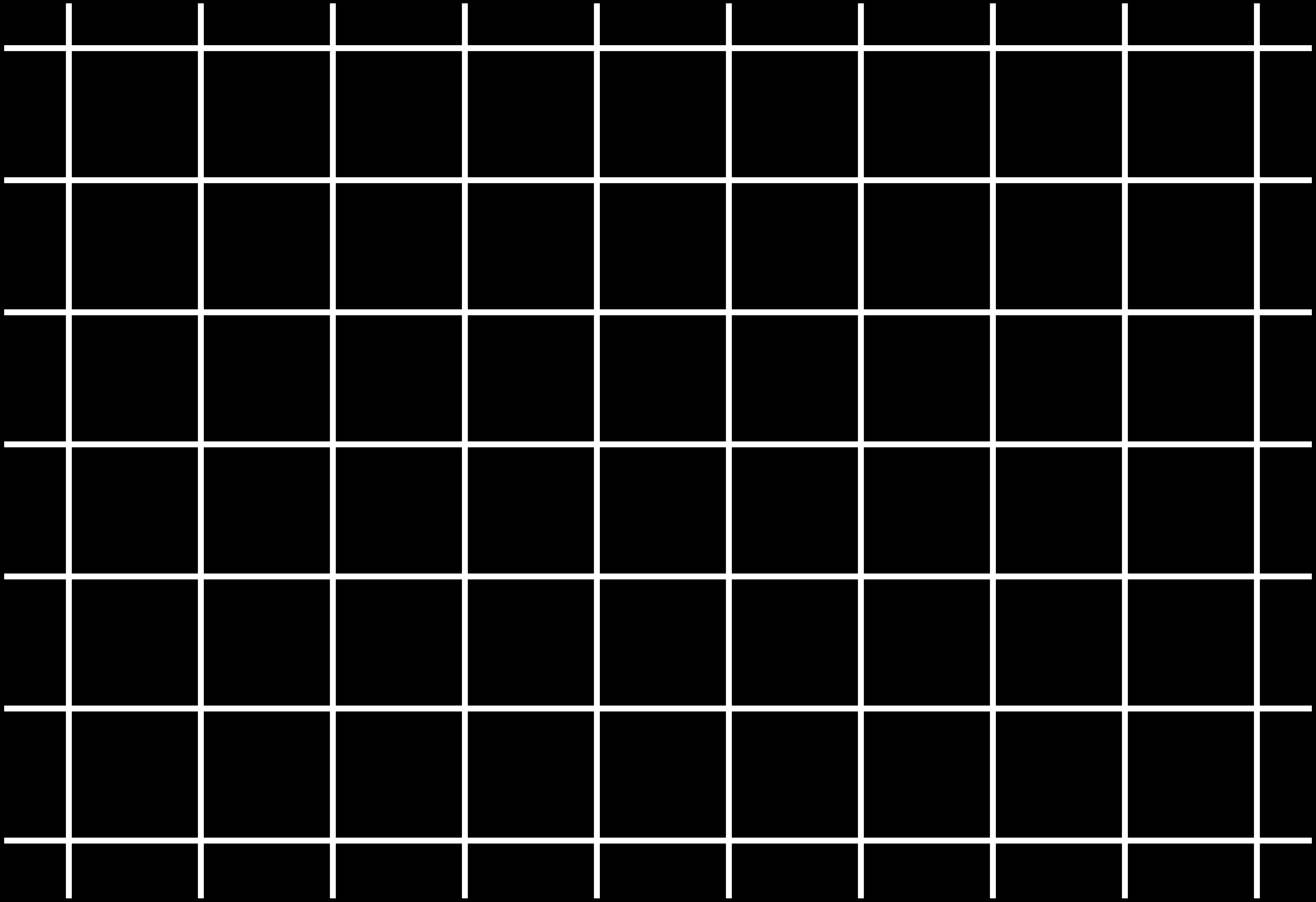
Folklore

The anyonic quasi-particle excitations of a topologically ordered state are described by a modular tensor category.

