Entanglement bootstrap and spurious topological entanglement entropy

Kohtaro Kato (Nagoya University)

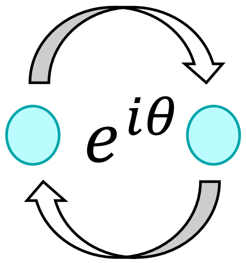
NAGOYA UNIVERSITY

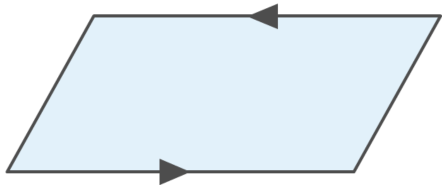
ICMAT QIT workshop, 3/17/2023

Topologically ordered phases (2D bosonic)

Non-trivial gapped quantum phases with the following properties:

- ✓ No local order parameter
- ✓ Topology-dependent degeneracy of ground states
- ✓ Anyonic excitations
- ✓ Protected gapless edge modes (if chiral)





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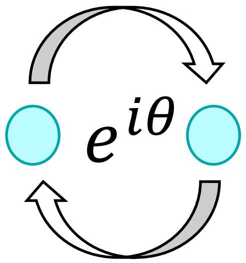

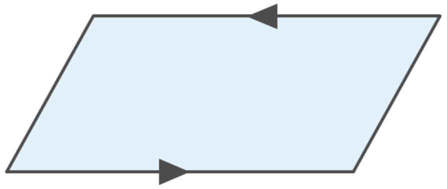

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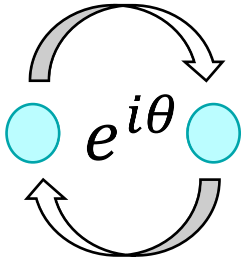
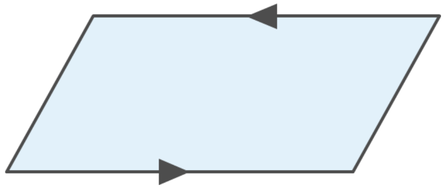




- Different topological orders are distinguished by algebraic theory of anyons

Unitary Modular Tensor Category (UMTC)

Anyon theory (UMTC)

➢ **Charges (Superselection sectors)**

possible types of quasiparticles (anyons)

$$\mathcal{L} = \{1, a, b, c, \ldots\}, \quad |\mathcal{L}| < \infty$$

➢ **Fusion rules**

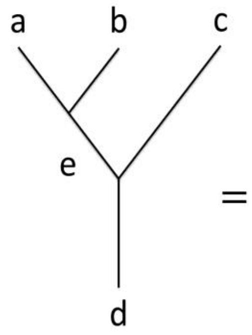
possible total charge of two charges

$$a \times b = \sum_c N_{ab}^c\, c, \quad N_{ab}^c \in \mathbb{Z}_{\geq 0}$$

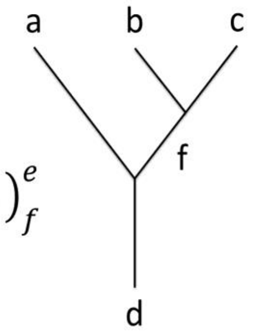
cf. fusion of SU(2)-spins $\frac{1}{2} \times \frac{1}{2} = 0 + 1$

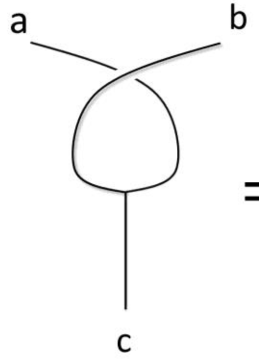
➢ ***F*** **and** ***R*** **matrices**

specify braiding statistics



$= \sum_f (F_{abc}^d)_f^e$





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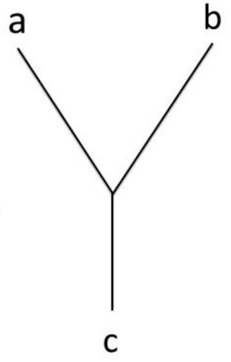
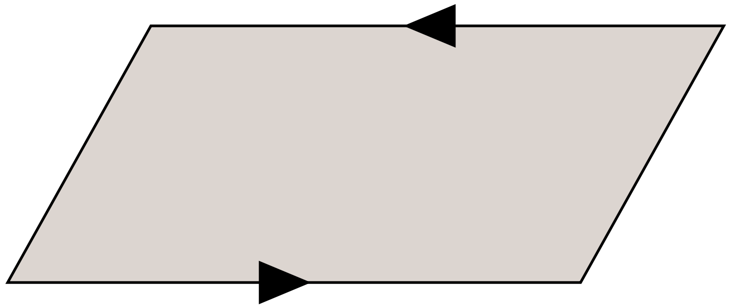


Chiral central charge

Chiral topological systems have gapless edge modes characterized by **chiral central charge c_-**



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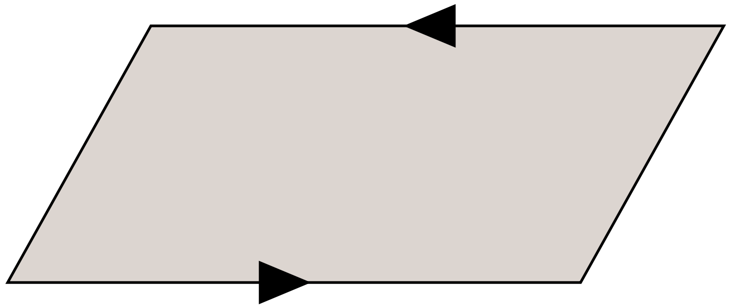

The gapless edge modes are **topologically protected** and described by CFT.

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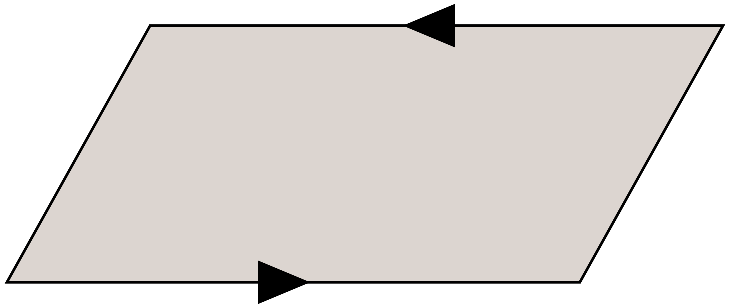

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c_- is another important topological data in addition to UMTC.

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Common belief:
2D bosonic topological orders are classified by $(UMTC, c_{-})$

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[Shi, Kato, Kim \`20]:

Extracting (a part of) UMTC from **a bulk GS** satisfying some entanglement properties.

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- An “axiomatic” approach for (finite) gapped spin systems.

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Problem:

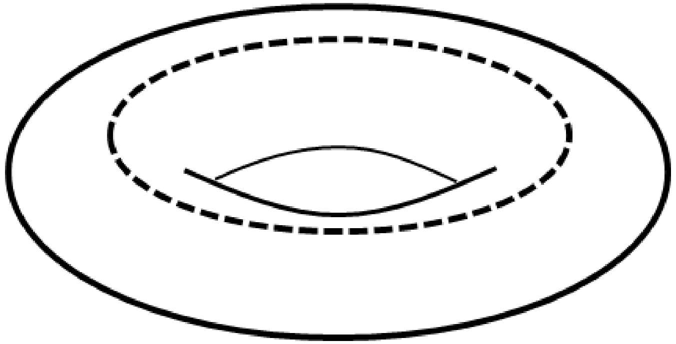
an axiom does not hold in some 2D gapped systems

(discussed in the second half of this talk)

Setting

Λ: a finite set of sites defined on a closed 2D manifold

$$\mathcal{H} = \bigotimes_{i\in\Lambda} \mathcal{H}_i\,, \qquad \dim\mathcal{H}_i < \infty.$$



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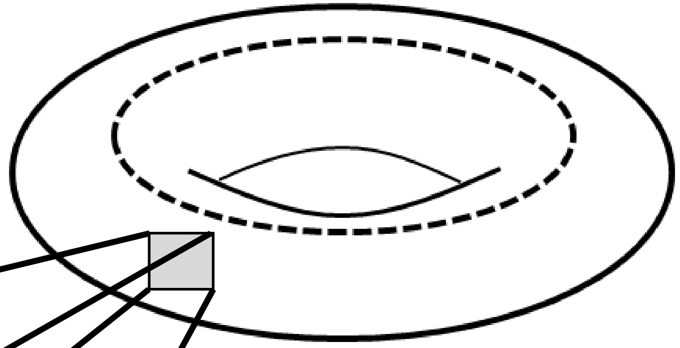
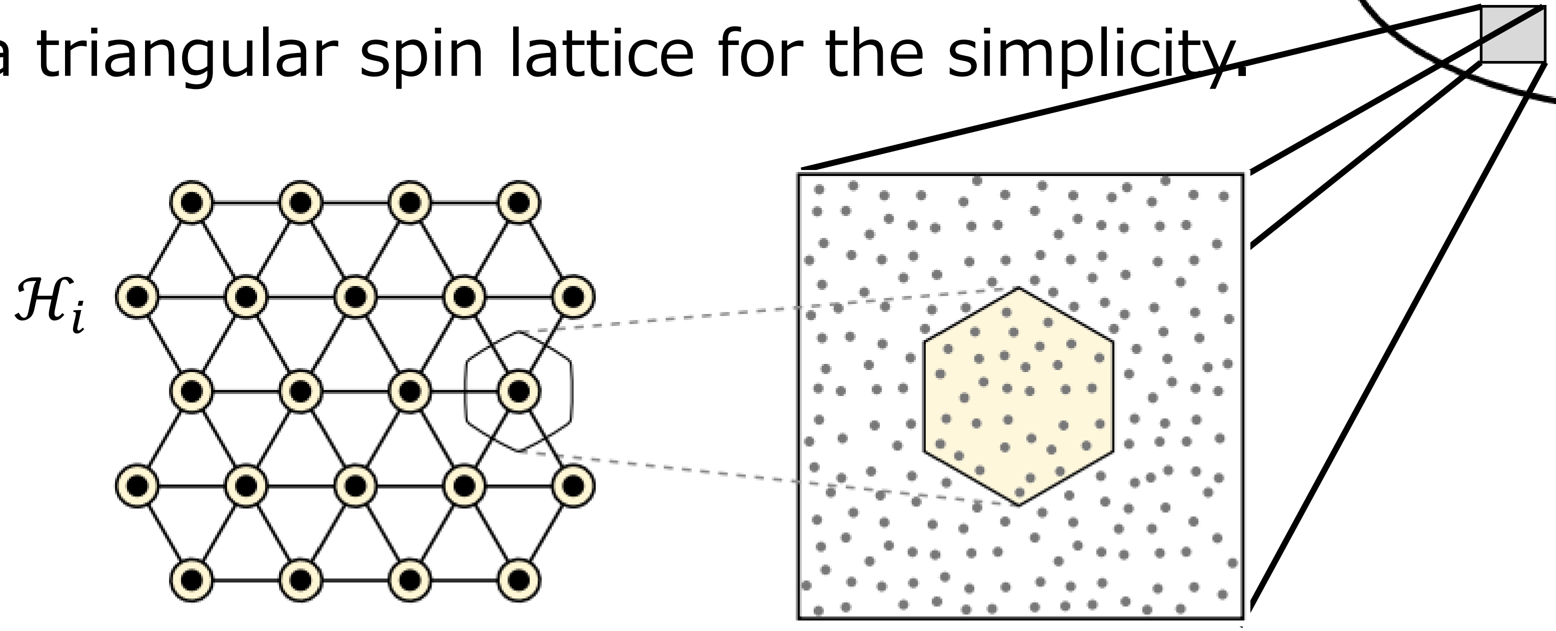
- Assume a triangular spin lattice for the simplicity.



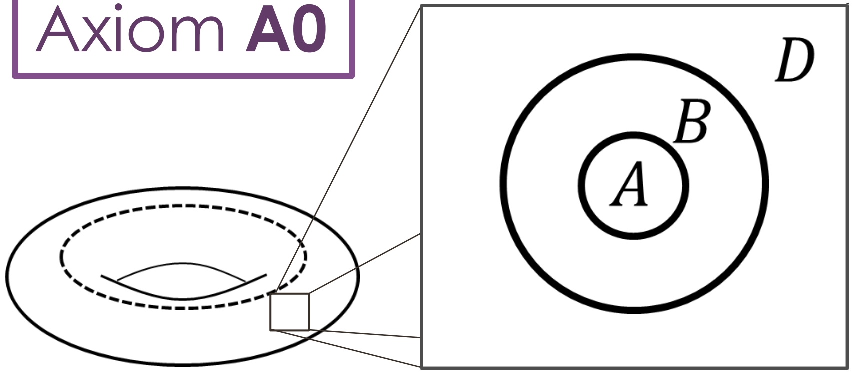


Two axioms

Consider a state ρ on Λ satisfying the following two "axioms" everywhere.

$$S(A)_\rho \coloneqq -\mathrm{tr}\rho_A \log \rho_A \qquad S(A|B)_\rho \coloneqq S(AB)_\rho - S(B)_\rho$$

Axiom **A0**



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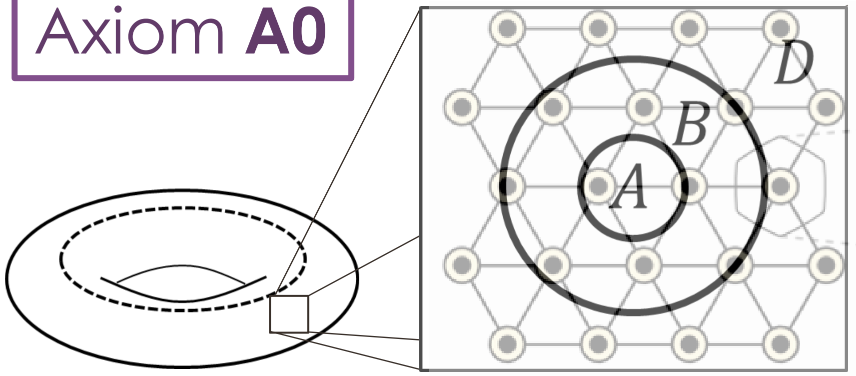

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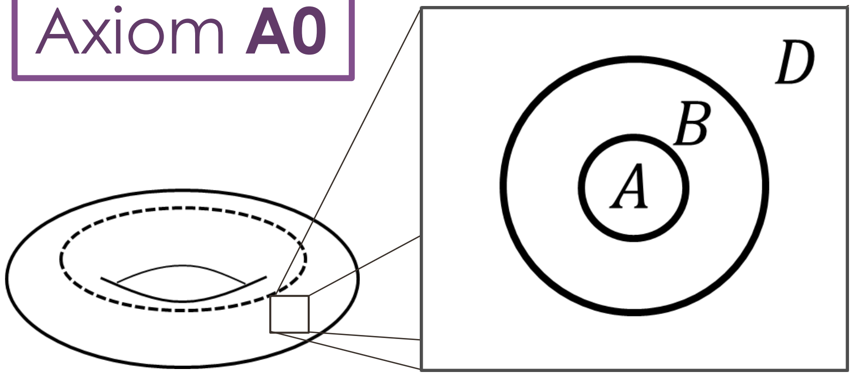

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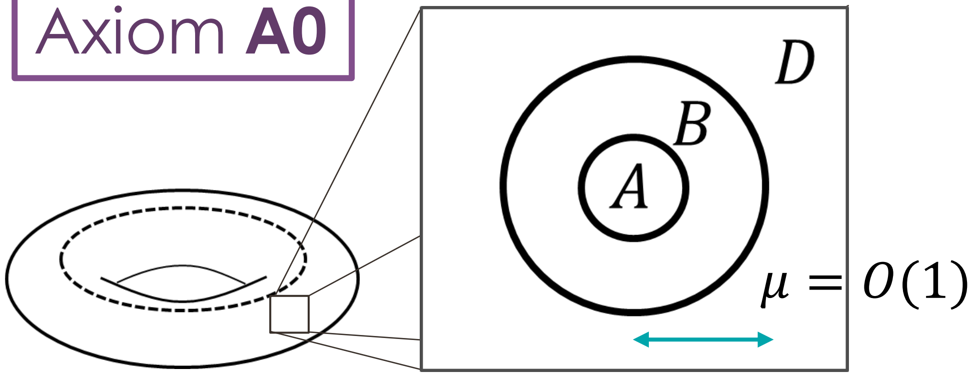

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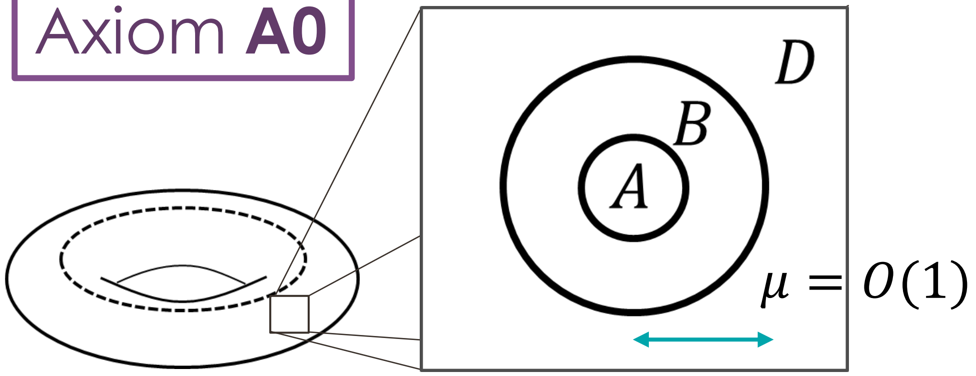

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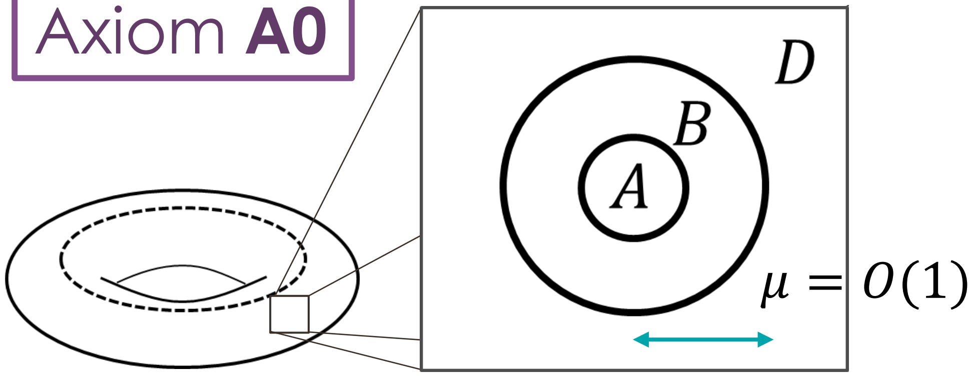

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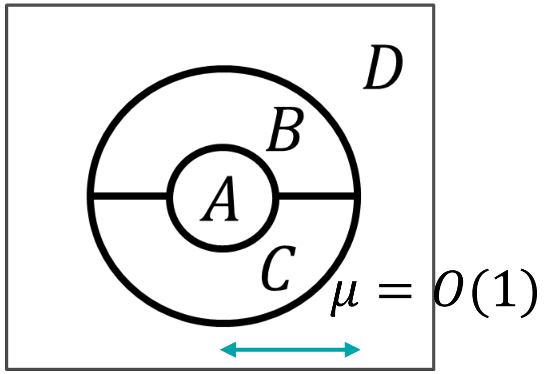

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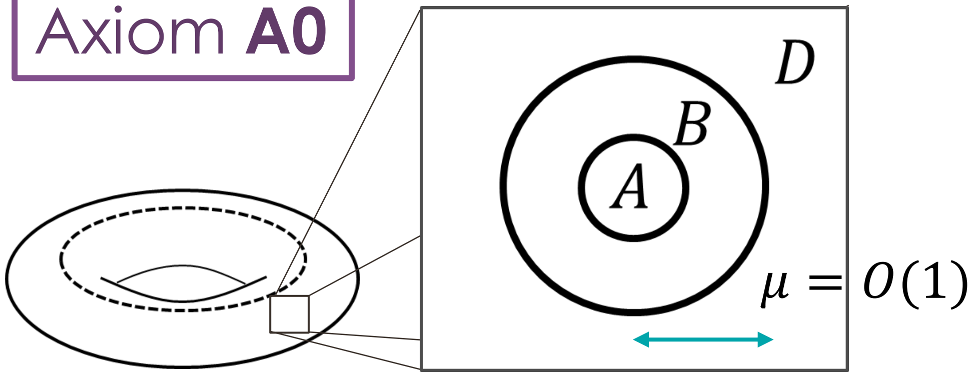

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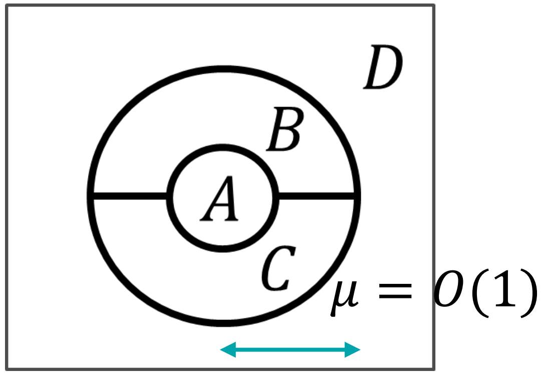

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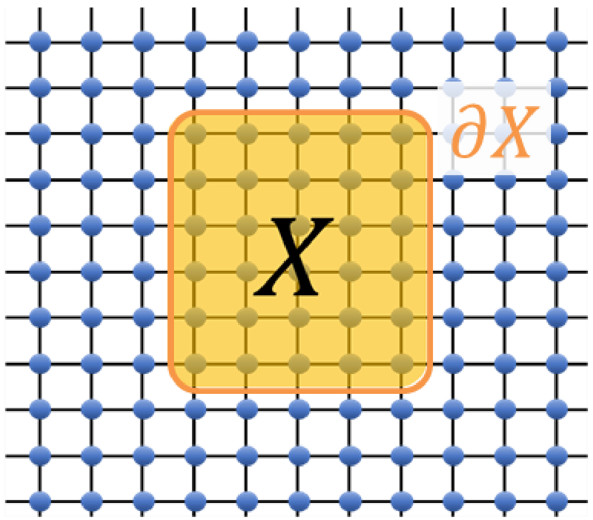
We call such ρ **a reference state**.

Area law

The axioms are motivated by area law of entanglement in gapped systems.

Area law (general form)

$$S(X)_\rho = O(|\partial X|)$$



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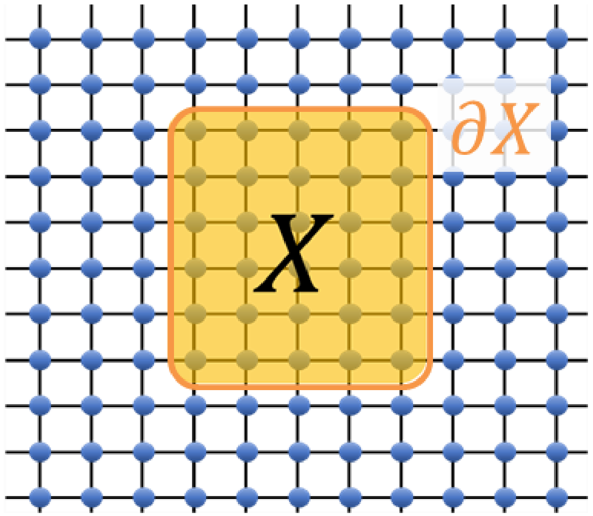

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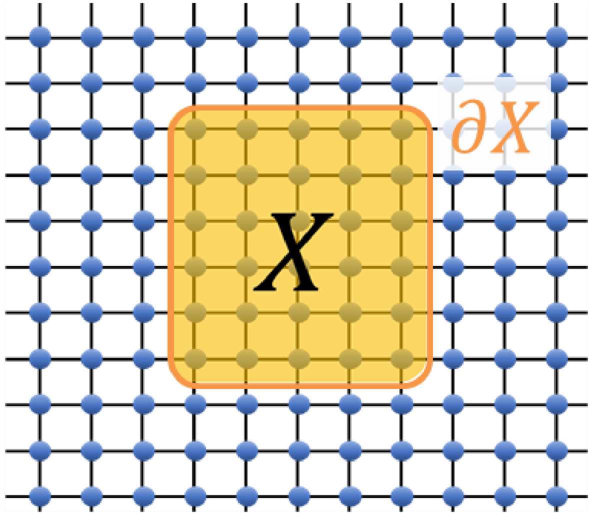

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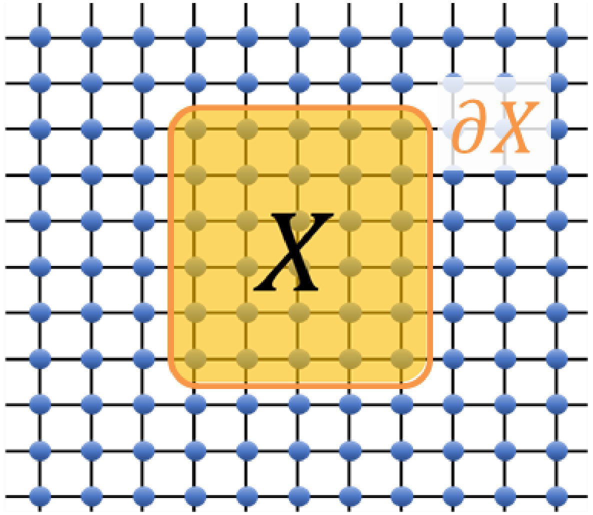

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γ: topological entanglement entropy

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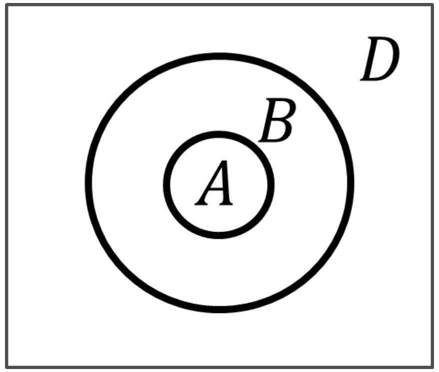

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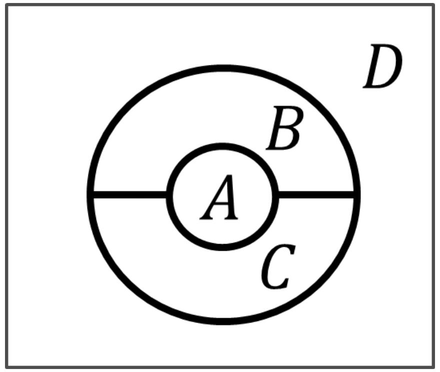

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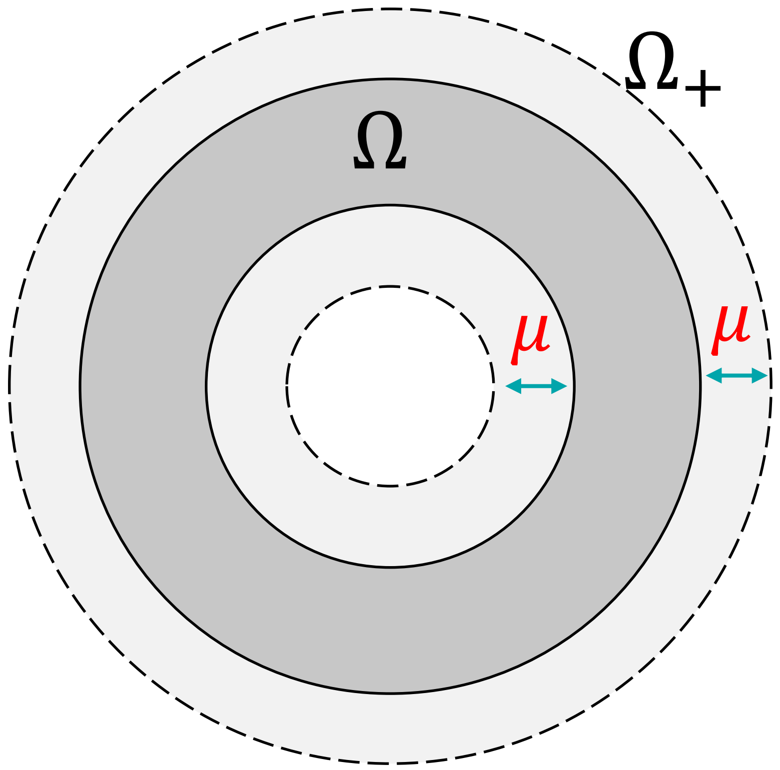