Kitaev, *Ann. Physics* **321**,2006

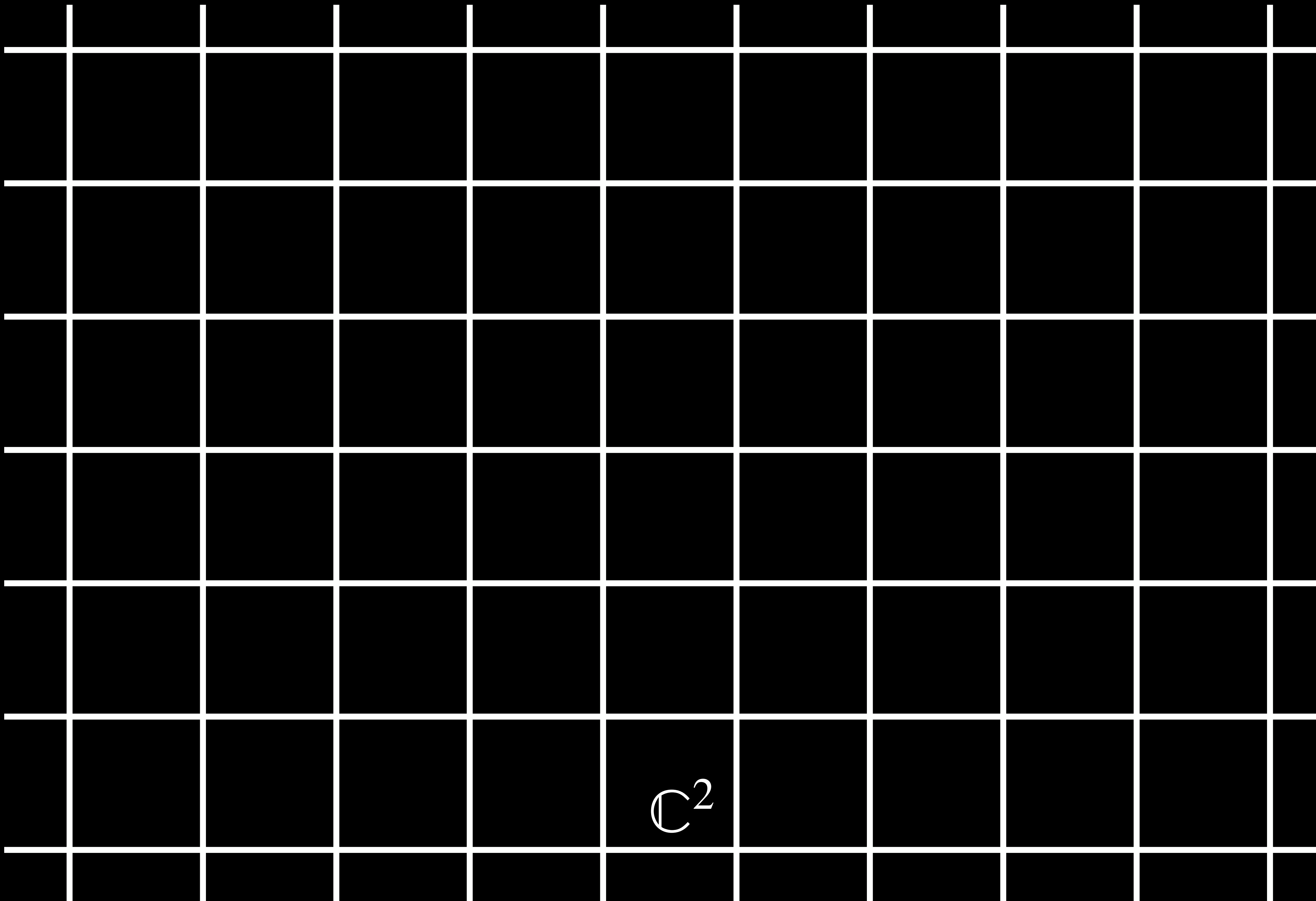
The toric code

Toric code



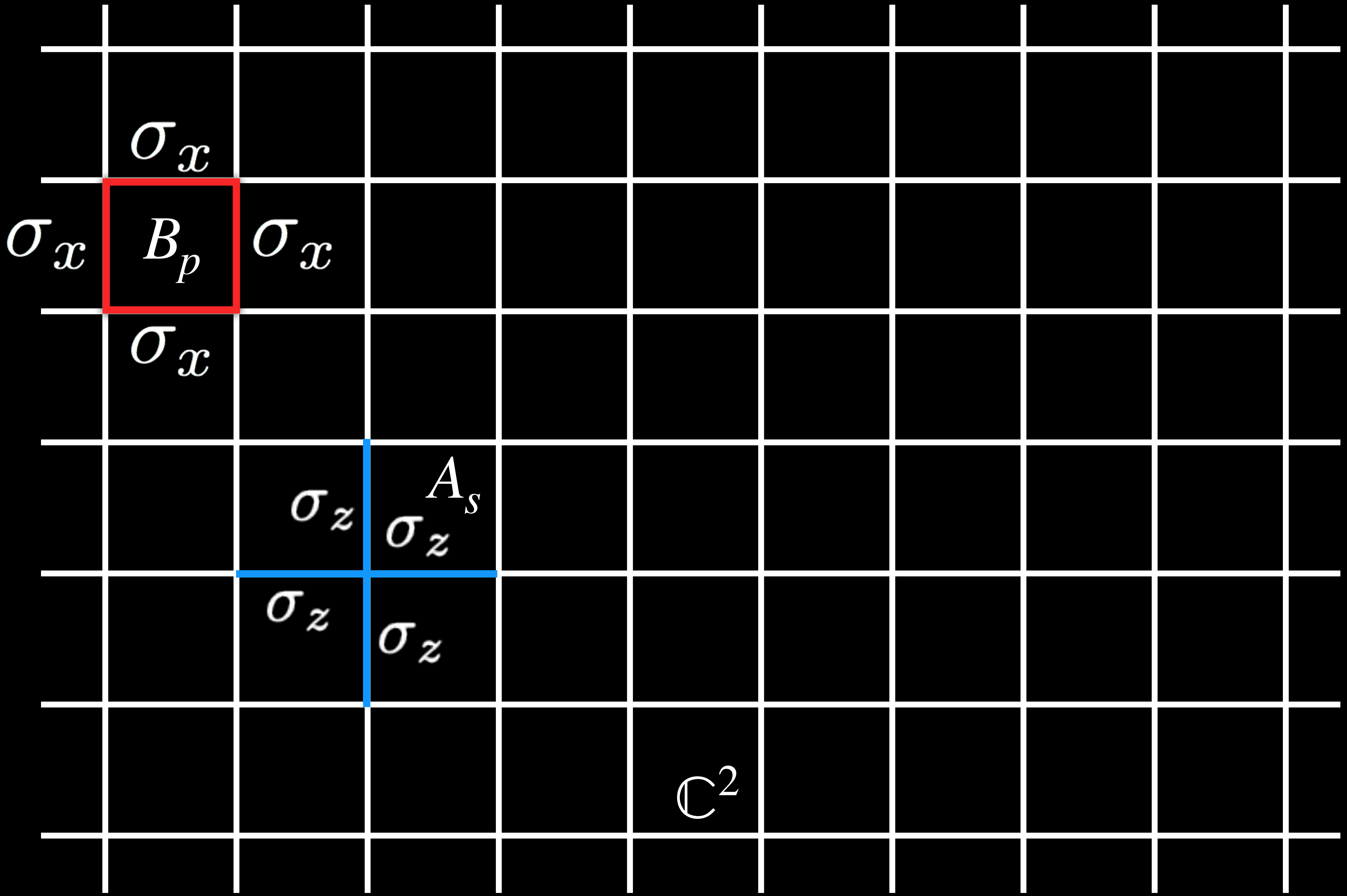
Kitaev, *Ann. Physics* **303**,2003

Toric code



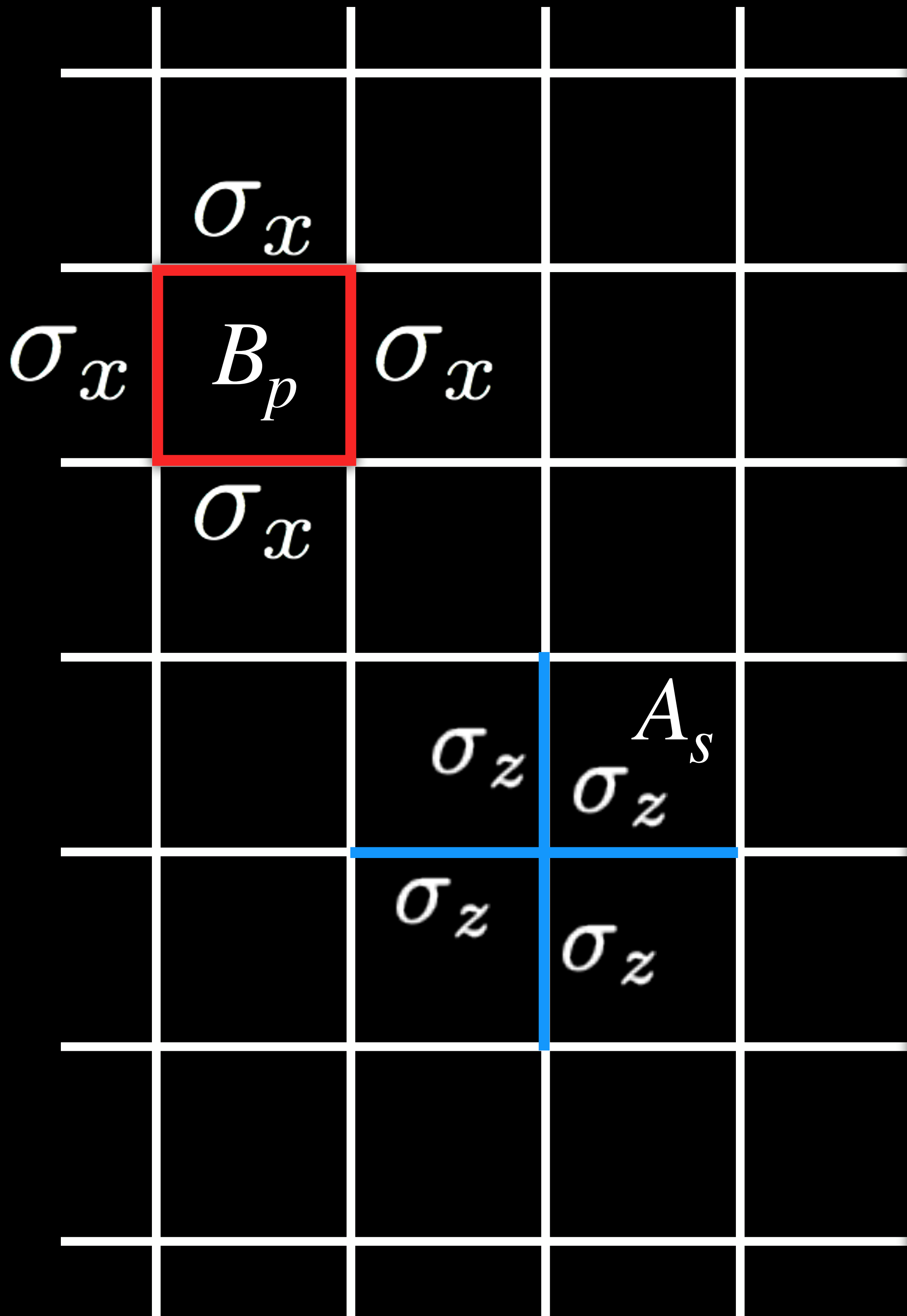
Kitaev, *Ann. Physics* **303**,2003

Toric code



Kitaev, *Ann. Physics* **303**,2003

Toric code



Kitaev, *Ann. Physics* **303**,2003

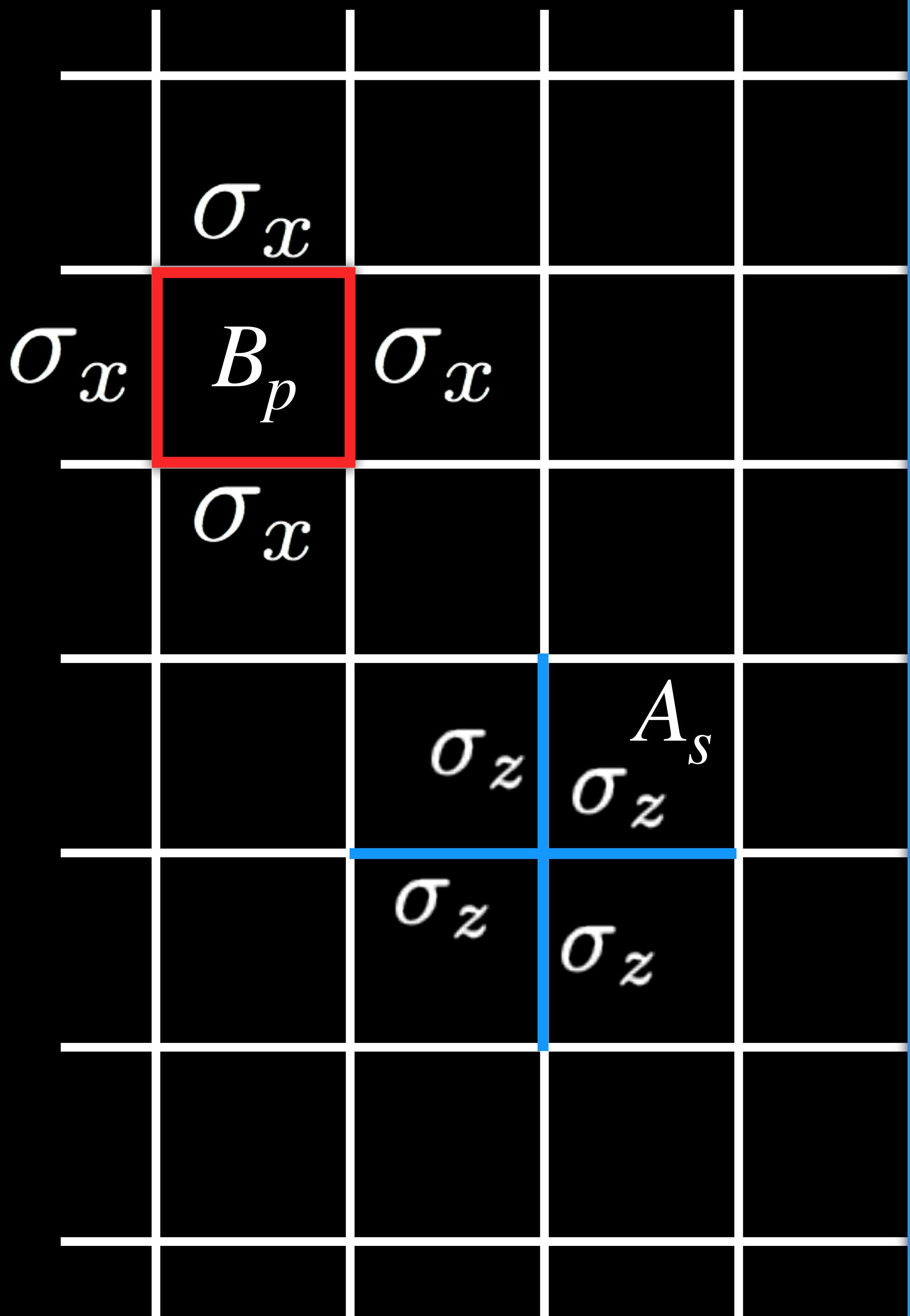
Hamiltonian:

$$H = -\sum_s A_s - \sum_p B_p$$

Ground state:

$$A_s\Omega = B_p\Omega = \Omega$$

Toric code



Kitaev, *Ann. Physics* **303**,2003

Hamiltonian:

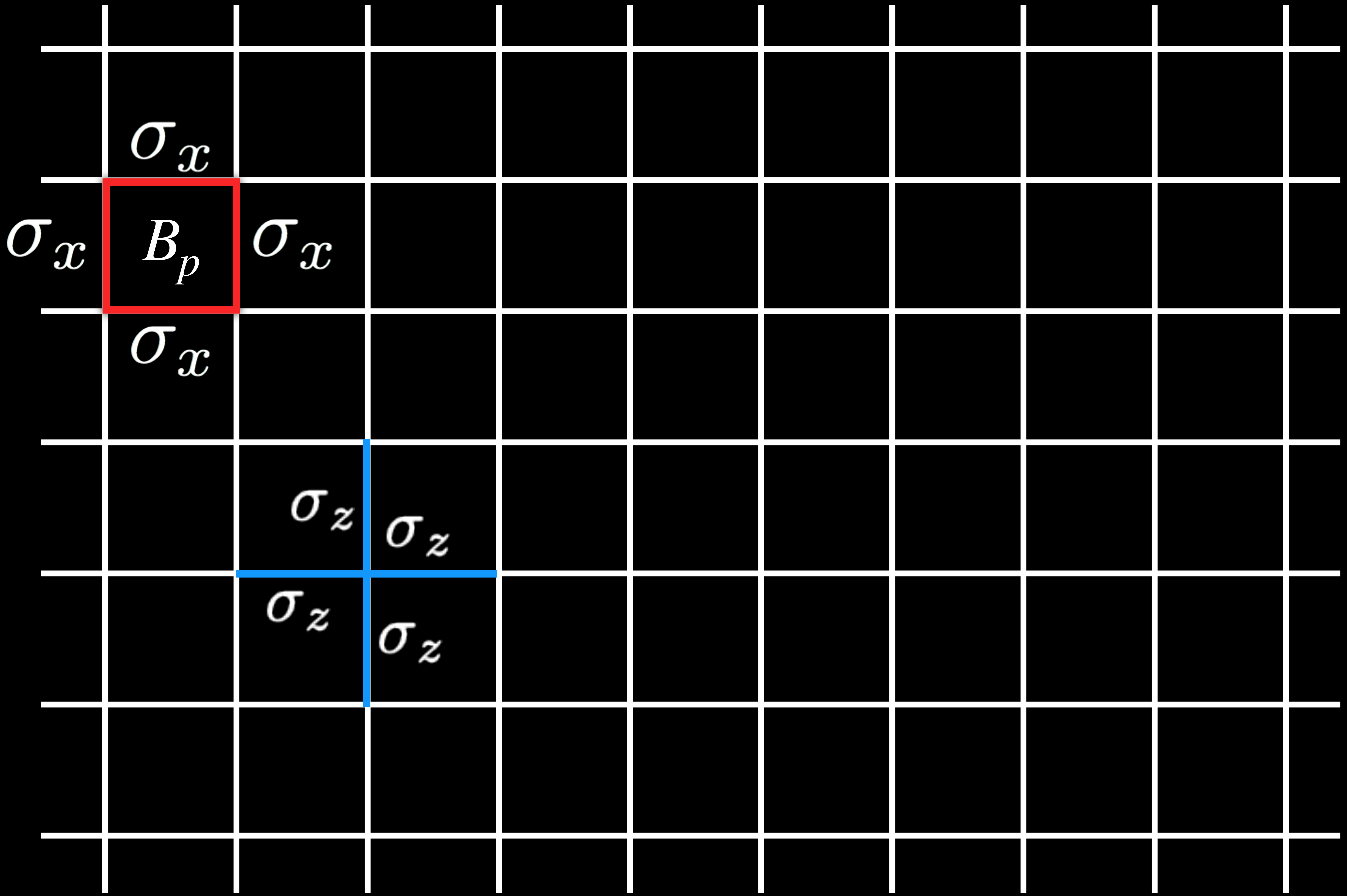
$$H = -\sum_s A_s - \sum_p B_p$$

Ground state:

$$A_s\Omega = B_p\Omega = \Omega$$

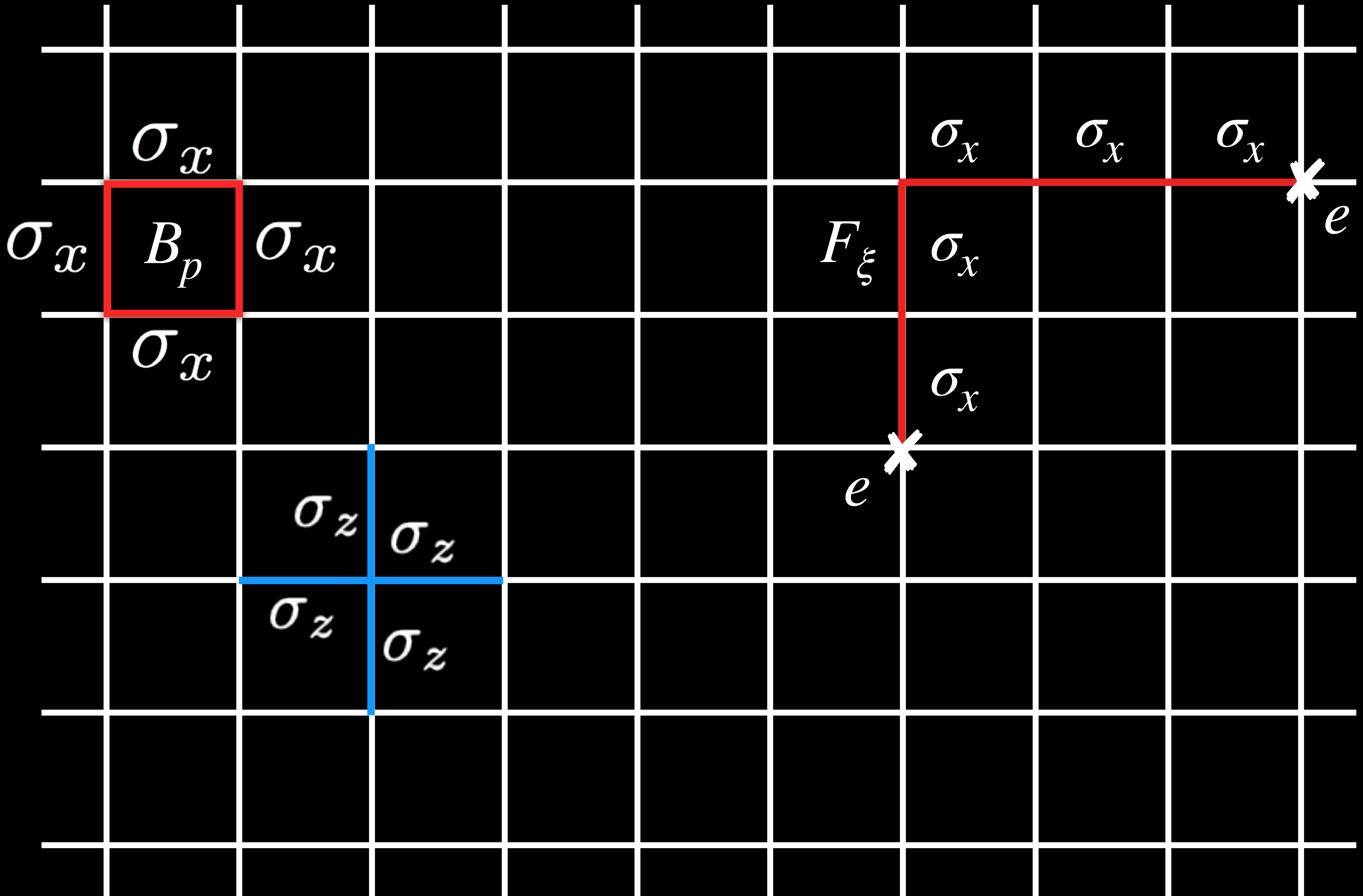
Remark: on compact surface, degeneracy depends on genus!