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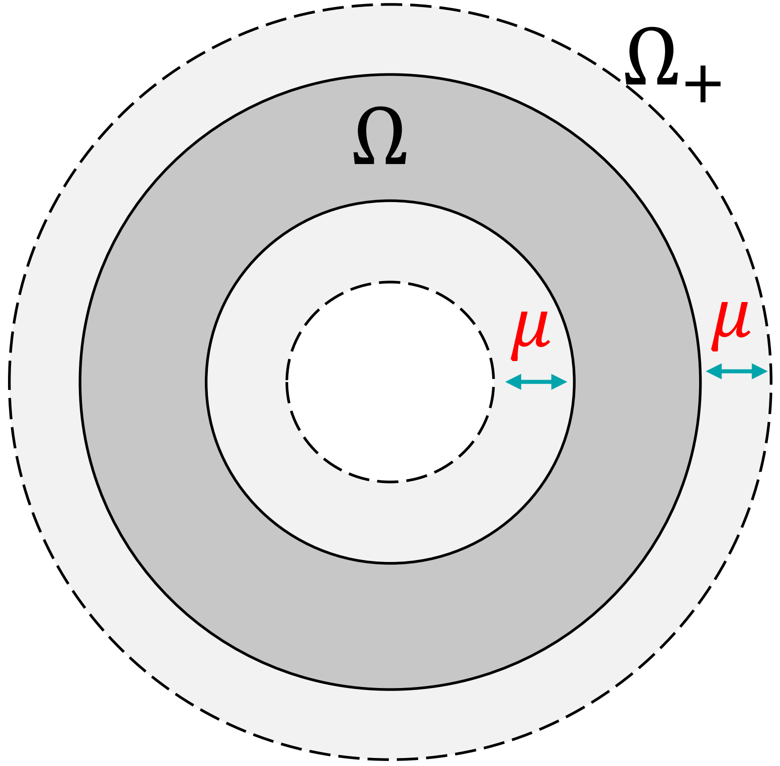

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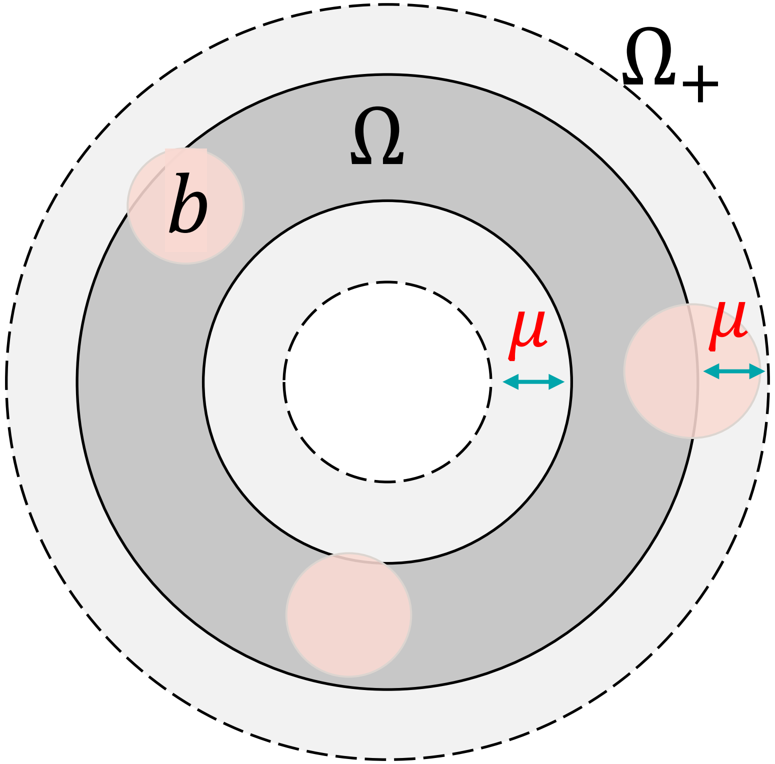

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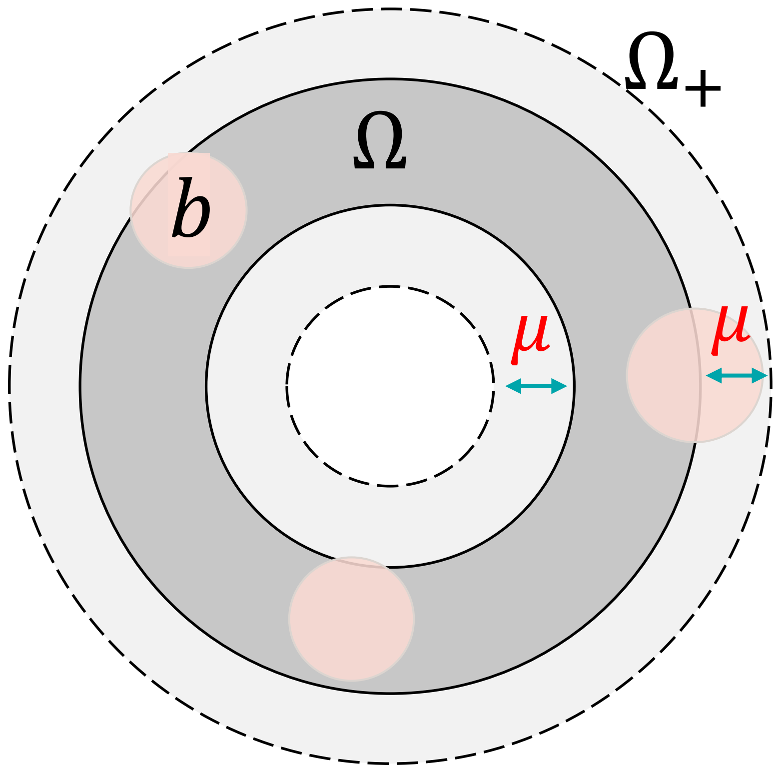

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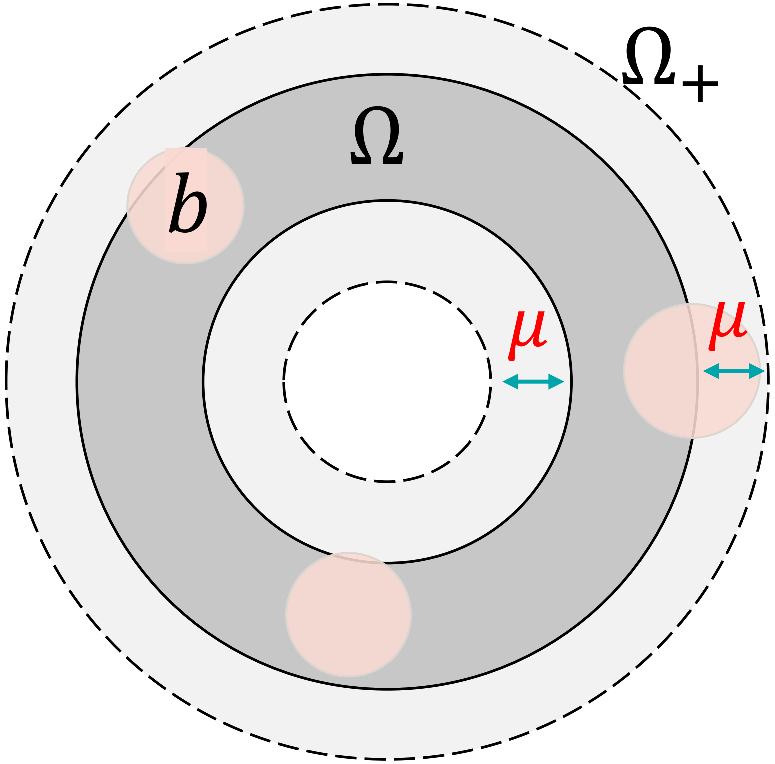
Tracing out $\Omega_+ \setminus \Omega$



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Information convex set [Kim,`15][Shi, KK, Kim '20]:

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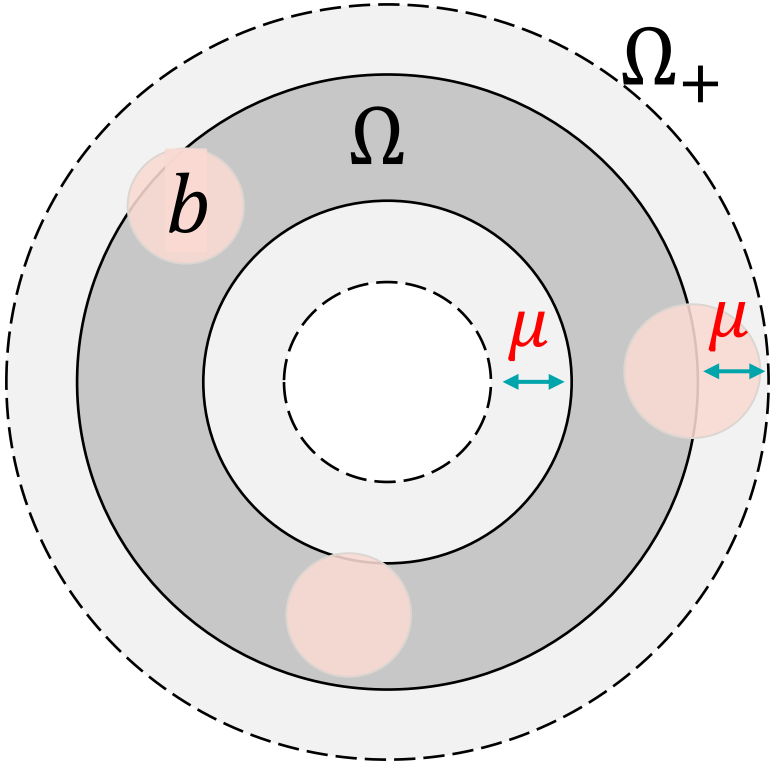

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Isomorphism theorem

How does the structure of $\Sigma(\Omega)$ depend on Ω?

Theorem (informal)

If Ω^0 and Ω^1 are connected by *local deformations* $\{\Omega^t\}$,there is a bijective CPTP-map

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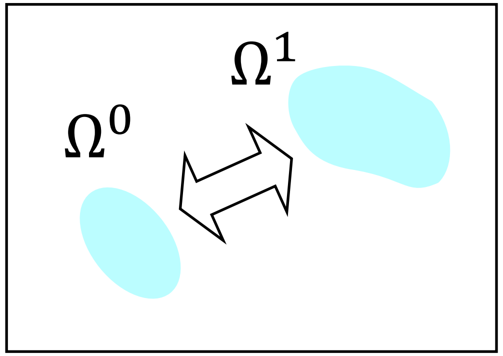

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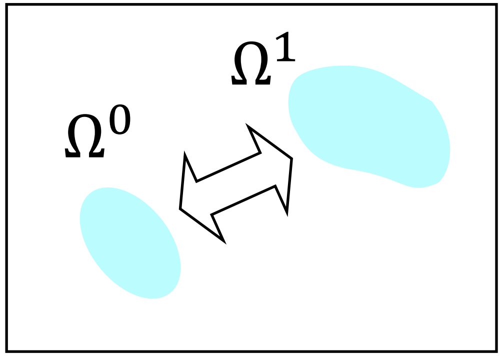

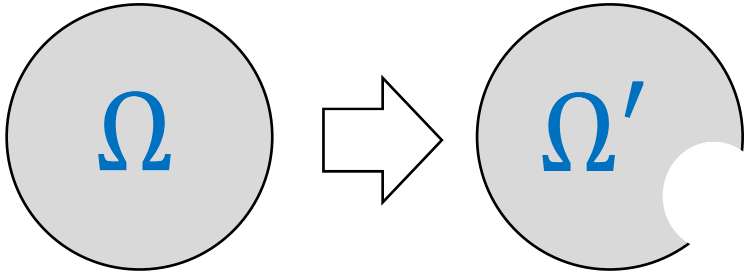

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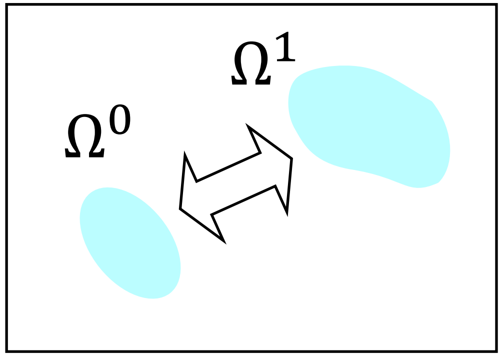
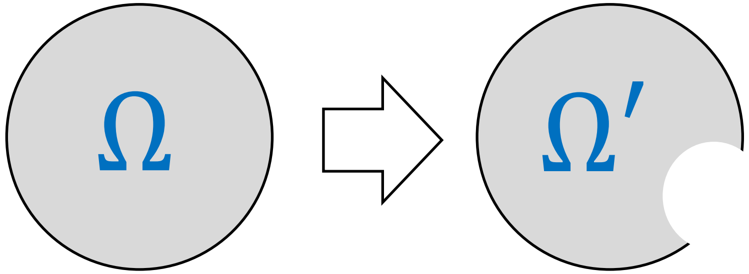
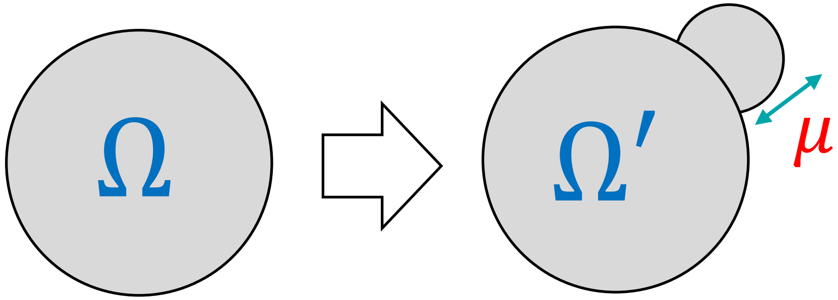






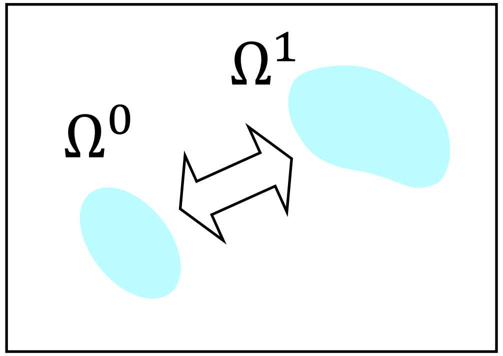
Isomorphism theorem

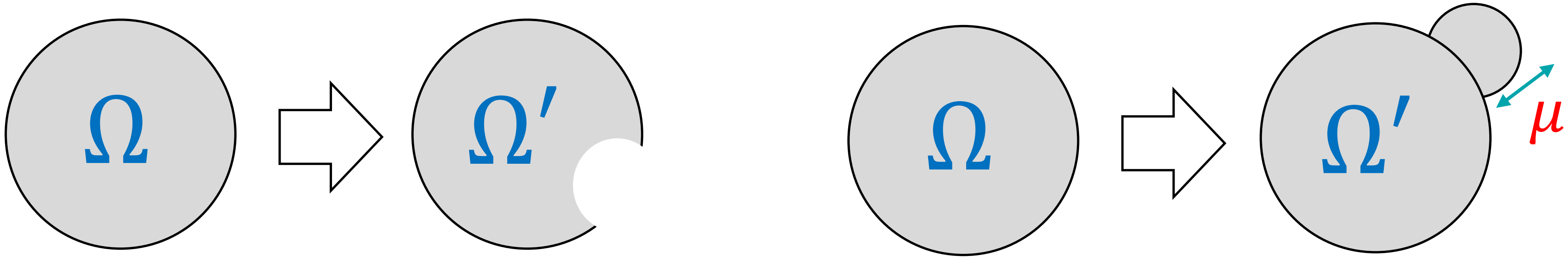
How does the structure of $\Sigma(\Omega)$ depend on Ω?

Theorem (informal)

If Ω^0 and Ω^1 are connected by *local deformations* $\{\Omega^t\}$,there is a bijective CPTP-map

$$\Phi_{\{\Omega^t\}}: \Sigma(\Omega^0) \to \Sigma(\Omega^1)$$





- The isomorphism preserves the entropy difference

$$S(\sigma) - S(\omega) = S\big(\Phi(\sigma)\big) - S\big(\Phi(\omega)\big) \quad \forall \rho, \sigma \in \Sigma(\Omega)$$

Key idea : quantum Markov states

A tripartite state ρ_{ABC} is called a **quantum Markov state (QMS)** if

$$I(A:C|B)_\rho = 0.$$

Conditional mutual information

$$I(A:C|B)_\rho := S(AB)_\rho + S(BC)_\rho - S(B)_\rho - S(ABC)_\rho \geq 0.$$

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ρ_{ABC} is a QMS iff there is a CPTP-map $\mathcal{R}_{B\to BC}$ s.t.

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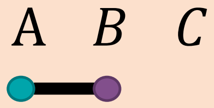

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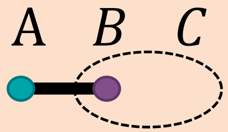

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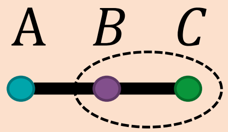

Axiom A1 and quantum Markov states

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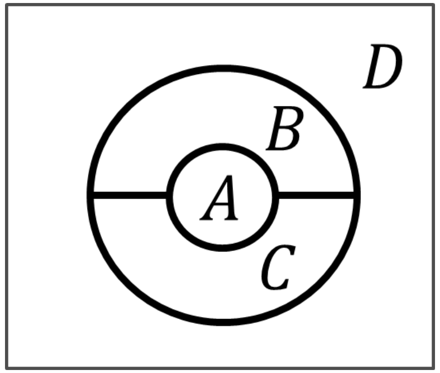

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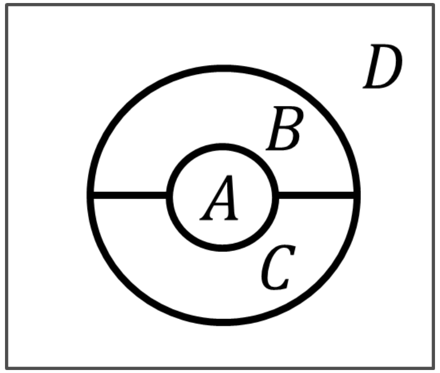

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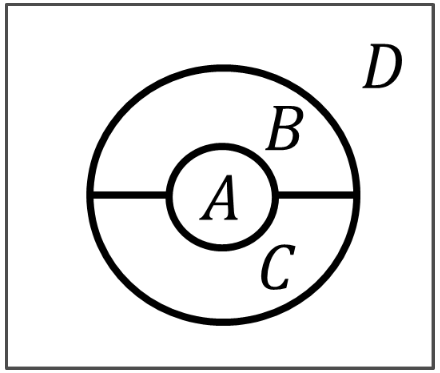

$$S(A|B)_\rho + S(A|C)_\rho = 0$$

$$I(A:D|B)_\rho = I(A:D|C)_\rho = 0$$

ρ_{ABD}, ρ_{ACD} are **QMS**

Key idea : merging QMS

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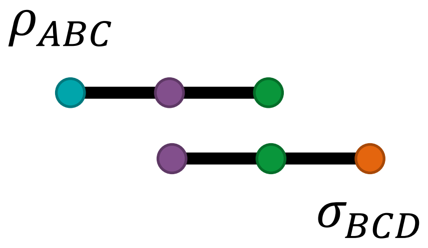
Two compatible quantum Markov states can be “merged” as a longer QMS !



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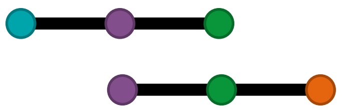

Merging lemma **[KK, Furrer, Murao ‘16][Shi, KK, Kim ‘20]:**

Consider a set of states $\mathcal{S} = \{\rho_{ABC}\}$ and σ_{BCD} such that $\rho_{BC} = \sigma_{BC}$ and

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Then, there exists a unique set of states $\{\tau^\rho_{ABCD}\}$ which satisfy

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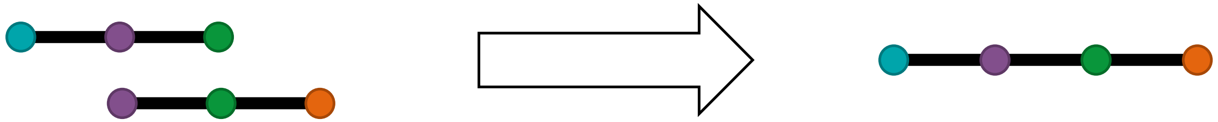
ρ_{ABC} $\quad\quad$ $\exists\tau_{ABCD} =: \rho_{A(BC)} \bowtie \sigma_{(BC)D}$



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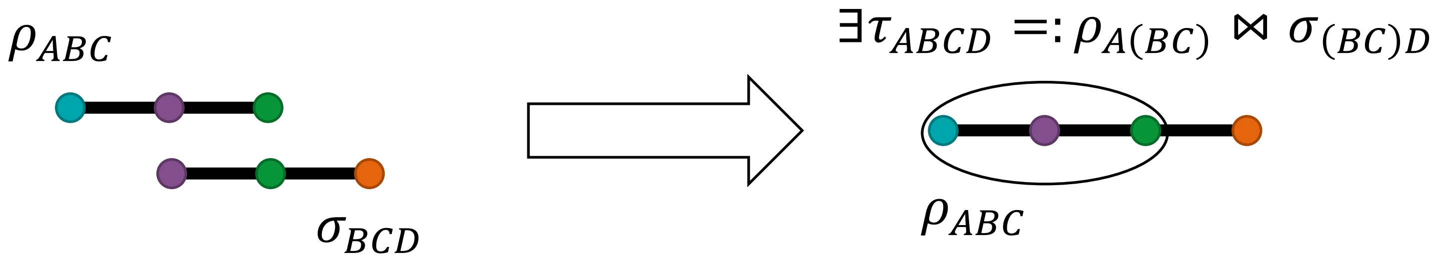

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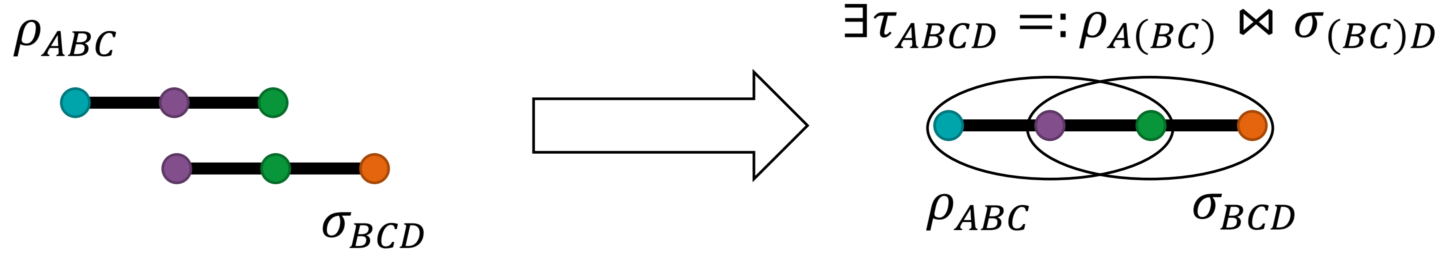

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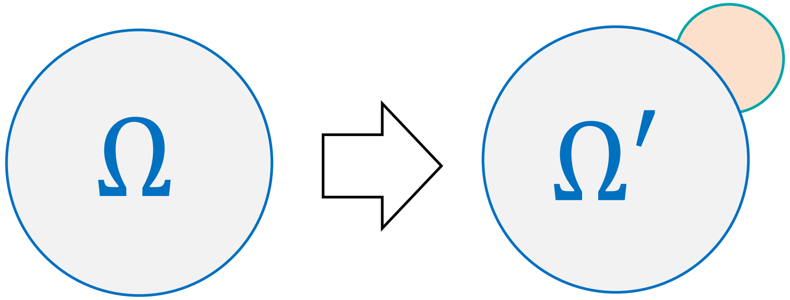

$\sigma_\Omega \in \Sigma(\Omega)$

ρ_{BCD}: the reference state

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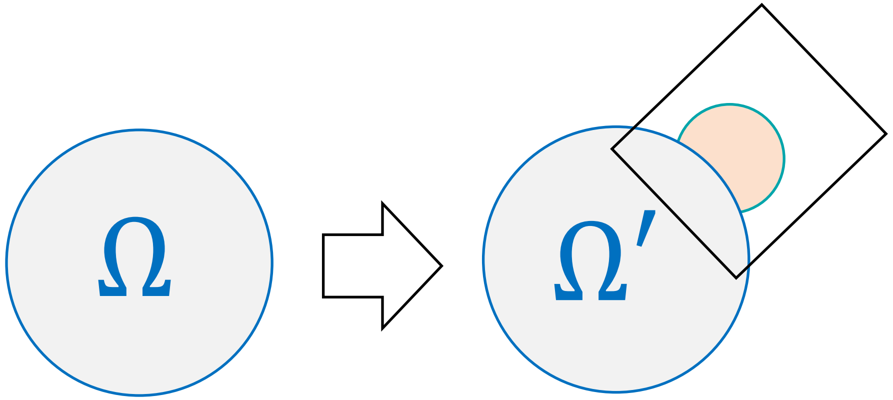

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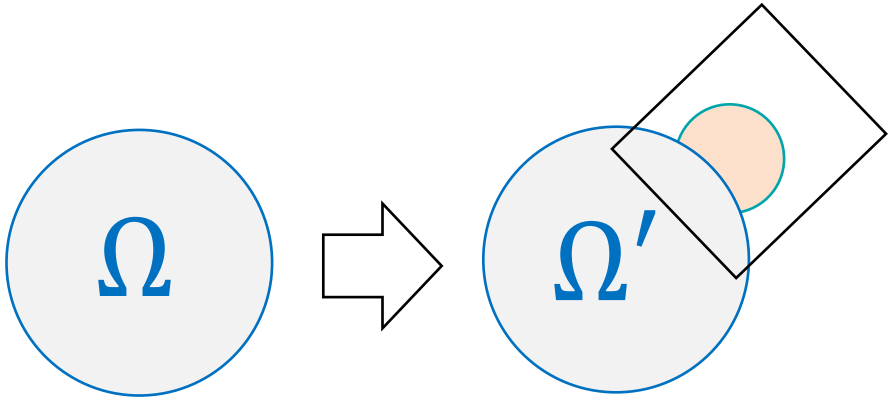

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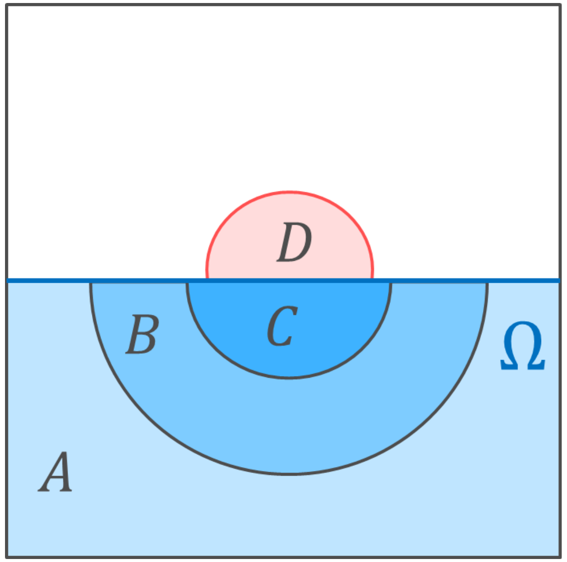

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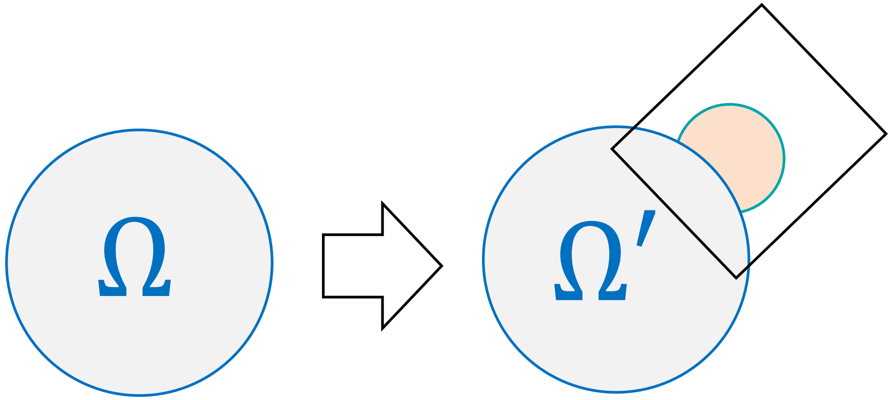

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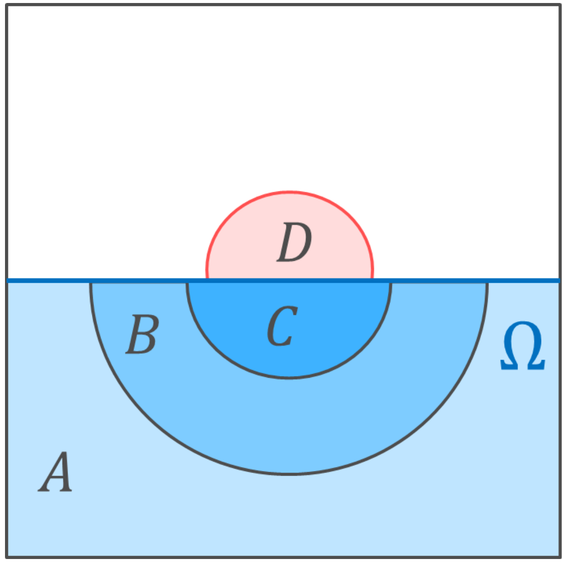

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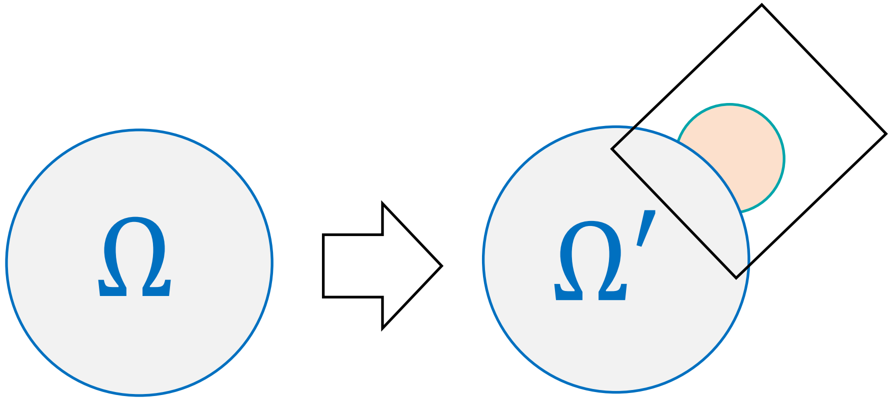

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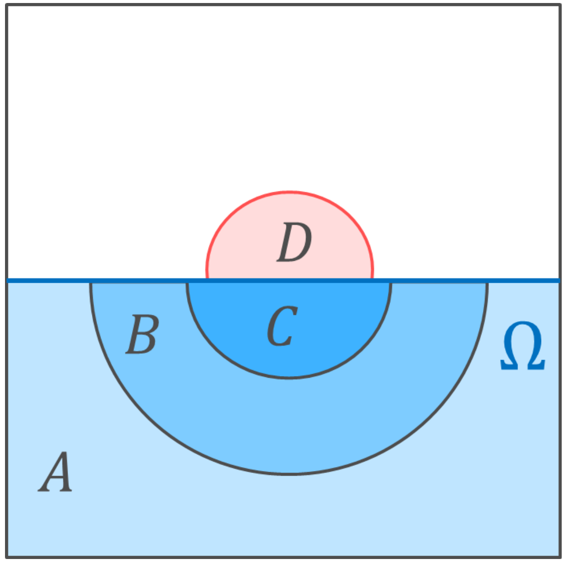

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The structure of $\Sigma(\Omega)$ only depends on the **topology** of Ω (if not winding).