Toric code: excitations



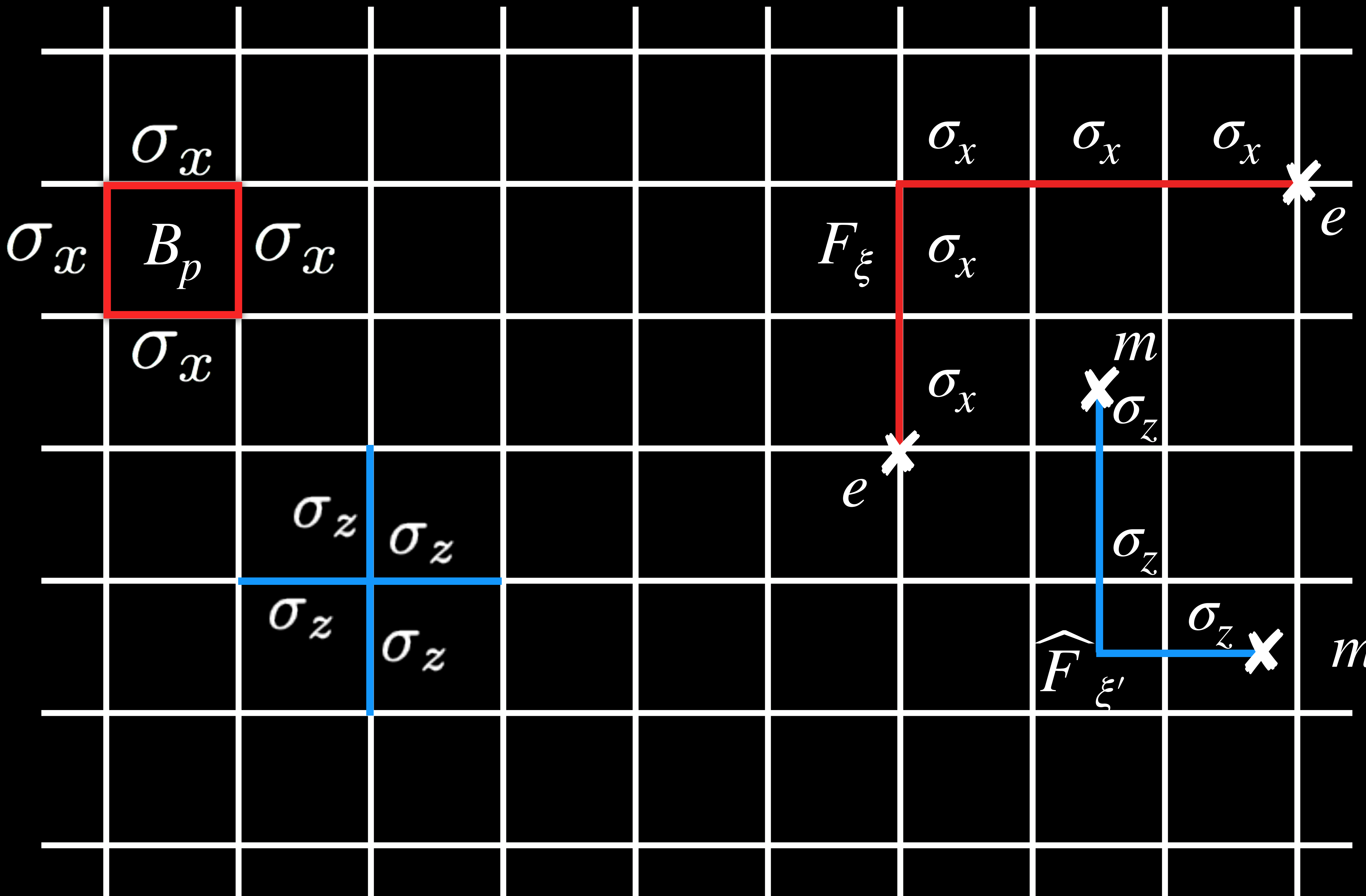
Kitaev, *Ann. Physics* **303**,2003

Toric code: excitations



Kitaev, *Ann. Physics* **303**,2003

Toric code: excitations



Kitaev, *Ann. Physics* **303**,2003

Modular tensor categories

Kitaev, *Ann. Physics* **303**,2003

Modular tensor categories

A careful analysis gives all algebraic properties of the anyons, leading to the MTC $\mathrm{Rep}\,\mathscr{D}(\mathbb{Z}_2)$

Kitaev, *Ann. Physics* **303**,2003

Modular tensor categories

A careful analysis gives all algebraic properties of the anyons, leading to the MTC $\mathrm{Rep}\,\mathscr{D}(\mathbb{Z}_2)$

Drawbacks:

- Only pairs of excitations: have to keep track

Kitaev, *Ann. Physics* **303**,2003

Modular tensor categories

A careful analysis gives all algebraic properties of the anyons, leading to the MTC $\mathrm{Rep}\,\mathscr{D}(\mathbb{Z}_2)$

Drawbacks:

- Only pairs of excitations: have to keep track
- Analysis requires deep understanding of microscopic details

Kitaev, *Ann. Physics* **303**,2003

Modular tensor categories

A careful analysis gives all algebraic properties of the anyons, leading to the MTC $\mathrm{Rep}\,\mathscr{D}(\mathbb{Z}_2)$

Drawbacks:

- Only pairs of excitations: have to keep track
- Analysis requires deep understanding of microscopic details
- Have to consider geometry, boundary conditions

Kitaev, *Ann. Physics* **303**,2003

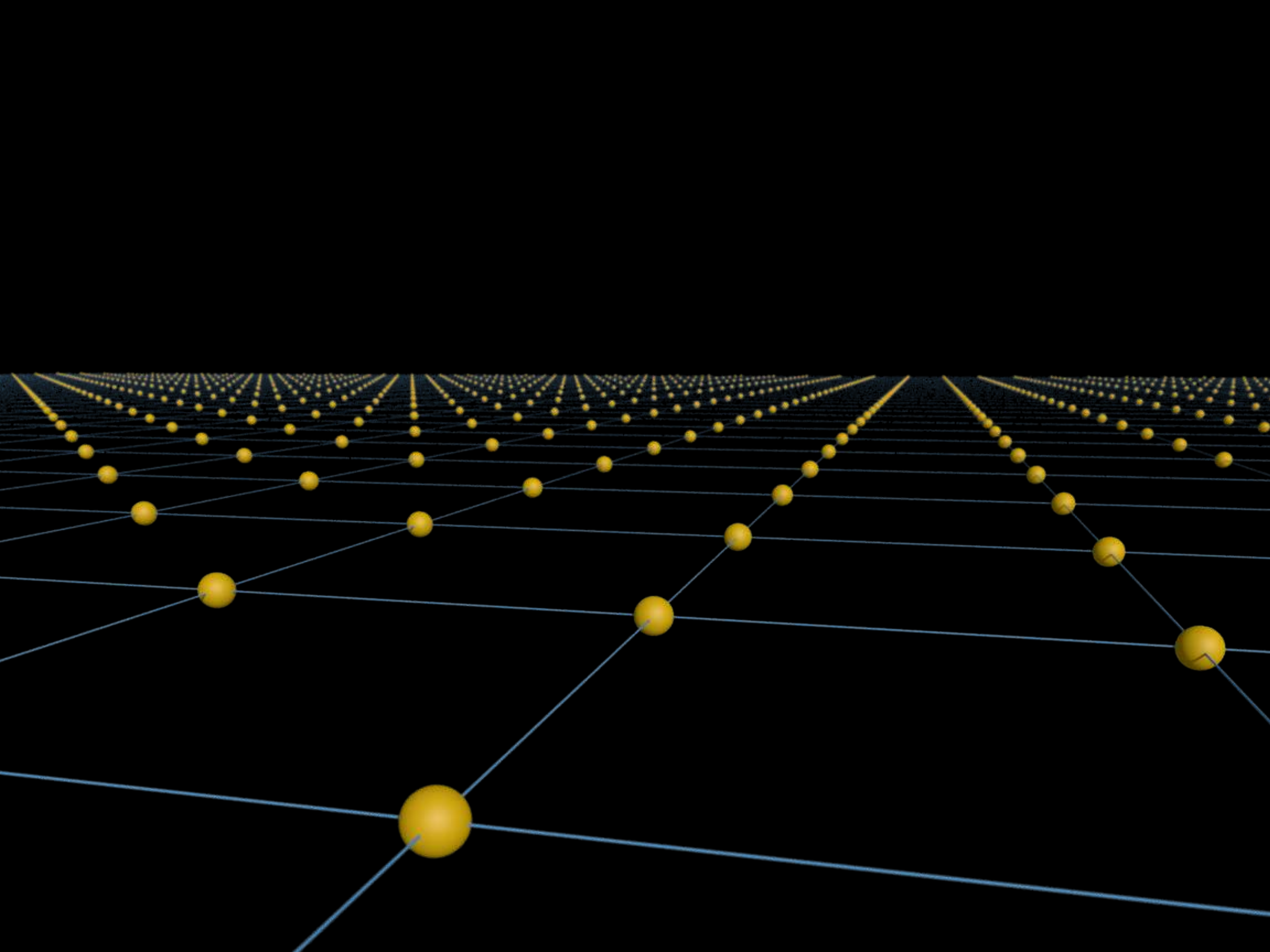
Problems:

How to get the MTC?

Is this an invariant?