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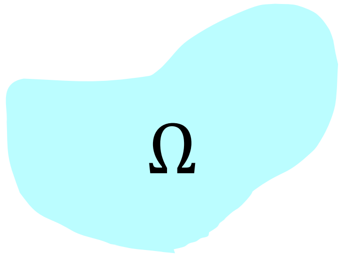

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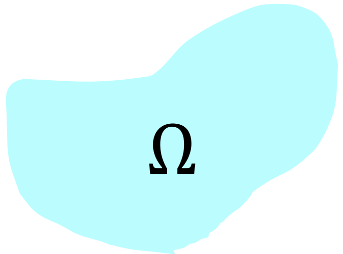

Only the reference state is allowed.

(cf. TQO condition**[Bravyi, Hastings '11]…**)

Structure of $\Sigma(\Omega)$: annulus

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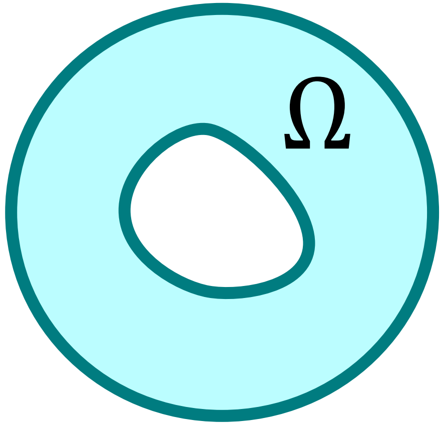
Theorem

If Ω is an annulus,

$$\sigma_\Omega = \bigoplus_a p_a \rho_\Omega^a, \qquad \forall \sigma_\Omega \in \Sigma(\Omega).$$

Each ρ_Ω^a is some state independent of σ_Ω.



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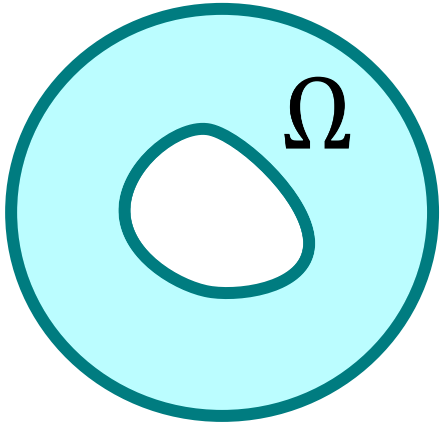

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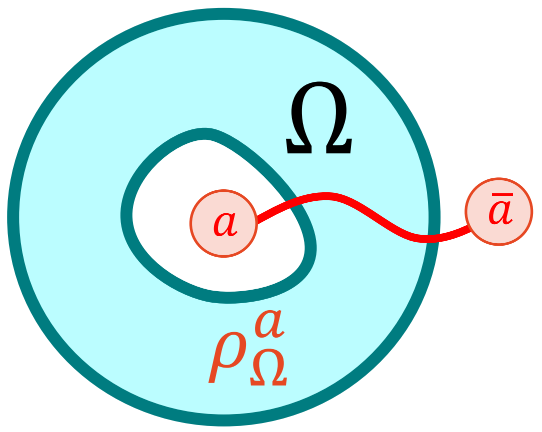

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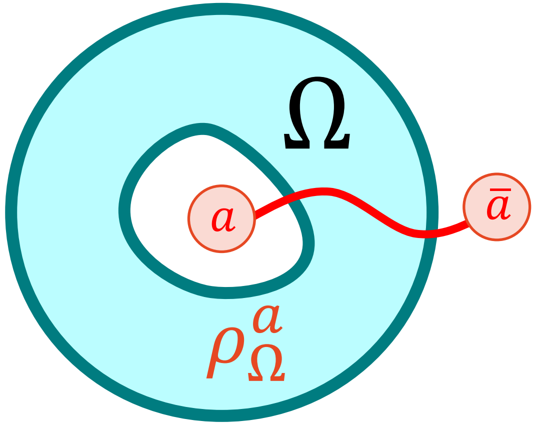

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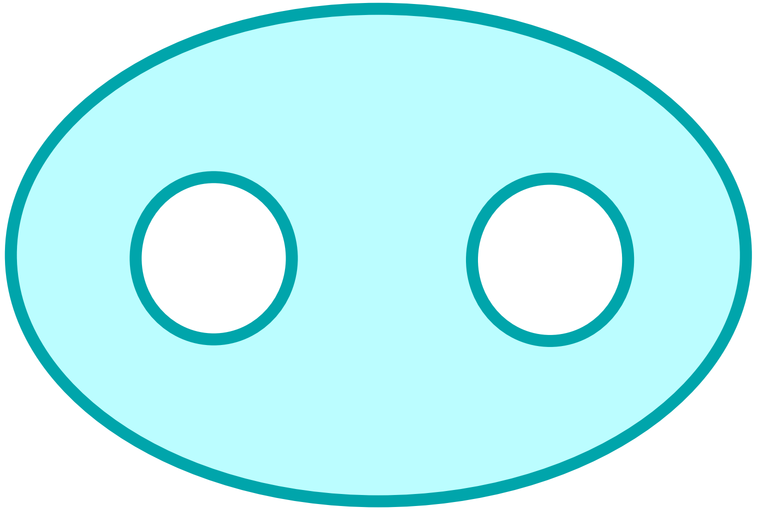
- The reference state ρ_Ω is an extreme point, defined as the "vacuum" $\rho_\Omega^1 \equiv \rho_\Omega \in \{\rho_\Omega^a\}$.

Structure of $\Sigma(\Omega)$: 2-hole disk

To define fusion rules, we consider a 2-hole disk Ω.



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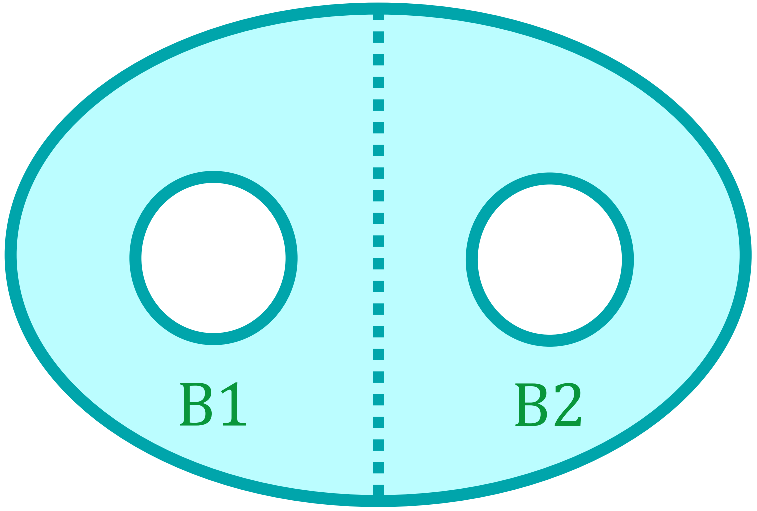

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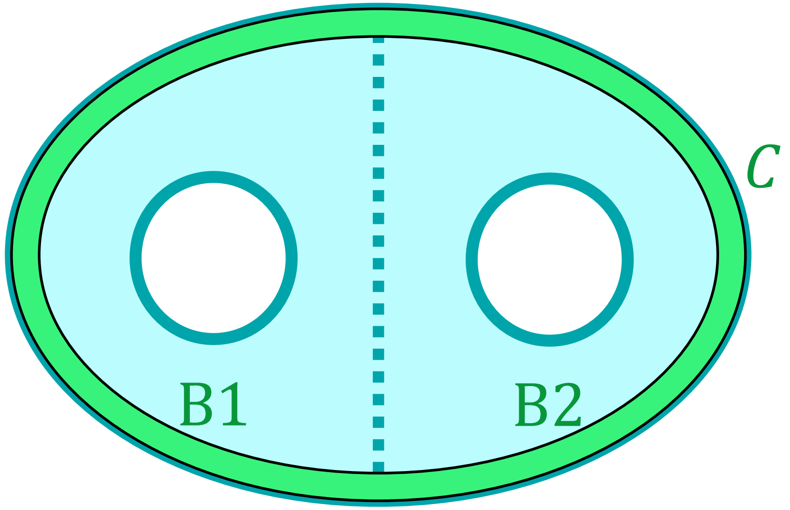

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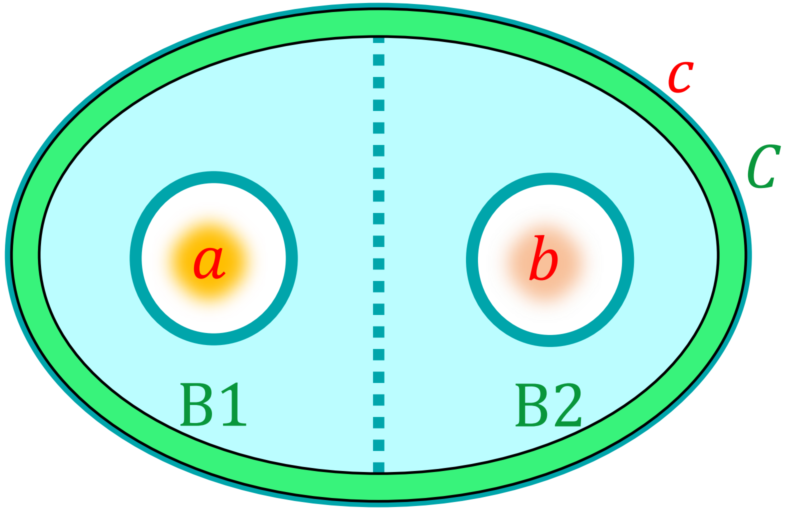

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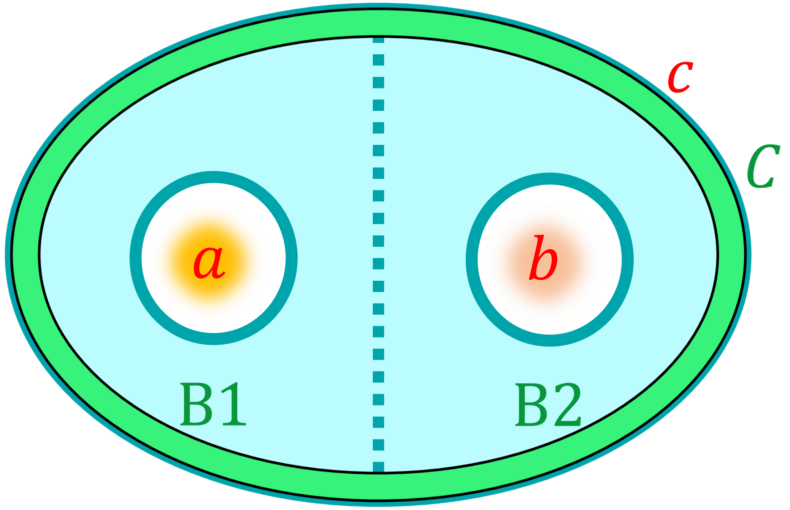

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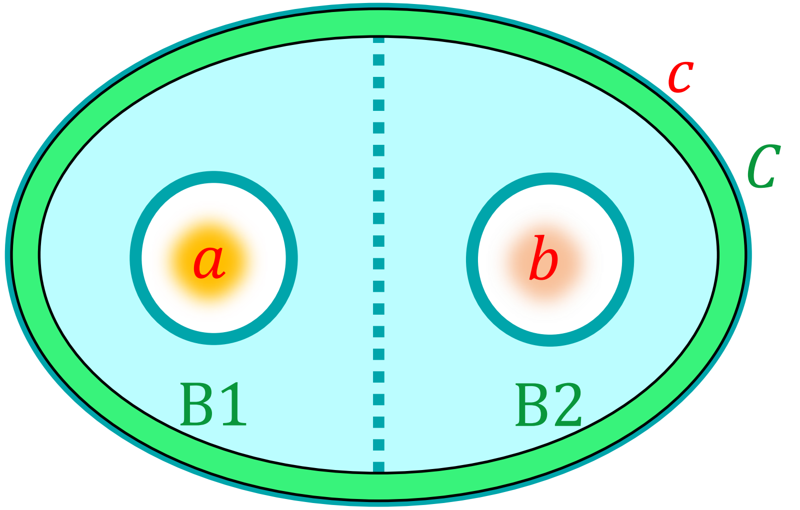

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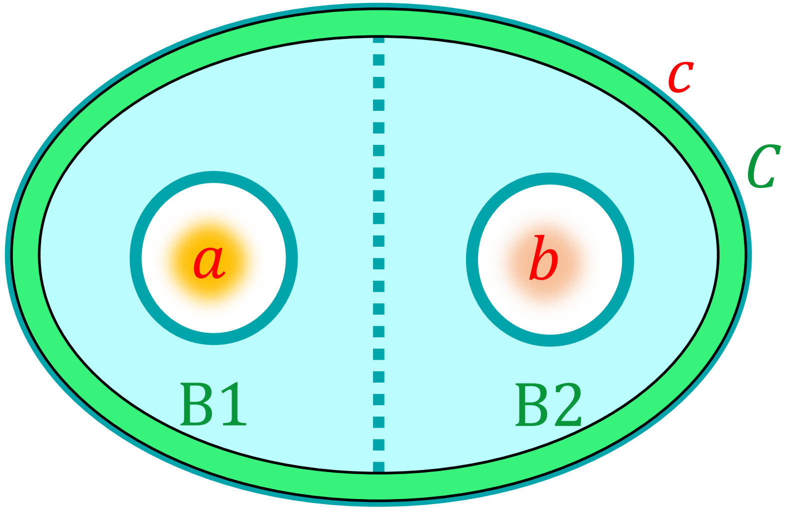

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We *define* the fusion multiplicity by $N_{ab}^{c} := \dim V_{ab}^{c}$

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^{c}\, c$$

Consistency with UMTC

In the anyon theory, the fusion multiplicities N_{ab}^{c} must satisfy the following rules.

1. $N_{ab}^{c} = N_{ba}^{c}$: commutativity of fusion rules
2. $N_{a1}^{c} = \delta_{a,c}$: vacuum
3. $N_{ab}^{1} = \delta_{b,\bar{a}}$: anticharge
4. $N_{ab}^{c} = N_{\bar{a}\bar{b}}^{\bar{c}}$: charge-anticharge duality
5. $\sum_i N_{ab}^{i} N_{ic}^{d} = \sum_j N_{aj}^{d} N_{bc}^{j}$: associativity

Consistency with UMTC

In the anyon theory, the fusion multiplicities N_{ab}^{c} must satisfy the following rules.

1. $N_{ab}^{c} = N_{ba}^{c}$: commutativity of fusion rules
2. $N_{a1}^{c} = \delta_{a,c}$: vacuum
3. $N_{ab}^{1} = \delta_{b,\bar{a}}$: anticharge
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Theorem

N_{ab}^{c} in our definition satisfies all the properties.

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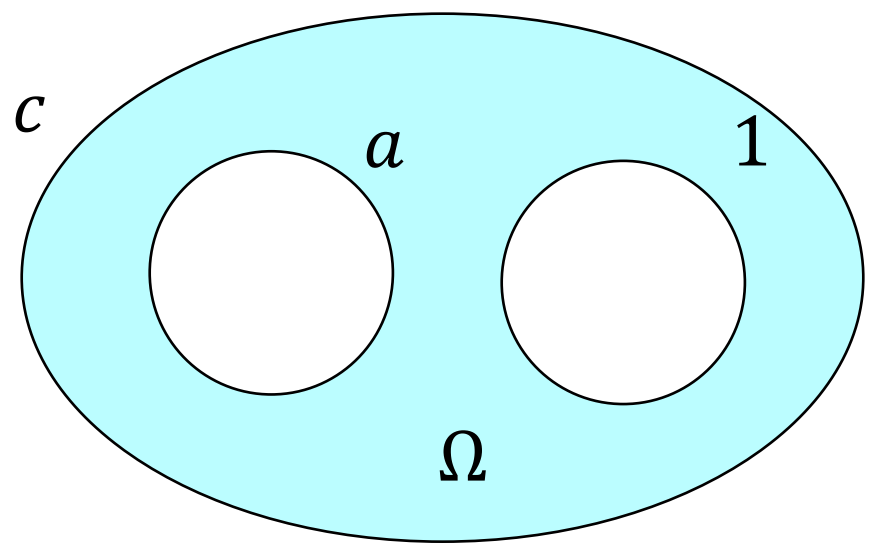
Consider a 2-hole disk with $(a, 1, c)$.

$$\sigma_{\Omega}^{(a,1,c)} \in \Sigma_{\mathrm{a1}}^{\mathrm{c}}(\Omega)$$



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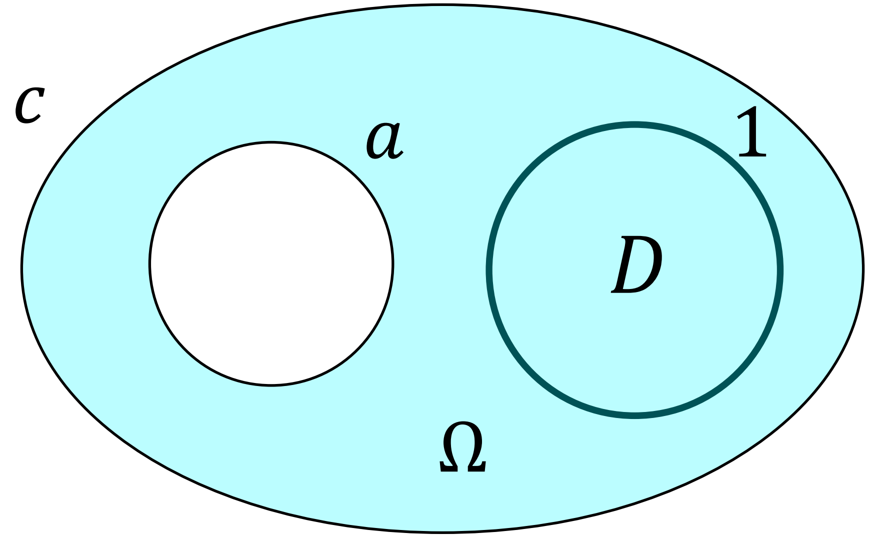

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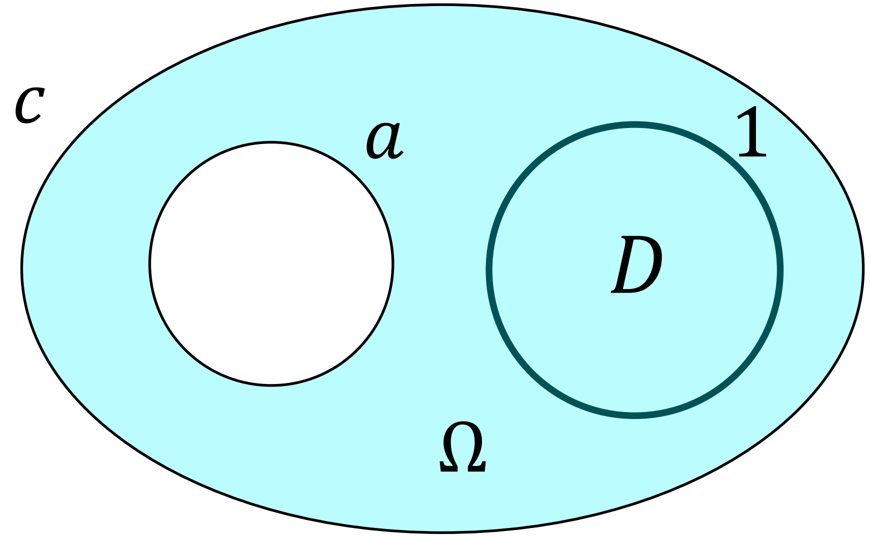

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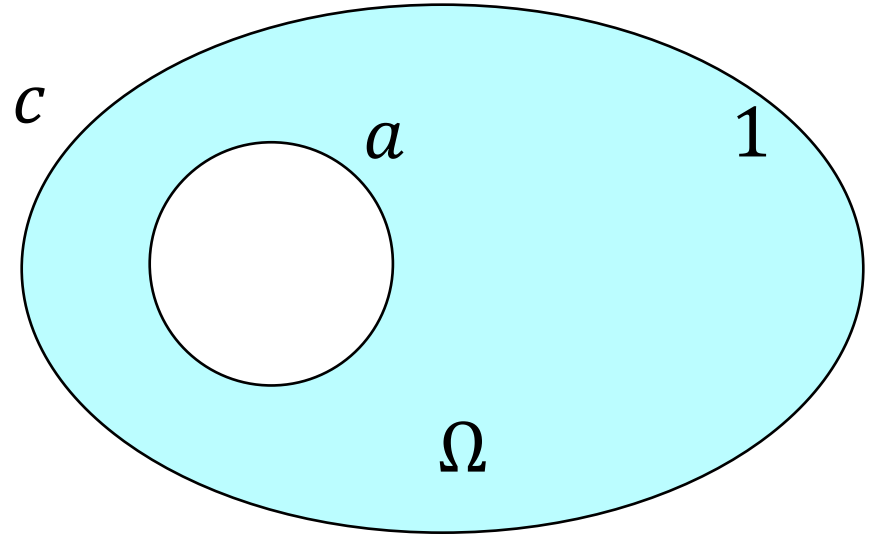

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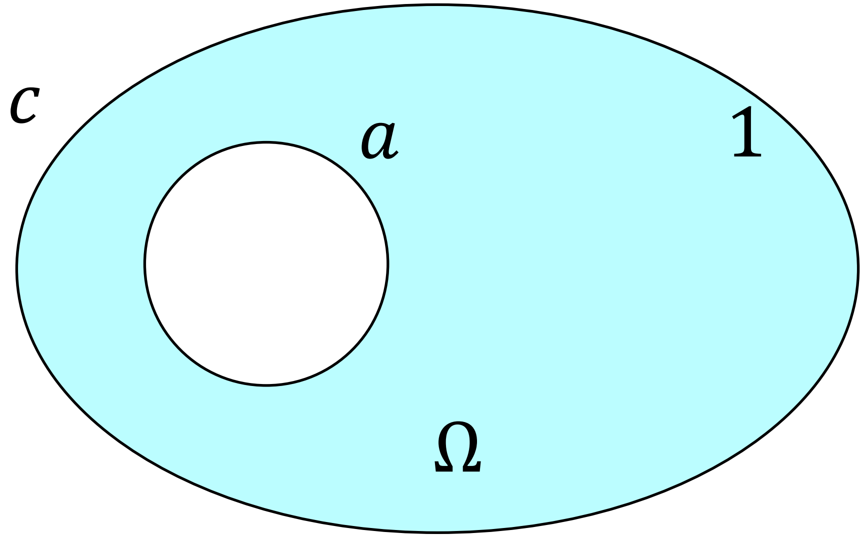
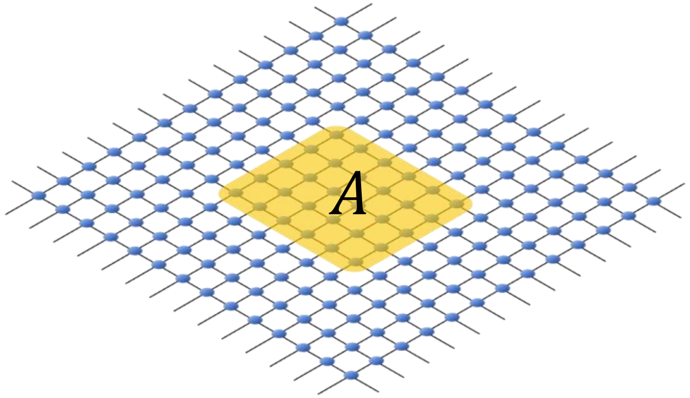


The value of TEE in entanglement bootstrap

The area law states that

$$S(A)_\rho = \alpha|\partial A| - \gamma \,.$$

γ: topological entanglement entropy (TEE)
[Kitaev, Preskill, ’06] [Levin, Wen ‘06]



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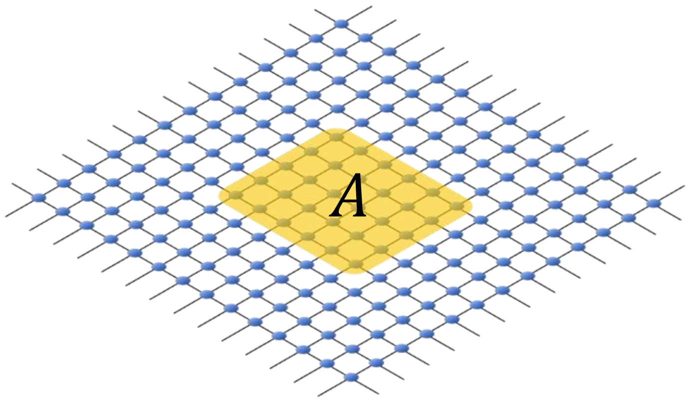

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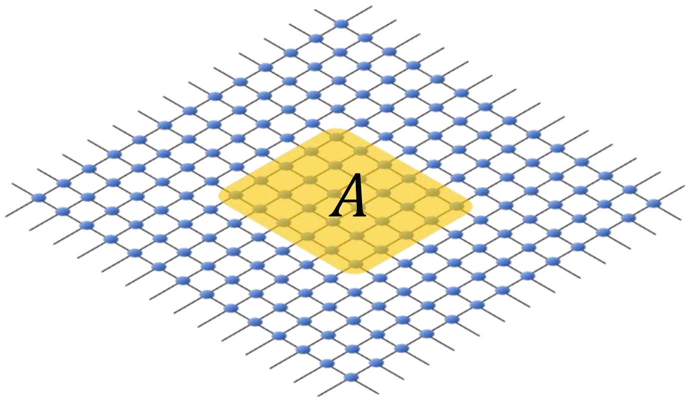

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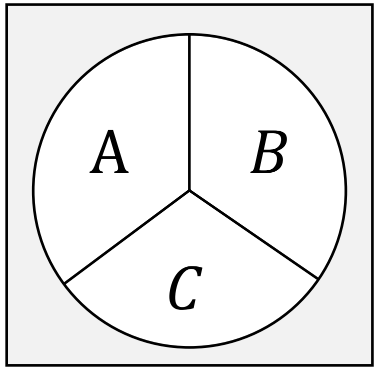

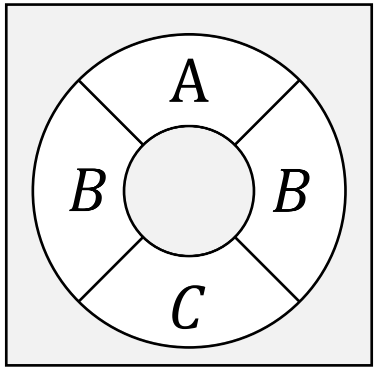

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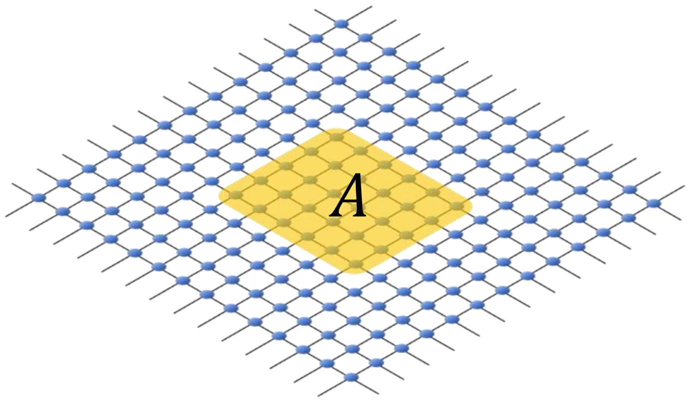

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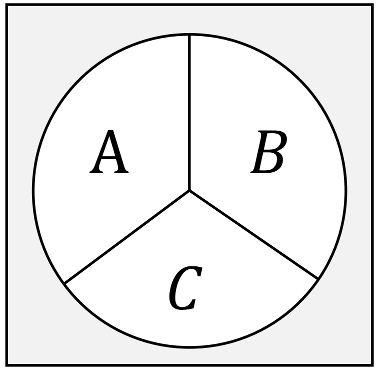