LRE and trivial phases

Quantum spin systems



Quantum spin systems

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$

- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics

Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states

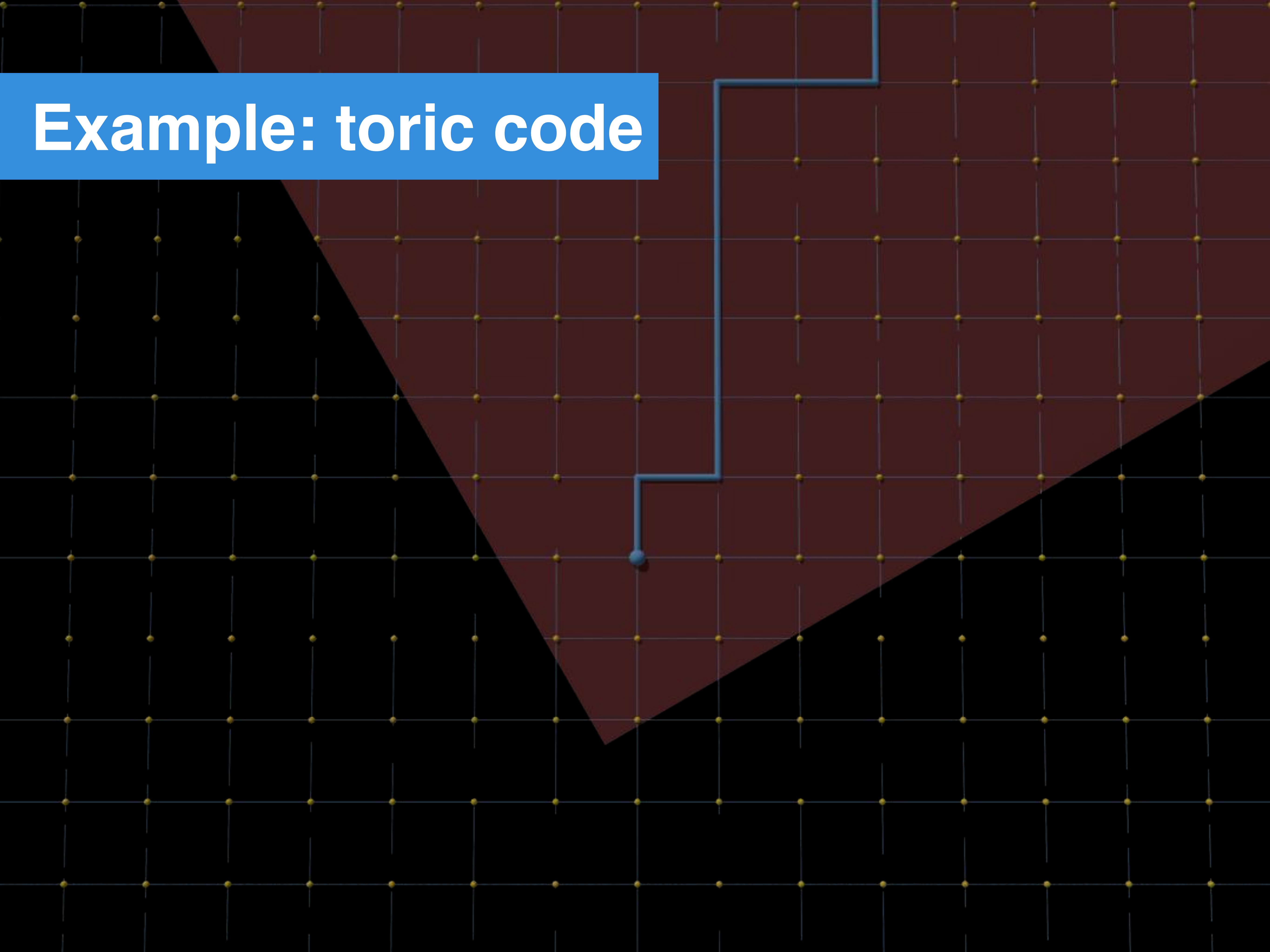
Quantum spin systems

Consider 2D quantum spin systems, e.g. on $\mathbb{Z}^2$:

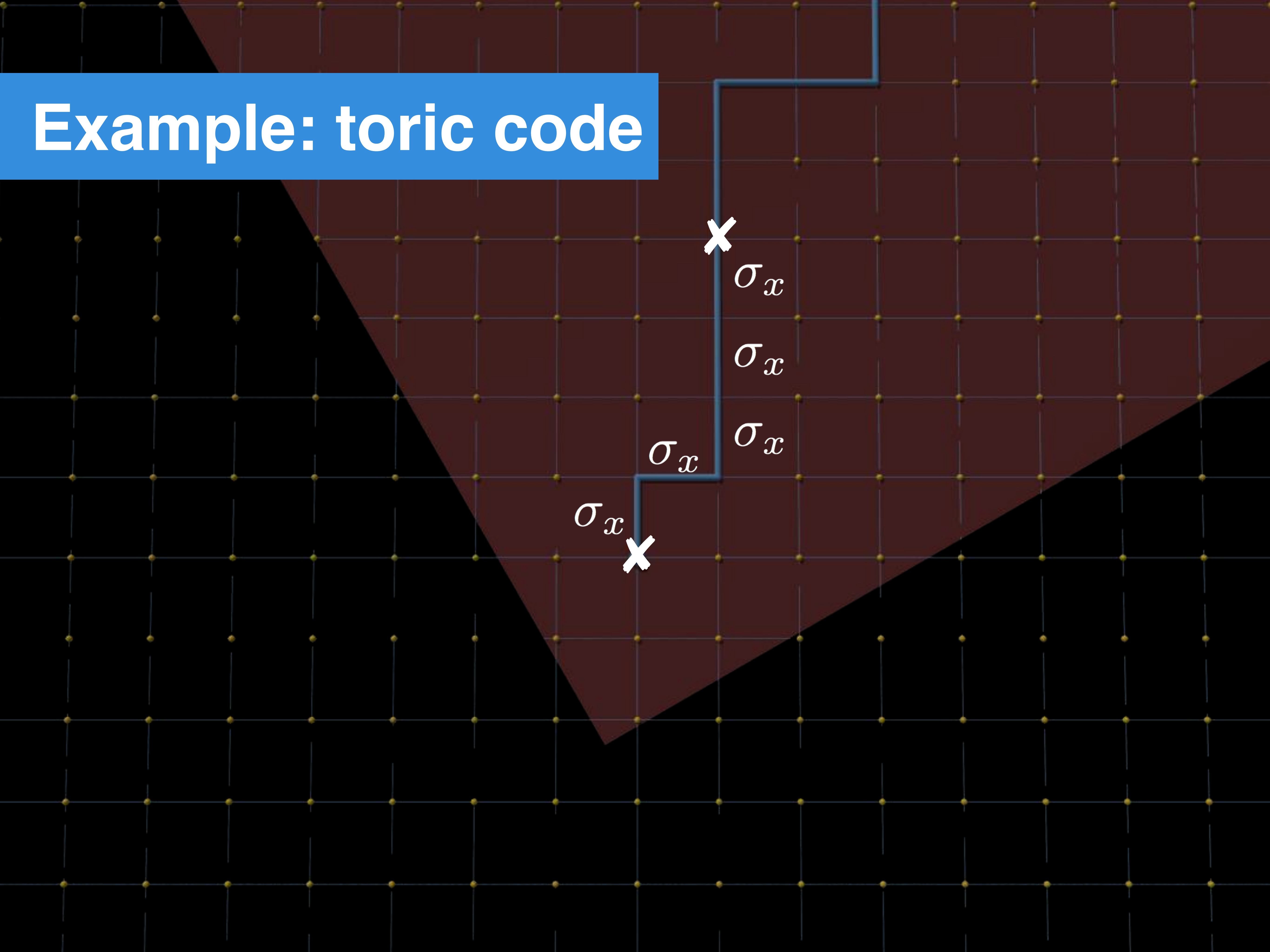
- local algebras $\Lambda \mapsto \mathfrak{A}(\Lambda) \cong \bigotimes_{x\in\Lambda} M_d(\mathbb{C})$
- quasilocal algebra $\mathfrak{A} := \overline{\bigcup \mathfrak{A}(\Lambda)}^{\|\cdot\|}$
- local Hamiltonians H_Λ describing dynamics
- gives time evolution α_t & ground states
- if ω a ground state, Hamiltonian H_ω in GNS repn.