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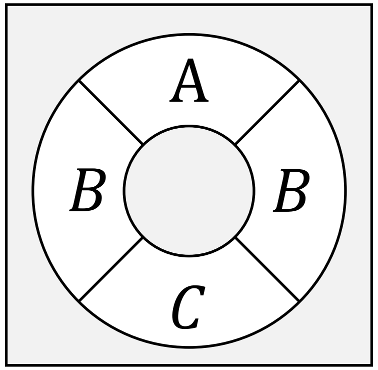

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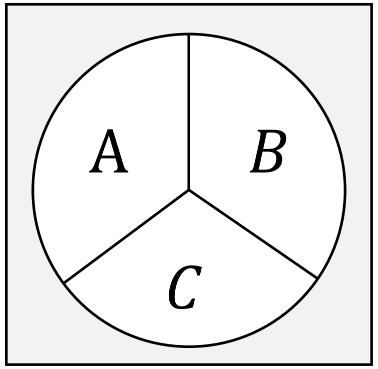
The value of TEE in entanglement bootstrap

KP partition



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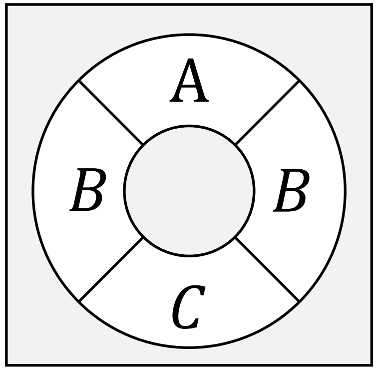

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$$\gamma = \log\mathcal{D}, \ \mathcal{D} := \sqrt{\textstyle\sum_{a\in\mathcal{L}} d_a^2}\,.$$

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$$d_a d_b = \sum_{c\in\mathcal{L}} N_{ab}^c d_c\,, \qquad d_a \in \mathbb{R}_{\geq 1}$$

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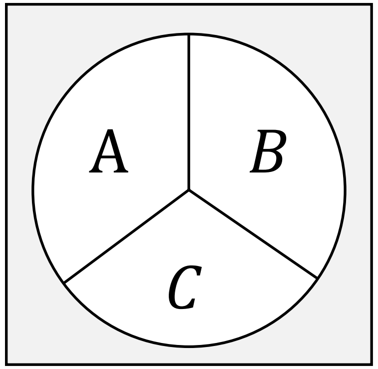

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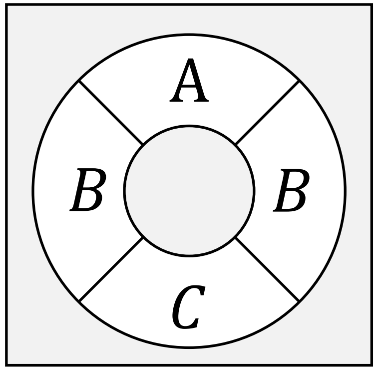

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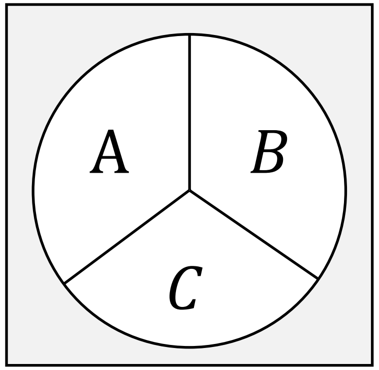
Theorem

$S_{\text{topo}} = \log\mathcal{D}$ for KP partition, $S_{\text{topo}} = \log\mathcal{D}^2$ for LW partition.

Chiral central charge from the bulk ?

[Shi, Kim, Kato, Albert, '22]



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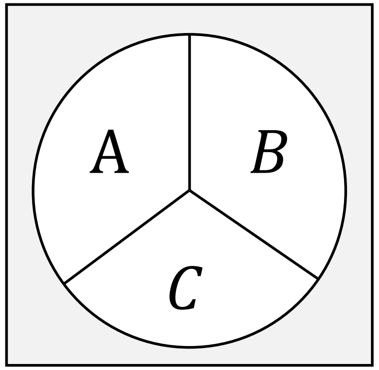

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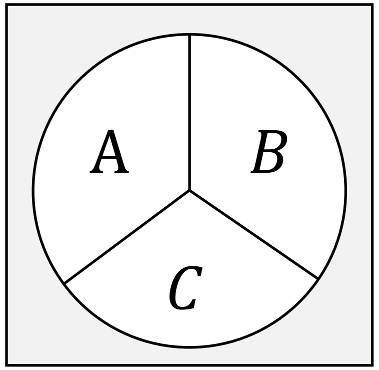

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1. The modular Hamiltonian $H^{\mathrm{mod}} := -\ln\rho$ is local.
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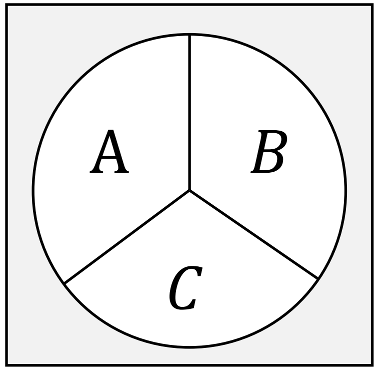

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Conjecture: For 2D gapped ground states,

$$J(A,B,C)_\rho = \frac{\pi}{3} c_- \,.$$

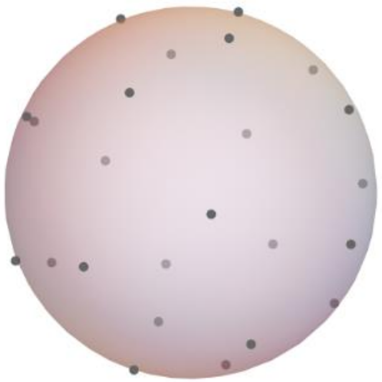
cf. Kane & Fisher

$$I_E = \frac{\pi}{12} c_- T^2$$

Numerical test of the conjecture

Discretized $\nu = \frac{1}{2}$ bosonic Laughlin state **[Nielsen, Cirac, Sierra, '12]**

$$\left|\Psi\left(N,\{z_j\}\right)\right\rangle = \sum_{\{s_i\}} c(\{s_i\})|s_1, s_2, \dots, s_N\rangle, \qquad c(\{s_i\}) = \delta_{\sum s_i, 0} \prod_{n<m}^{N} (z_n - z_m)^{\frac{1}{2}s_n s_m}.$$



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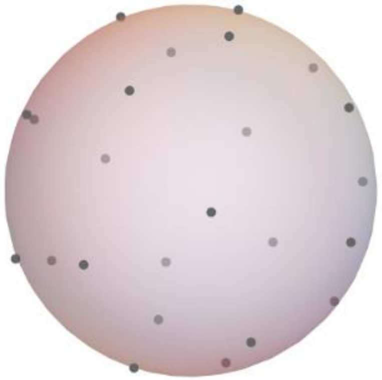

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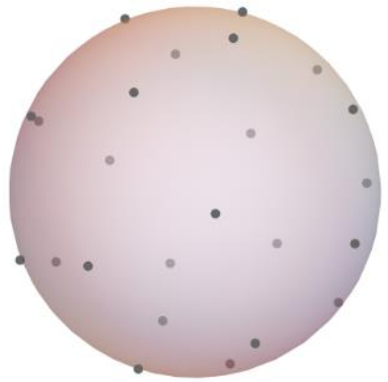

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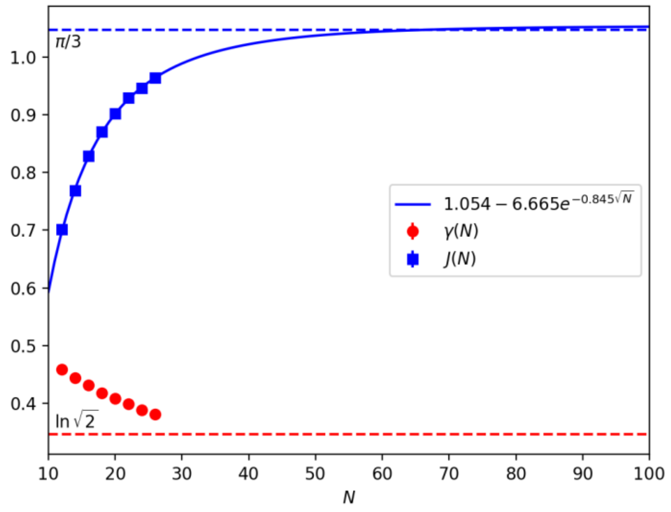

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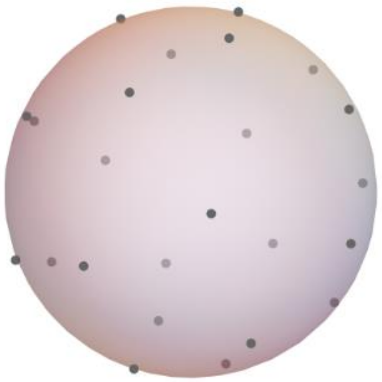

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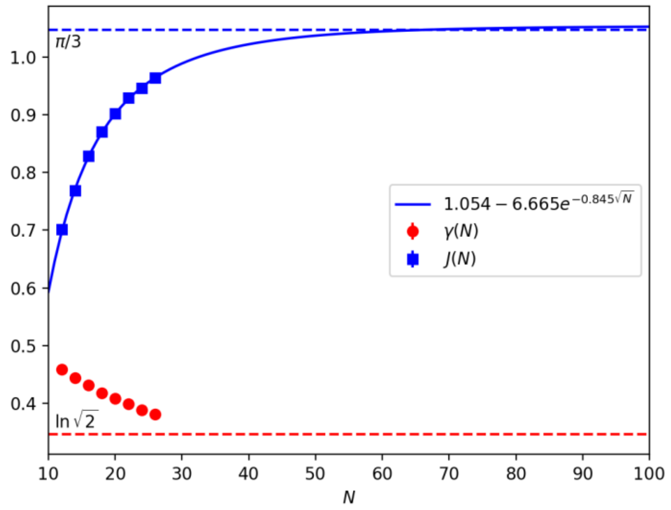

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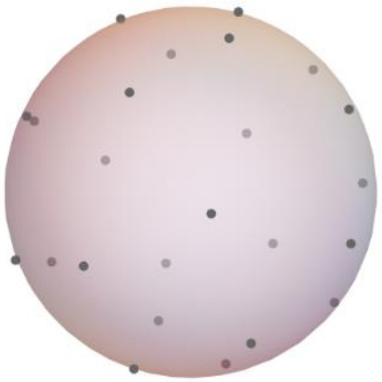

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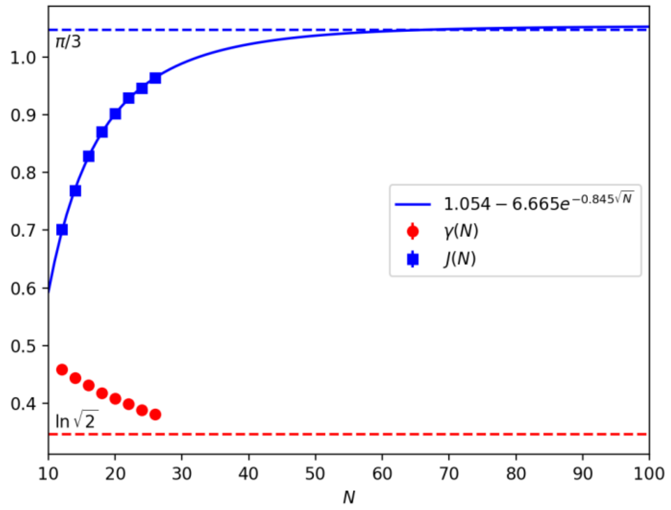

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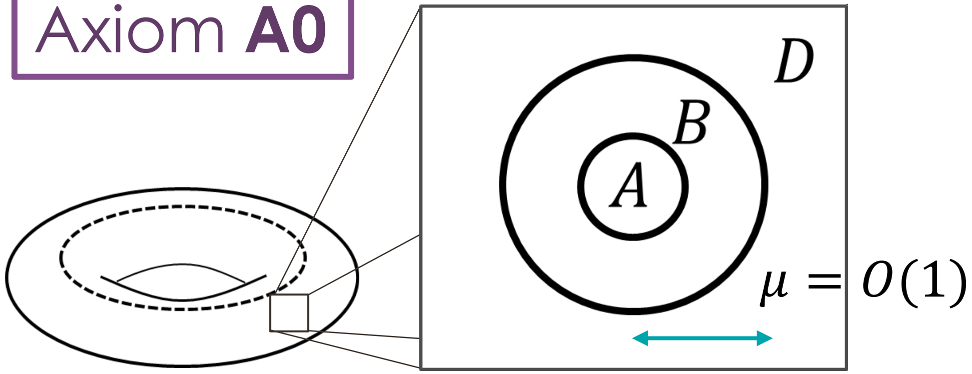
The conjectured formula also holds for free fermion models (p+ip SC). **[Fuji, private communication]**

Spurious TEE

Obstacle for a stability proof

Axiom **A0**



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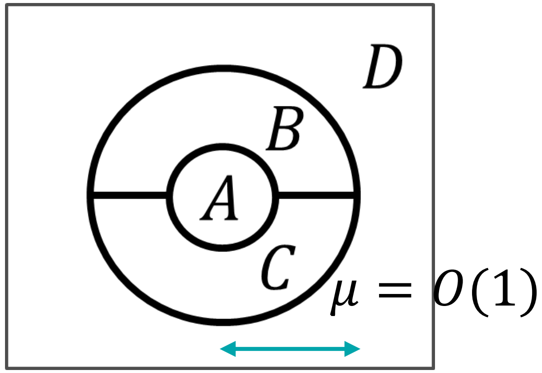

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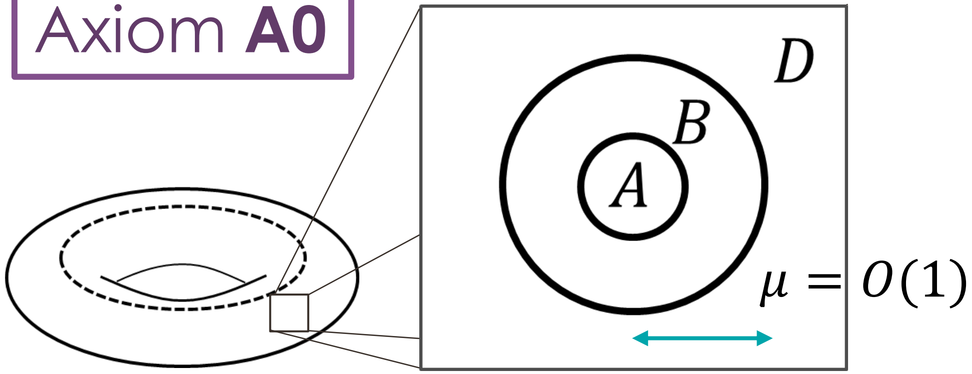