Sector theory

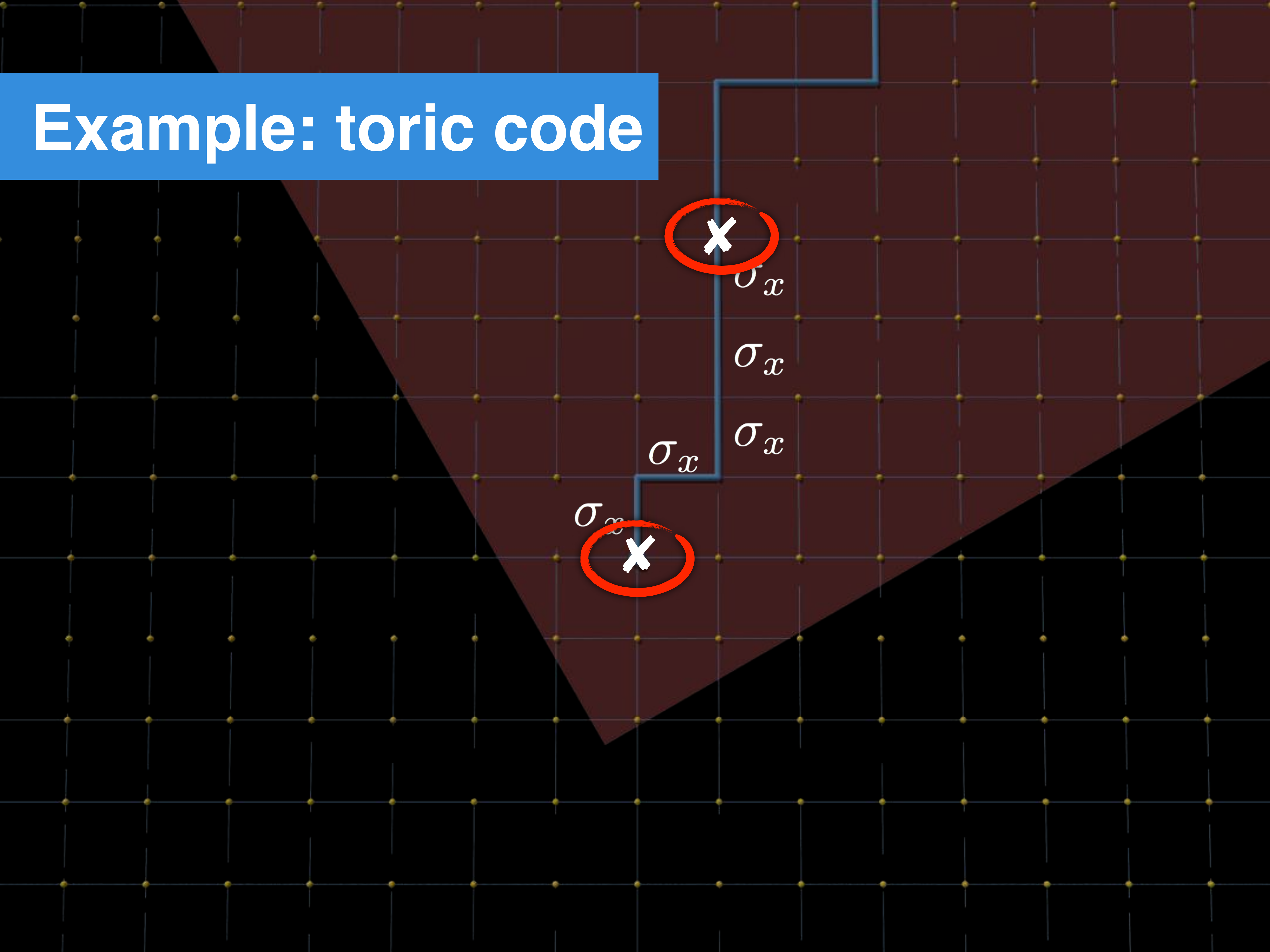
Example: toric code



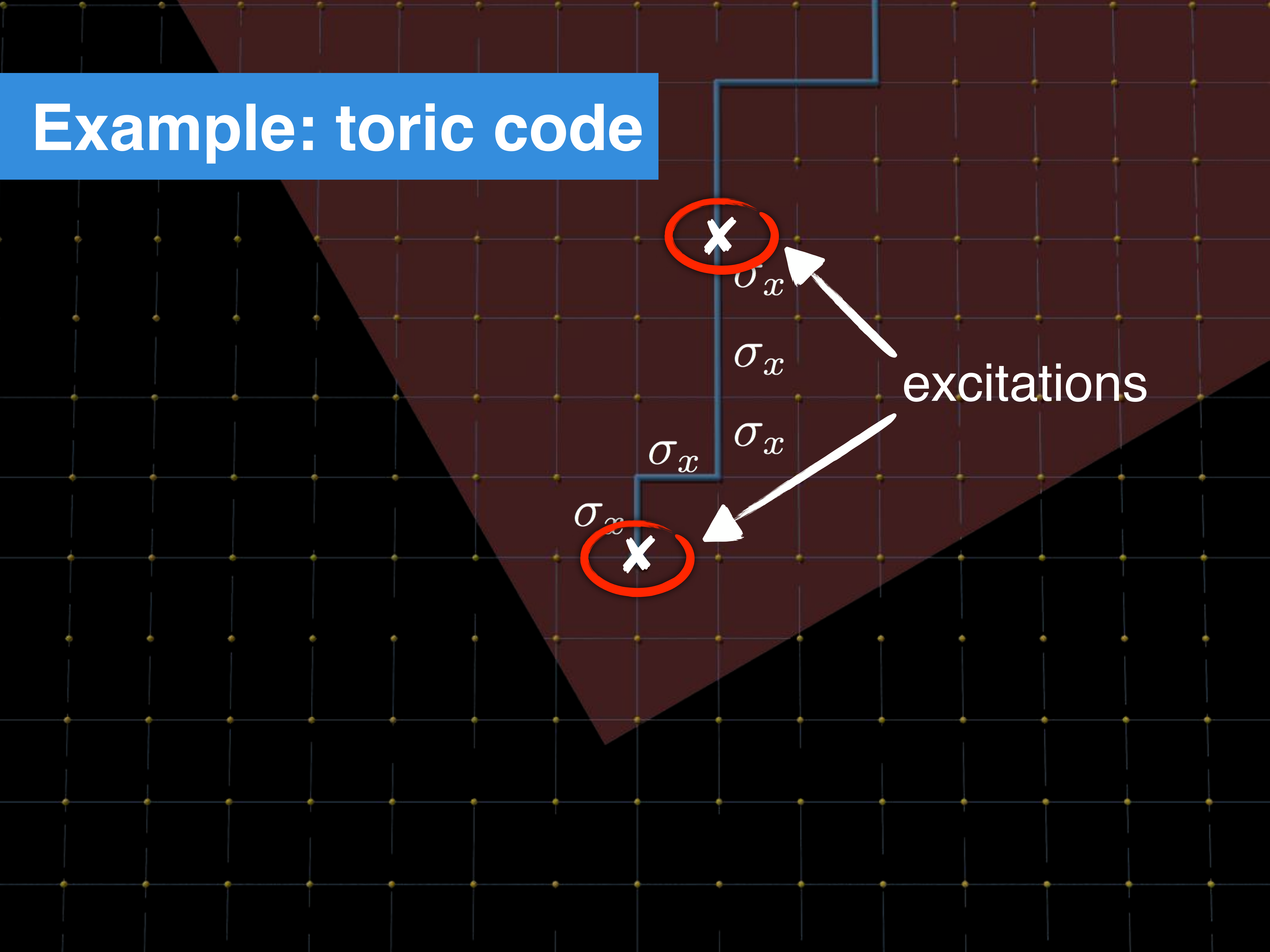
Example: toric code



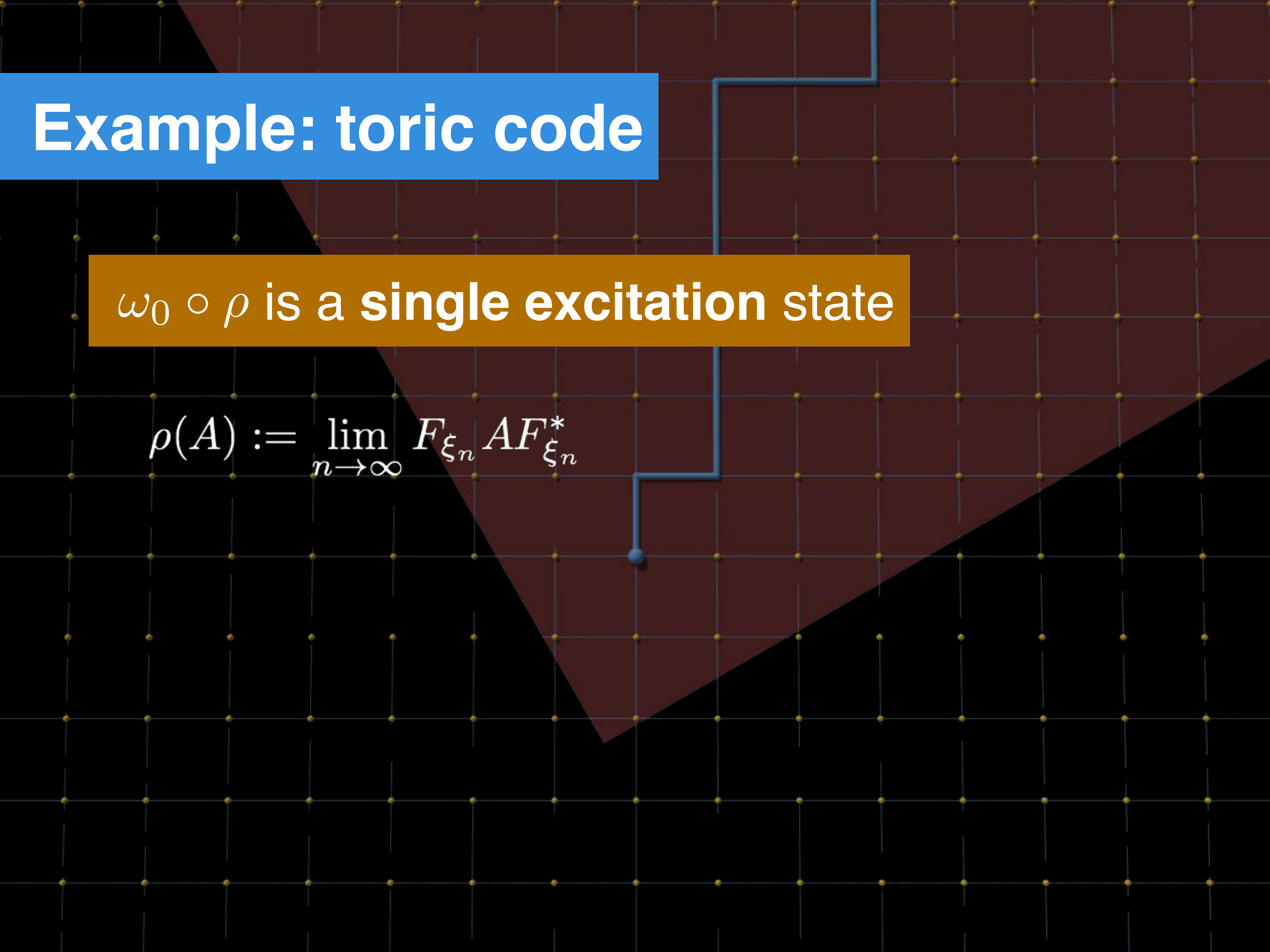
Example: toric code



Example: toric code



Example: toric code



$\omega_0 \circ \rho$ is a **single excitation** state

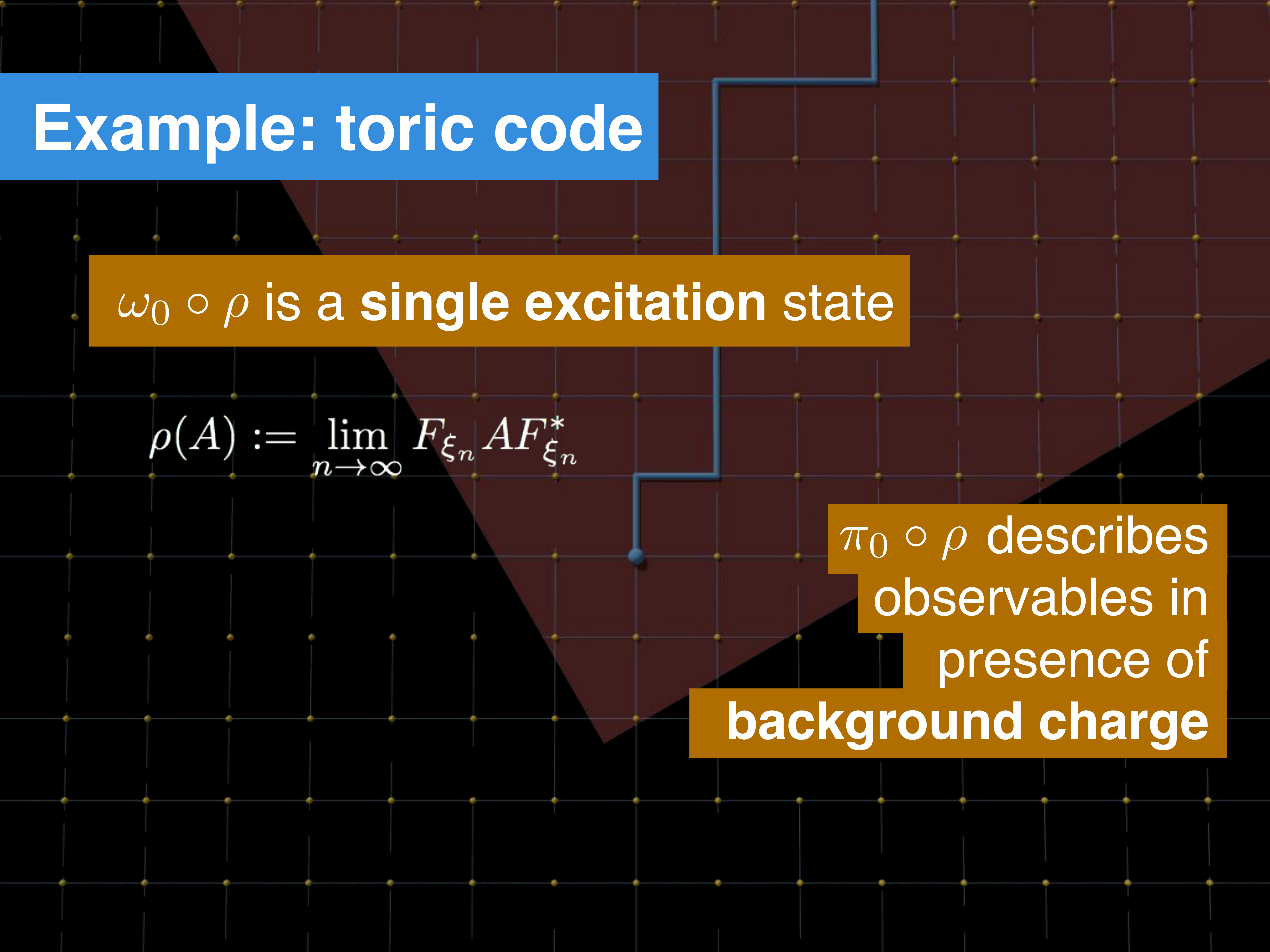
$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*$$

Example: toric code

$\omega_0 \circ \rho$ is a **single excitation** state

$$\rho(A) := \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*$$

$\pi_0 \circ \rho$ describes observables in presence of **background charge**



Localised and transportable morphisms

The endomorphism ρ has the following properties:

Localised and transportable morphisms

The endomorphism ρ has the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$

Localised and transportable morphisms

The endomorphism ρ has the following properties:

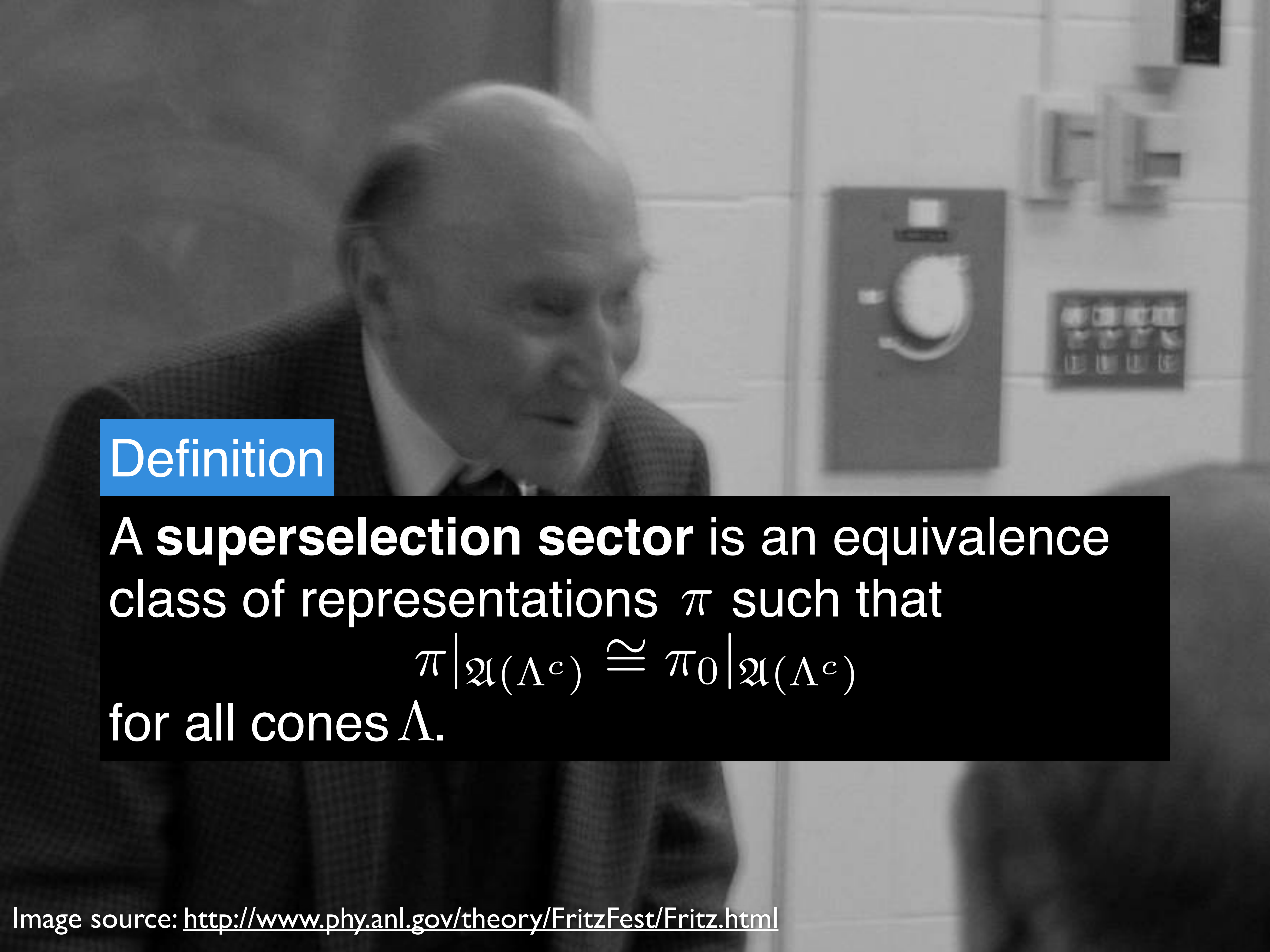
- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Localised and transportable morphisms

The endomorphism ρ has the following properties:

- **localised:** $\rho(A) = A \quad \forall A \in \mathfrak{A}(\Lambda^c)$
- **transportable:** for Λ' there exists σ localised and $V\pi_0(\rho(A))V^* = \pi_0(\sigma(A))$

Can study all endomorphisms with these properties



Definition

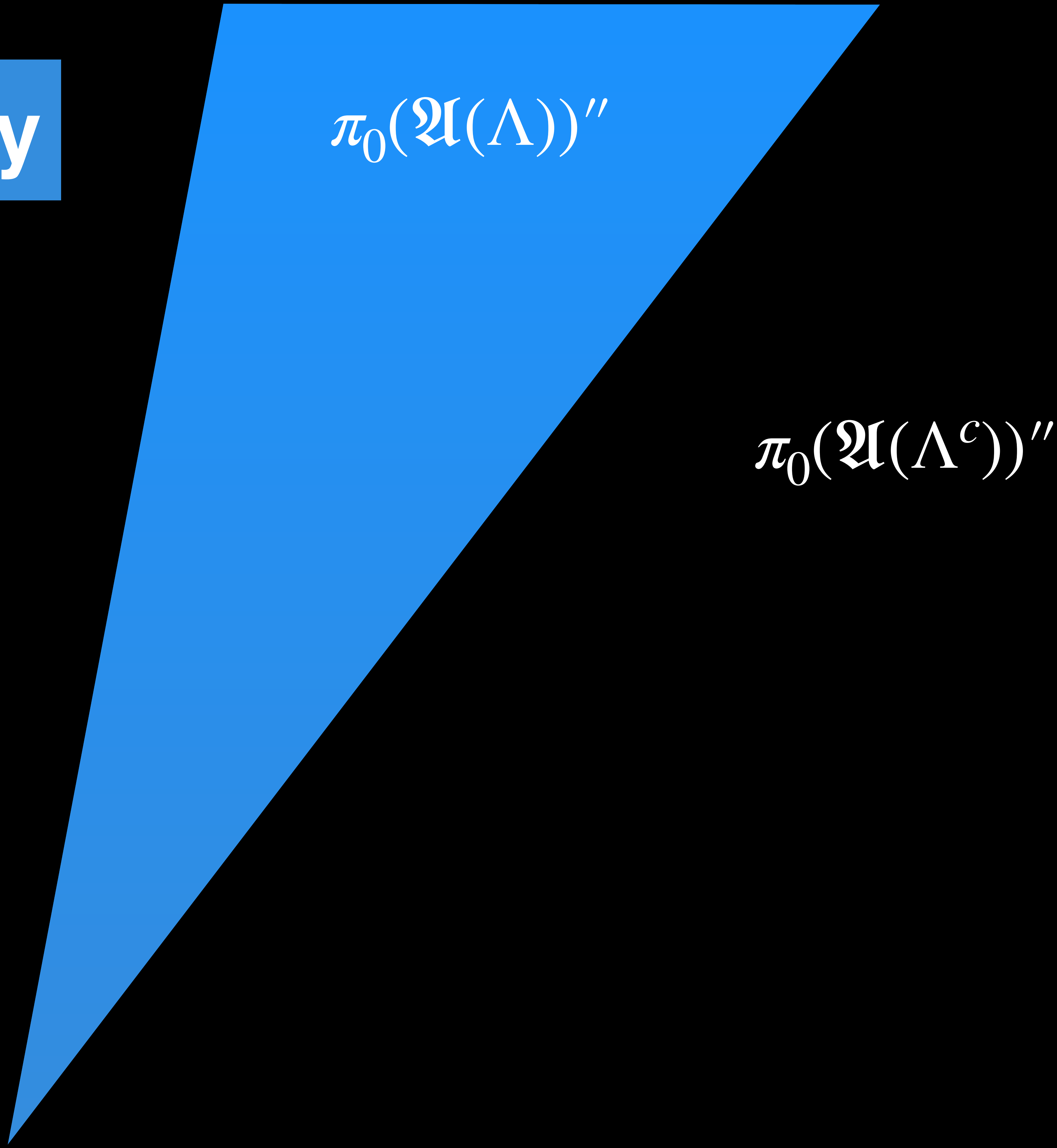
A **superselection sector** is an equivalence class of representations π such that

$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

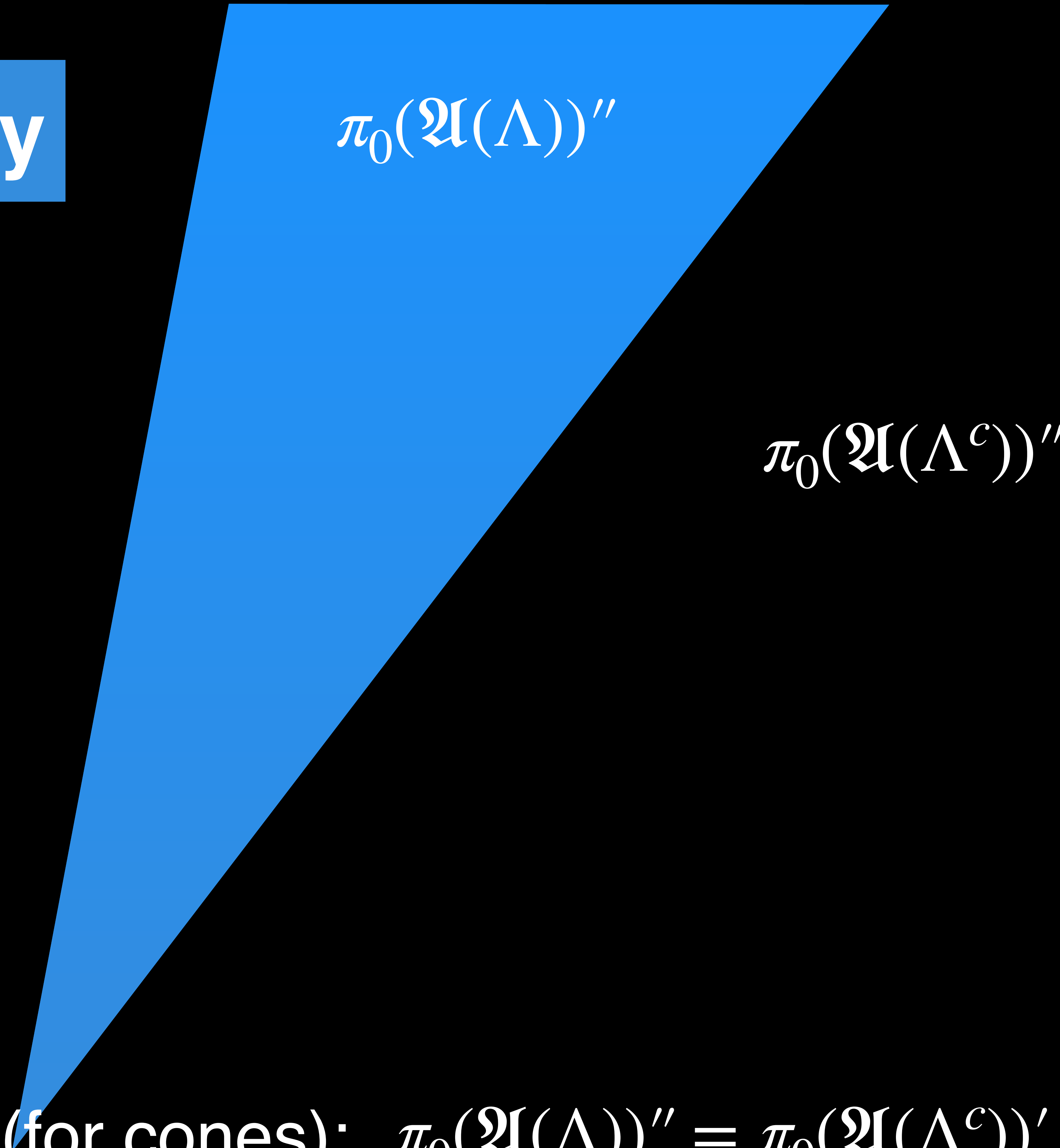
for all cones Λ.

Image source: http://www.phy.anl.gov/theory/FritzFest/Fritz.html

Haag duality



Haag duality



Haag duality (for cones): $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$

Theorem

Suppose Haag duality for cones hold. Then the set of representations satisfying the superselection criterion has the structure of a braided tensor *C^**-category.

Doplicher, Haag, Roberts, *Commun. Math. Phys.* **23** (1971)
Doplicher, Haag, Roberts, *Commun. Math. Phys.* **35** (1974)
Buchholz, Fredenhagen, *Commun. Math. Phys.* **84** (1982)

Theorem (Fiedler, PN)

Let G be a finite abelian group and consider Kitaev's quantum double model. Then Haag duality holds and the set of superselection sectors can be endowed with the structure of a modular tensor category. This category is equivalent to $\operatorname{Rep} D(G)$.

Rev. Math. Phys. **23** (2011)
J. Math. Phys. **54** (2013)
Rev. Math. Phys. **27** (2015)

Automorphic equivalence

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

See also: Nachtergaele, *arXiv:2205.10460* (2022)

Quantum phases of ground states

Two ground states ω_0 and ω_1 are said to be *in the same phase* if there is a continuous path $s \mapsto H(s)$ of gapped local Hamiltonians, such that ω_s is a ground state of $H(s)$.

(Chen, Gu, Wen, *Phys. Rev. B* **82**, 2010)

Alternative definition: ω_0 can be transformed into ω_1 with a *finite depth local quantum circuit.*

See also: Nachtergaele, *arXiv:2205.10460* (2022)

Classification of phases

Classification of phases

- Does the gap stay open under small perturbations?

Bravyi & Hastings, *J. Math. Phys.* **51** (2010)
Michalakis & Zwolak, *Commun. Math. Phys.* **322** (2013)
Nachtergaele, Sims & Young, arXiv:2102.07209
and many others…

Classification of phases

- Does the gap stay open under small perturbations?

  Bravyi & Hastings, *J. Math. Phys.* **51** (2010)
  Michalakis & Zwolak, *Commun. Math. Phys.* **322** (2013)
  Nachtergaele, Sims & Young, arXiv:2102.07209
  and many others…

- How are the states related?

Classification of phases

- Does the gap stay open under small perturbations?

  Bravyi & Hastings, *J. Math. Phys.* **51** (2010)
  Michalakis & Zwolak, *Commun. Math. Phys.* **322** (2013)
  Nachtergaele, Sims & Young, arXiv:2102.07209
  and many others…

- How are the states related?

  Hastings, *Phys. Rev. B* **69** (2004)
  Hastings & Wen, *Phys. Rev. B* **72** (2005)
  Bachmann, Michalakis, Nachtergaele & Sims, *Commun. Math. Phys.* **309** (2012)
  Nachtergaele, Sims & Young, *J. Math. Phys.* **60** (2019)
  Moon & Ogata, *J. Funct. Anal.* **278** (2020)

Can we find invariants?

Theorem (Bachmann, Michalakis, Nachtergaele, Sims)

Let $s \mapsto H_\Lambda + \Phi(s)$ be a family of gapped Hamiltonians. Then there is a family $s \mapsto \alpha_s$ of automorphisms such that the weak-* limits of ground states (with open boundary conditions) are related via

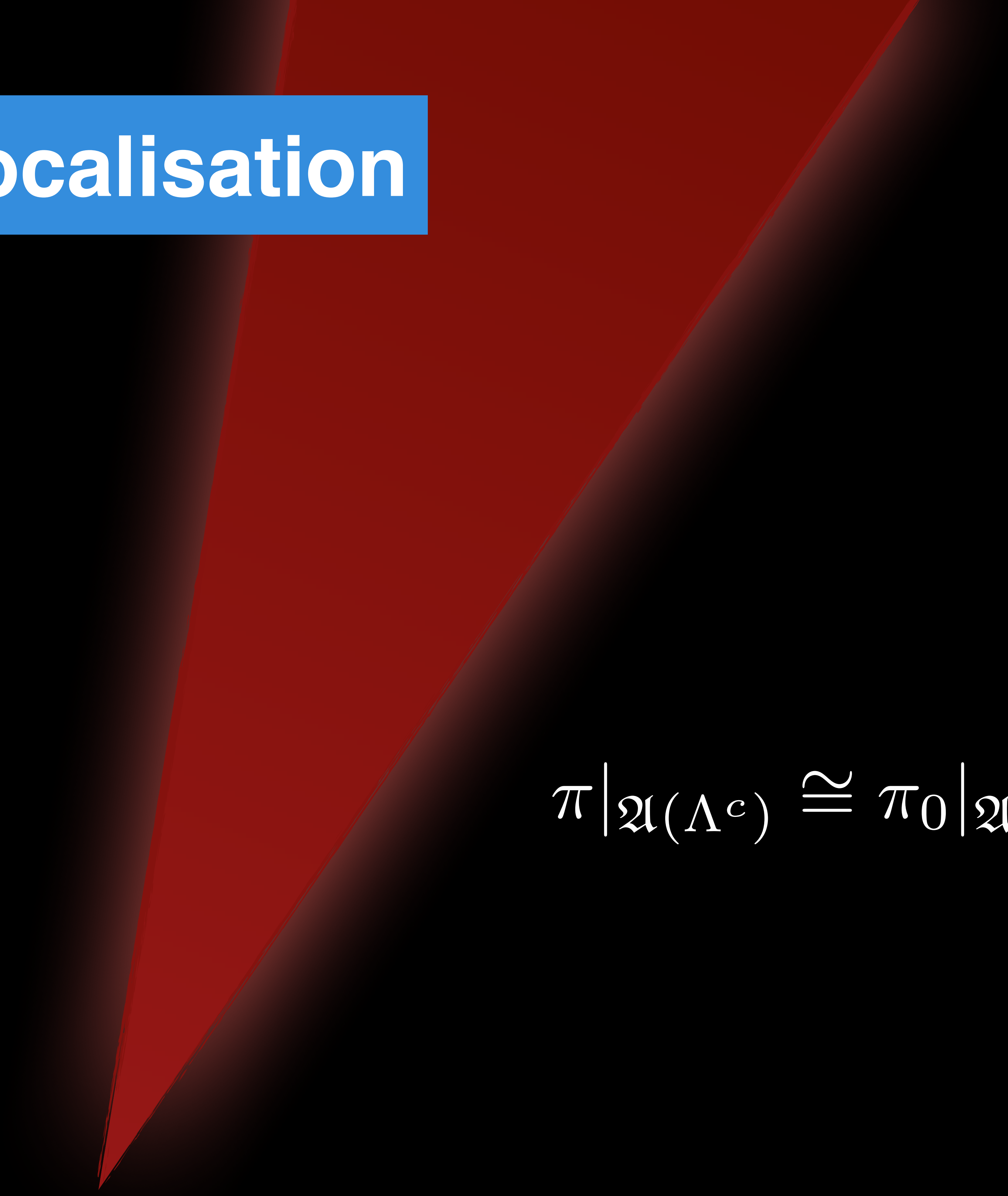
$$\mathcal{S}(s) = \mathcal{S}(0) \circ \alpha_s$$

Commun. Math. Phys. **309** (2012)

This is not enough to conclude stability of the superselection structure!

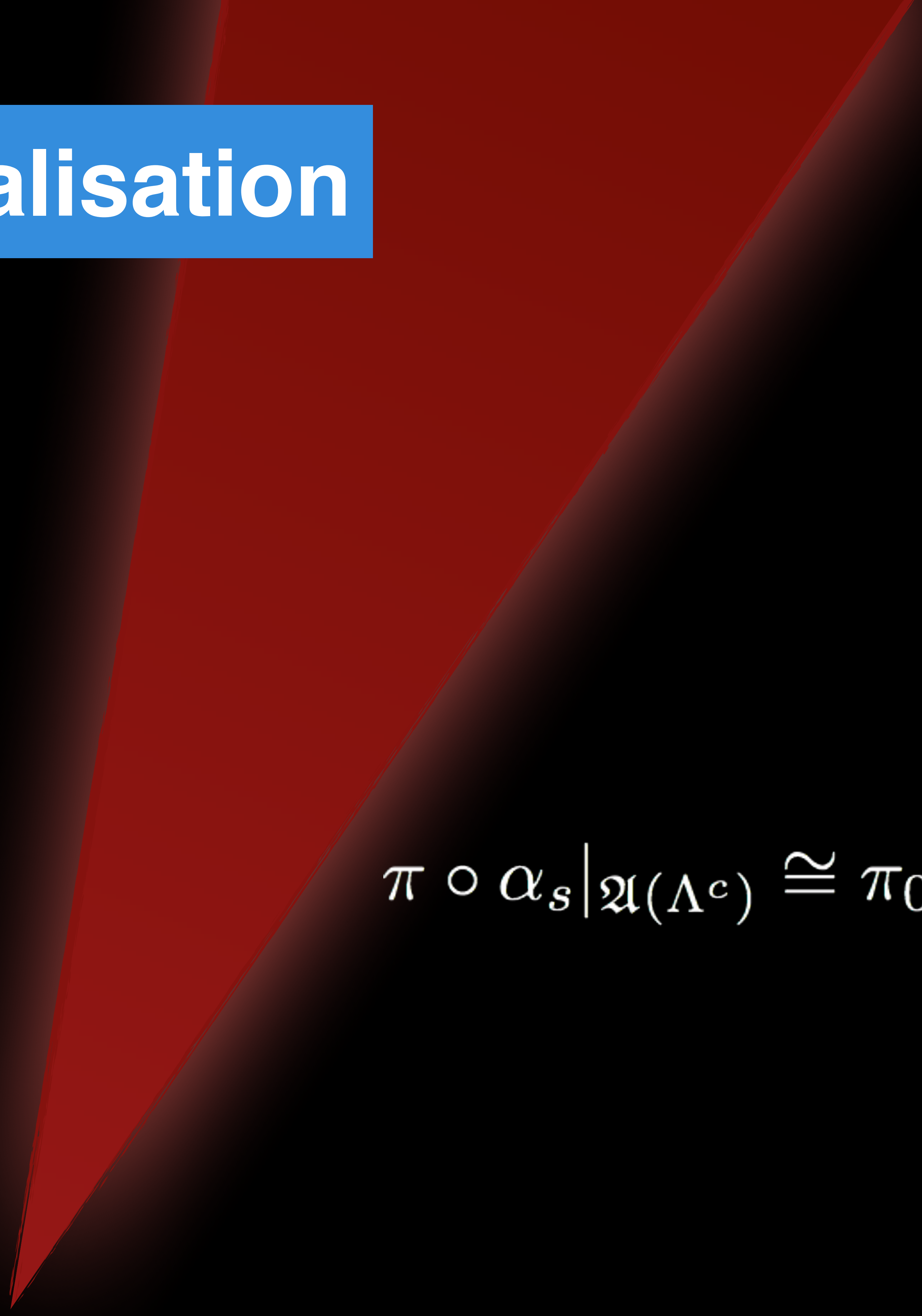
No strict localisation

No strict localisation



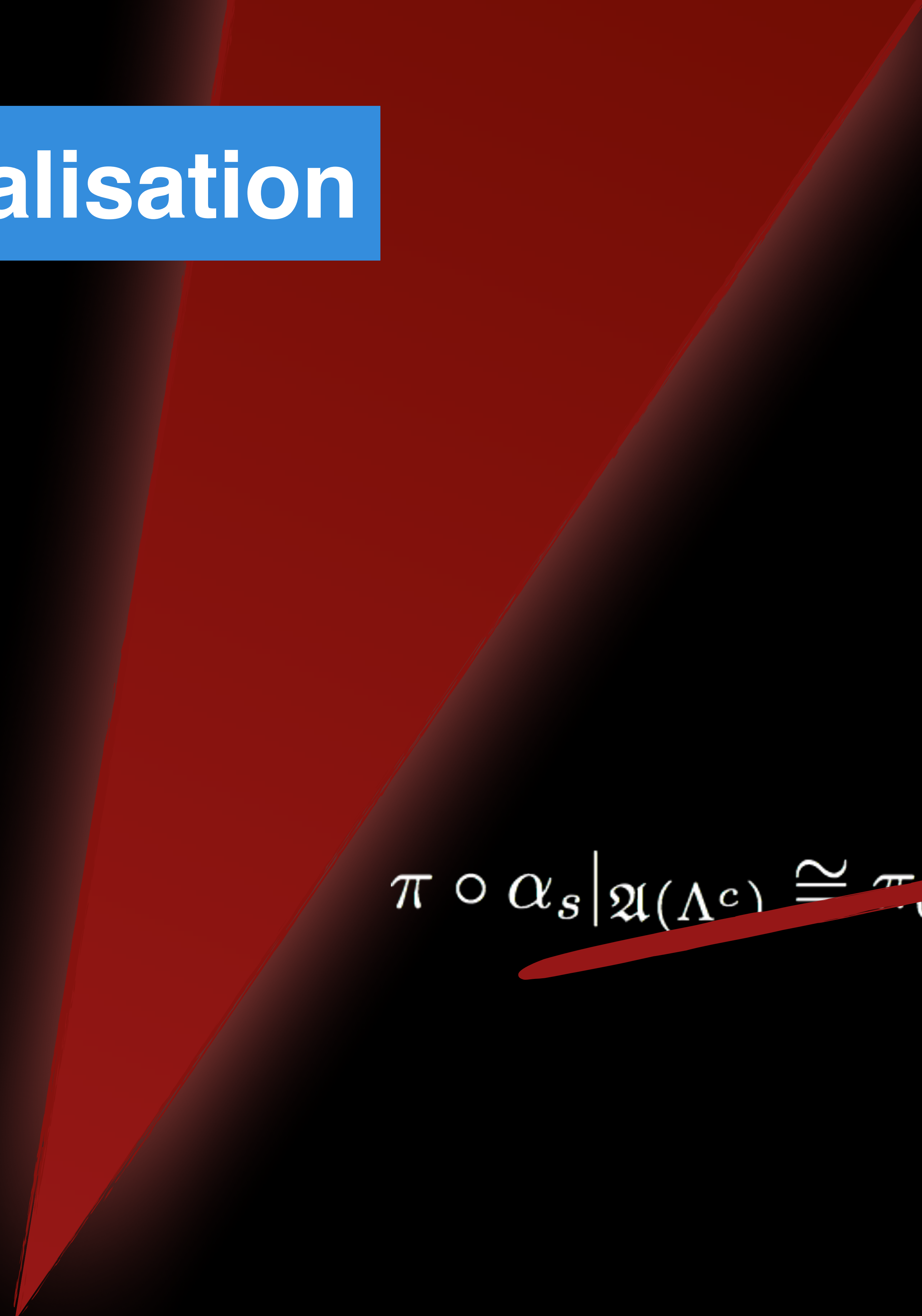
$$\pi|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0|_{\mathfrak{A}(\Lambda^c)}$$

No strict localisation



$$\pi \circ \alpha_s\big|_{\mathfrak{A}(\Lambda^c)} \cong \pi_0 \circ \alpha_s\big|_{\mathfrak{A}(\Lambda^c}$$

No strict localisation



$$\pi \circ \alpha_s|_{\mathfrak{A}(\Lambda^c)} \cancel{\cong \pi_0 \circ \alpha_s}|_{\mathfrak{A}(\Lambda^c)}$$

First approach: replace selection criterion

Almost localised endomorphisms

An endomorphism ρ of $\mathcal{A}$ is called *almost localised* in a cone Λ_α if

$$\sup_{A \in \mathcal{A}(\Lambda^c_{\alpha+\epsilon}+n)} \frac{\|\rho(A) - A\|}{\|A\|} \leq f_\epsilon(n)$$

where $f_\epsilon(n)$ is a non-increasing family of absolutely continuous functions which decay faster than any polynomial in *n*.

Almost localised endomorphisms

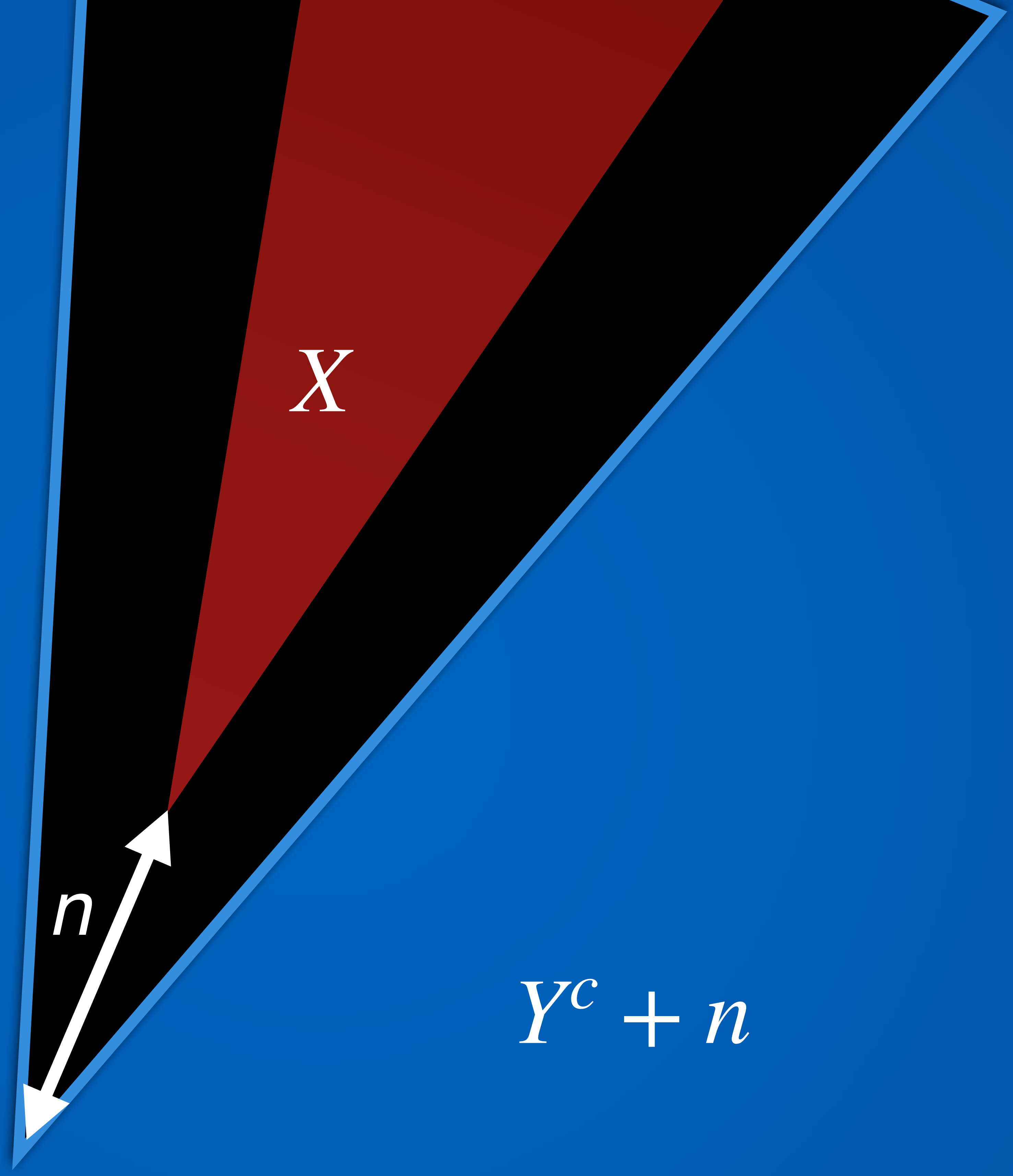
An endomorphism ρ of $\mathcal{A}$ is called *almost localised* in a cone Λ_α if

$$\sup_{A \in \mathcal{A}(\Lambda^c_{\alpha+\epsilon}+n)} \frac{\|\rho(A) - A\|}{\|A\|} \leq f_\epsilon(n)$$

where $f_\epsilon(n)$ is a non-increasing family of absolutely continuous functions which decay faster than any polynomial in *n*.

$\Rightarrow$ Combine with transportability & **asymptopia**

Buchholz, Doplicher, Morchio, Roberts & Strocchi. In: *Rigorous quantum field theory* (2007)



X
n
Y^c + n

Theorem

Let G be a finite abelian group and consider the perturbed Kitaev's quantum double model. Then for each s in the unit interval, the category $\Delta^{qd}(s)$ category is braided tensor equivalent to $\mathrm{Rep}\, D(G)$.

Cha, PN, Nachtergaele, *Commun. Math. Phys.* **373** (2020)

Approximate localisation

Advantage:

Approximate localisation

Advantage:

- Does not require Haag duality

Approximate localisation

Advantage:

- Does not require Haag duality

Disadvantage:

- Needs extra condition to show no new sectors are introduced

Second approach: replace Haag duality

Approximate Haag duality

For all $\epsilon > 0$ and cone Λ, we have that there is a unitary $U_{\Lambda,\epsilon}$ such that

$$\pi_0(\mathfrak{A}(\Lambda^c))' \subset U_{\Lambda,\epsilon}\, \pi_0(\mathfrak{A}(\Lambda_\epsilon))''\, U^*_{\Lambda,\epsilon}$$

(+ some technical approximation properties).

Ogata, *J. Math. Phys.* **63** (2022)

Theorem

Let π_0 be the GNS representation of a gapped ground state, and suppose that *approximate* Haag duality for cones hold. Then the set of representations satisfying the superselection criterion has the structure of a braided tensor *C**-category. This category is stable under applying approximately factorisable automorphisms.

Ogata, *J. Math. Phys.* **63** (2022)

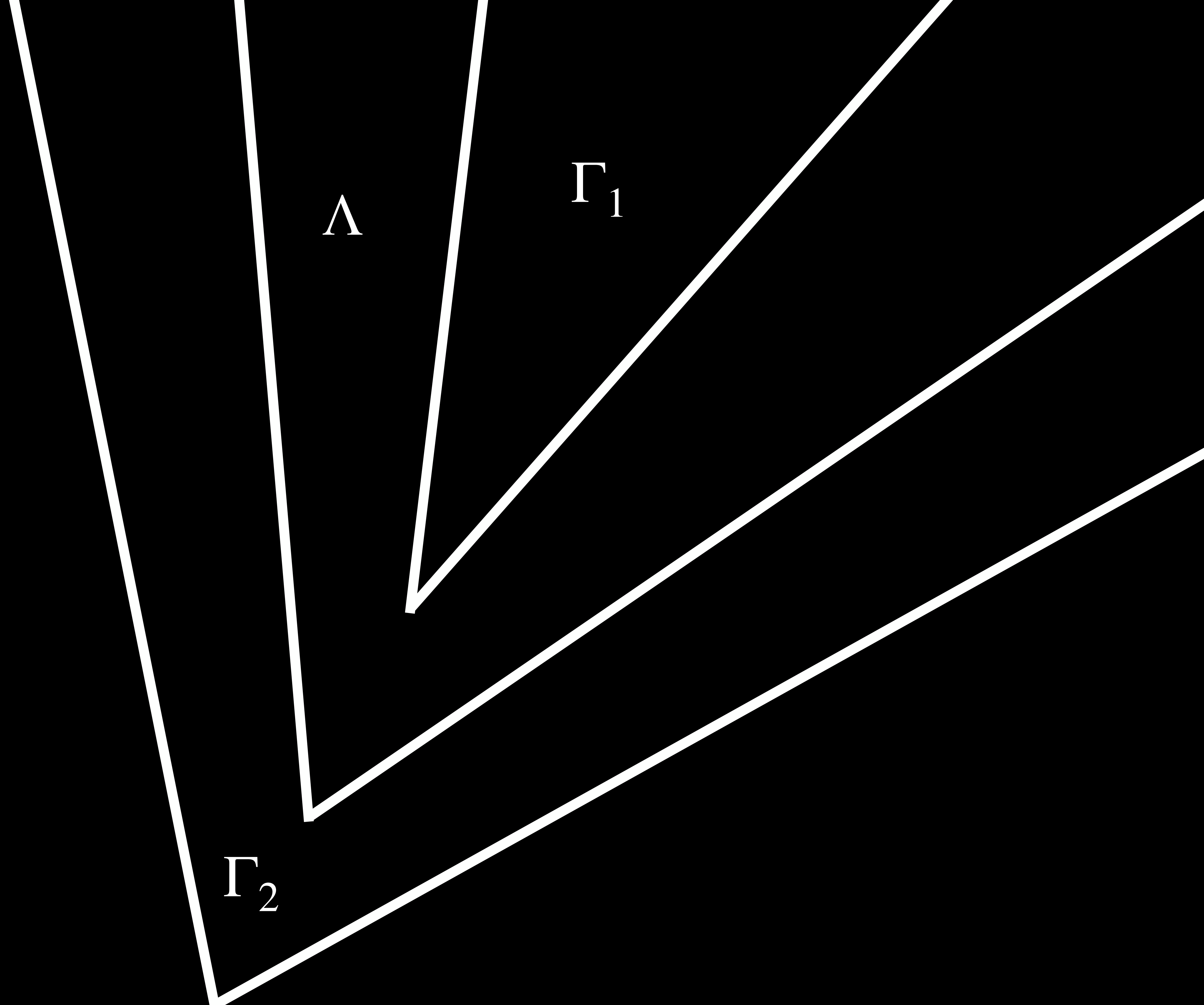
Definition

Consider an inclusion $\Gamma_1 \subset \Lambda \subset \Gamma_2$ of cones. Then $\alpha \in \mathrm{Aut}(\mathfrak{A})$ is called *quasi-factorisable* if:

$$\alpha = \mathrm{Ad}(u) \circ \Xi \circ (\alpha_\Lambda \otimes \alpha_{\Lambda^c})$$

for some unitary $u \in \mathfrak{A}$ and "local" automorphisms (see picture).

PN & Y. Ogata, *Commun. Math. Phys.* **392**, 921-950 (2022)



Λ
Γ1
Γ2

Approximate localisation

Advantage:

Approximate localisation

Advantage:

- Does not change selection criterion

Approximate localisation

Advantage:

- Does not change selection criterion
- No extra conditions

Approximate localisation

Advantage:

- Does not change selection criterion
- No extra conditions

Disadvantage:

- Need to prove approximate Haag duality

The trivial phase

Long range entanglement

- Bipartite system $\mathfrak{A}_\Lambda \otimes \mathfrak{A}_{\Lambda^c}$
- Product states $\omega = \omega_\Lambda \otimes \omega_{\Lambda^c}$ have only classical correlations
- LRE: $\omega \circ \alpha$ is not quasi-equivalent to a product state for any quasi-local automorphism
- In 1D, gapped ground states are not LRE, in 2D this can be different!

Folklore

Topological order (and in particular anyonic excitations) are due to long range entanglement

A new superselection criterion

We can relax the superselection criterion:

$$\pi \mid \mathfrak{A}_{\Lambda^c} \sim_{qe} \pi_\omega \mid \mathfrak{A}_{\Lambda^c}$$

That is, *quasi* instead of *unitary* equivalence

Remark: can be constructed naturally in non-abelian theories using amplimorphisms!

Szlachányi &Vecsernyés, CMP **156**, 1993

Theorem

Let ω be a pure state such that its GNS representation is quasi-equivalent to $\pi_\Lambda \otimes \pi_{\Lambda^c}$ for some cone Λ. Then the corresponding superselection structure is trivial. This is also stable under quasi-factorisable automorphisms.

PN & Y. Ogata, *Commun. Math. Phys* **932**:921-950

Some recent results

Theorem

Let ω be the frustration free ground state of the quantum double model for an abelian group G. Then for any convex cone Λ, the von Neumann algebra $\pi_\omega(\mathfrak{A}(\Lambda))''$ is a factor of Type II_∞.

Y. Ogata, arXiv:2212.09036

Toric code with boundary

Can consider models with a *gapped boundary*.

Theorem

The fusion category of boundary excitations —more precisely, the fusion category of superselection sectors localized in a fixed cone along the boundary—is a module tensor category over the category of sectors for the bulk toric code.

Kitaev & Kong, *Comm. Math. Phys.* **313**:351– 373 (2012)
D. Wallick, arXiv:2212.01952 (2022)

Open problems

More examples

Explicit results for small class of models.

Can we extend this to other models, in particular *non-abelian* models?

Work in Progress:
Hamdan & PN
Jones, PN, Penneys, Wallick

What are generic features?

Many interesting properties can be proven in concrete models (and sometimes that they are stable). Are these generic?

What are generic features?

Many interesting properties can be proven in concrete models (and sometimes that they are stable). Are these generic?

- Type of cone algebra

What are generic features?

Many interesting properties can be proven in concrete models (and sometimes that they are stable). Are these generic?

- Type of cone algebra
- (Approximate) Haag duality

What are generic features?

Many interesting properties can be proven in concrete models (and sometimes that they are stable). Are these generic?

- Type of cone algebra
- (Approximate) Haag duality
- Approximate split property

What are generic features?

Many interesting properties can be proven in concrete models (and sometimes that they are stable). Are these generic?

- Type of cone algebra
- (Approximate) Haag duality
- Approximate split property
- Modularity of tensor category

Relation to TEE

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

Relation to TEE

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

but:

$$\mathcal{R}_{AB} \subsetneqq \widehat{\mathcal{R}}_{AB}$$

Relation to TEE

Locality: $\mathcal{R}_{AB} \subset \widehat{\mathcal{R}}_{AB}$

but:

$$\mathcal{R}_{AB} \subsetneq \widehat{\mathcal{R}}_{AB}$$

$$[\widehat{\mathcal{R}}_{AB} : \mathcal{R}_{AB}]$$