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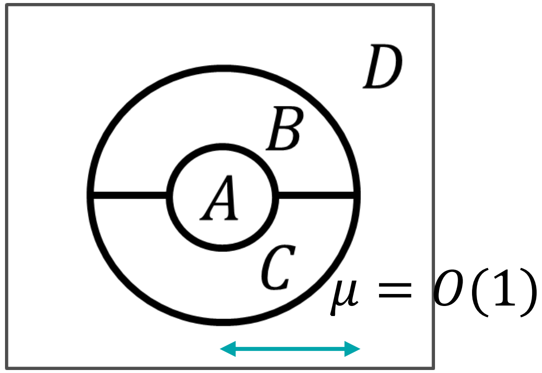

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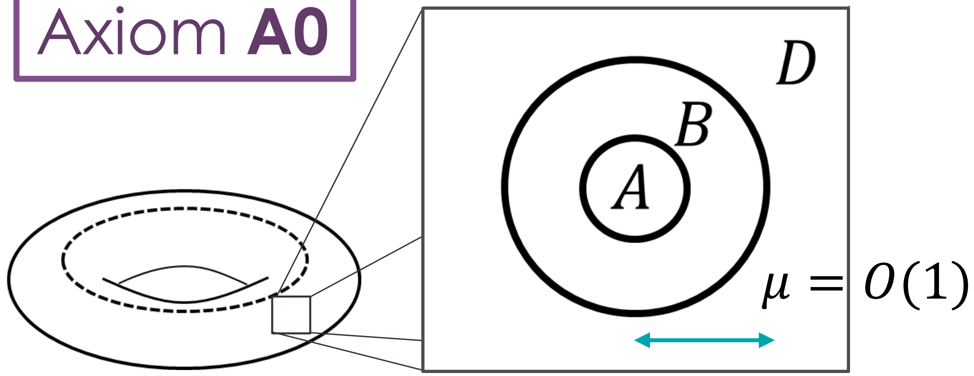

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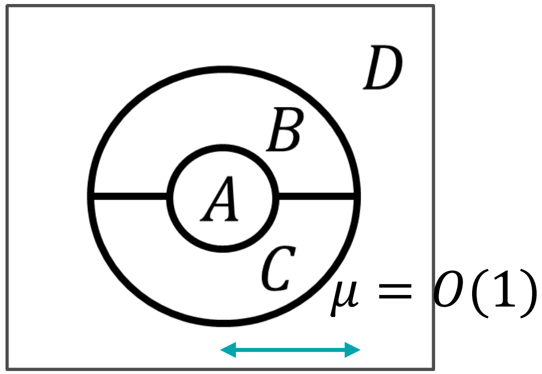

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States with **spurious topological entanglement entropy**

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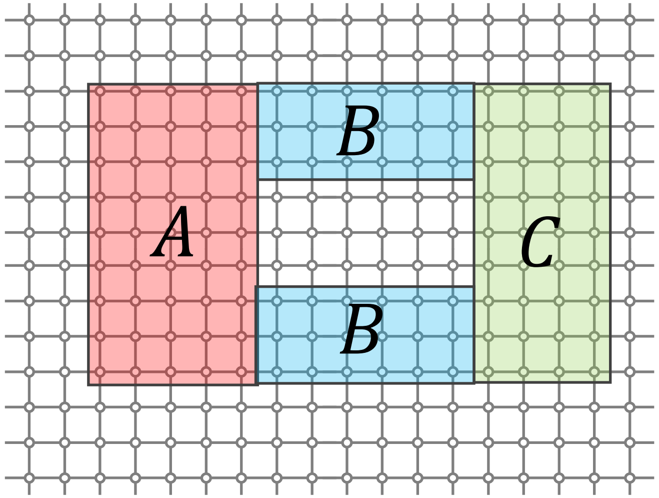
[Bravyi '08]:
There exists a ground state which is **not** topo. ordered $(\boldsymbol{\mathcal{D}} = \mathbf{1})$, but $S_{\text{topo}} = \boldsymbol{I}(\boldsymbol{A}\text{:}\,\boldsymbol{C}|\boldsymbol{B})_{\boldsymbol{\rho}} > \mathbf{0}$ for particular ABC.



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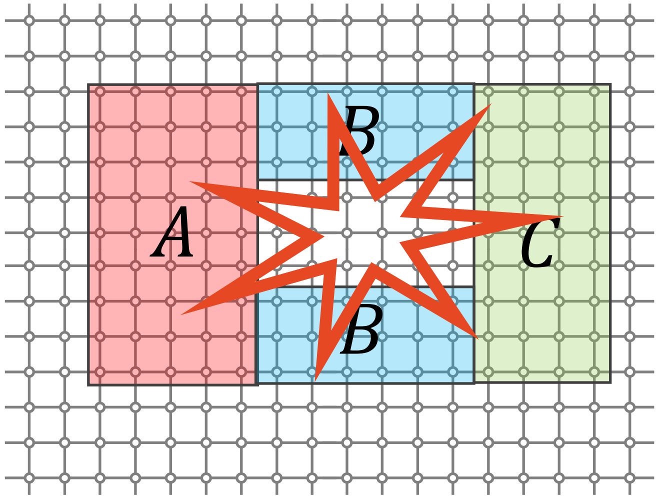
(1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT phase)



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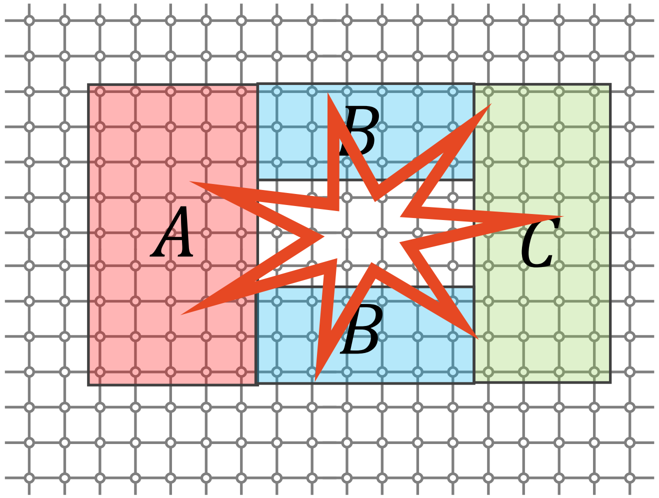

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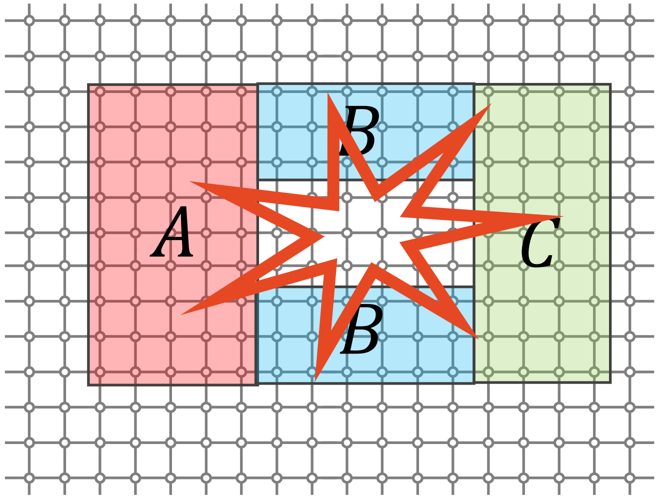

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[Williamson, Dua, Cheng, '19]:
homogeneous 2D model in subsystem SPT phases (2D cluster state)

Subsystem Symmetry-Protected Topological phase

Symmetry-Protected Topological (SPT) phase

- Ground state is unique and constructed by a const.-depth circuit (no topo. order)
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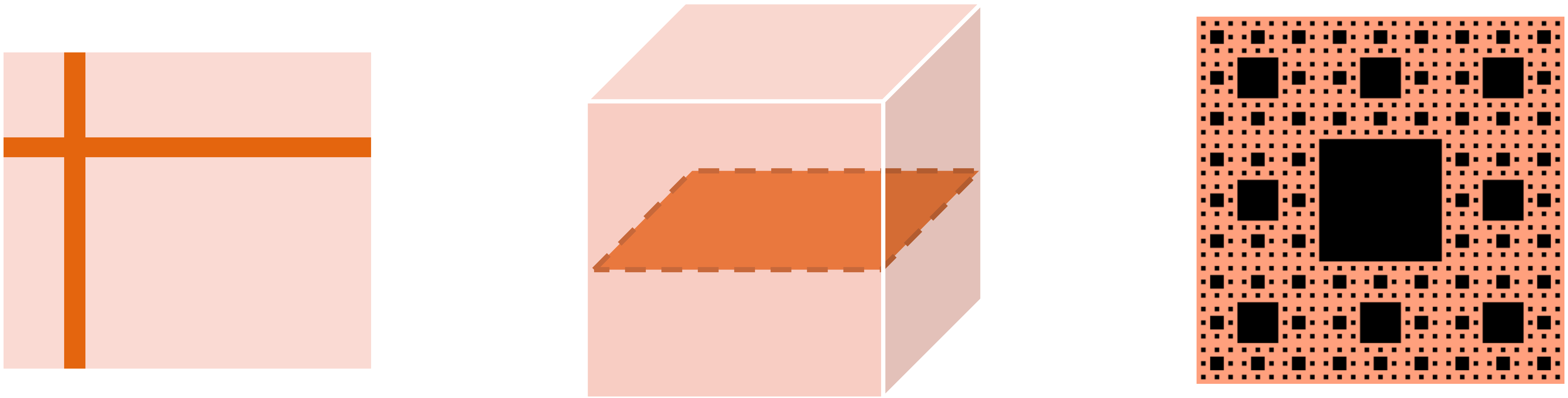
Subsystem symmetry

generators of the symmetry act on lower-dimensional subsystems (lines, planes, fractals)



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Subsystem SPT = SPT phases under subsystem symmetries

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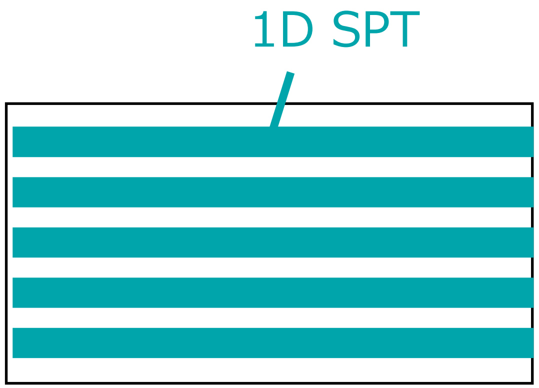

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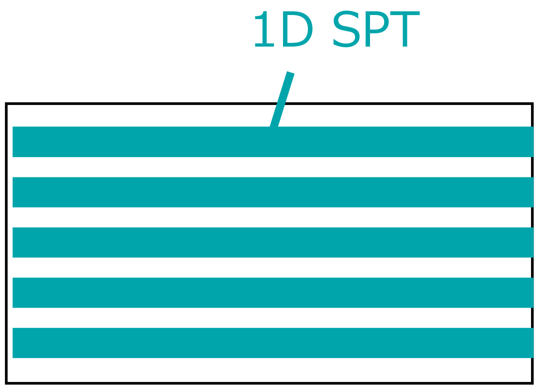

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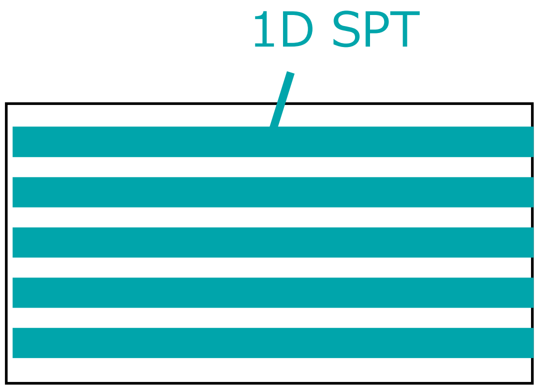

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weak subsystem SPT => spurious TEE

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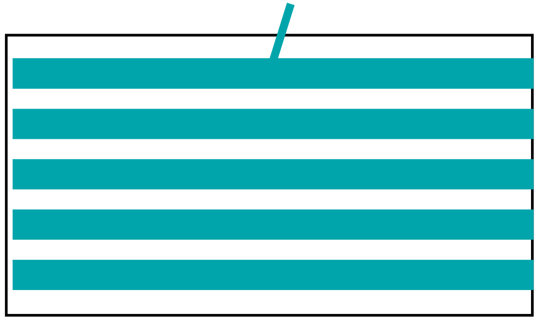

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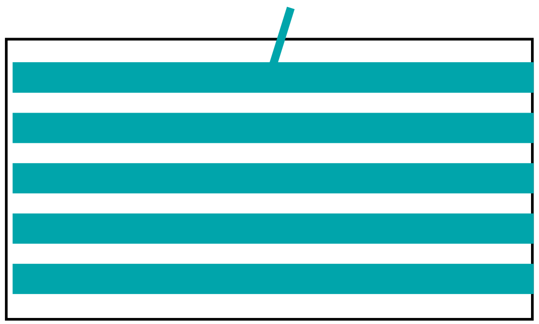

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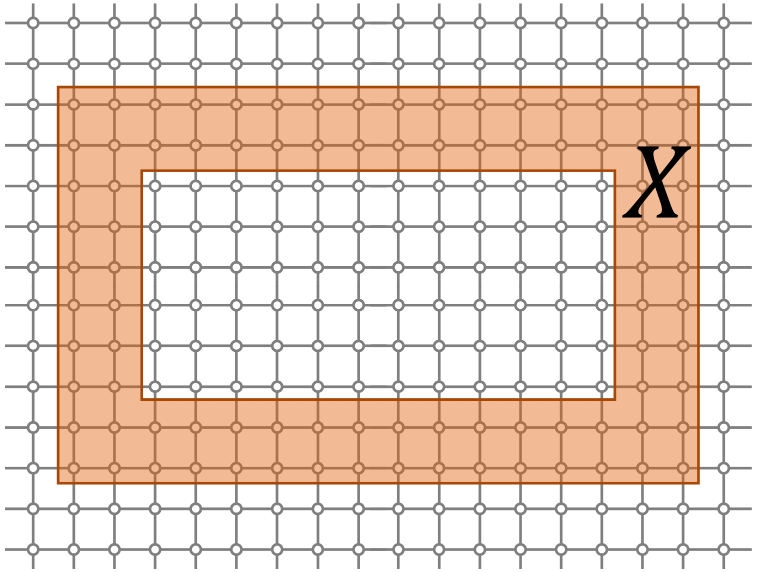
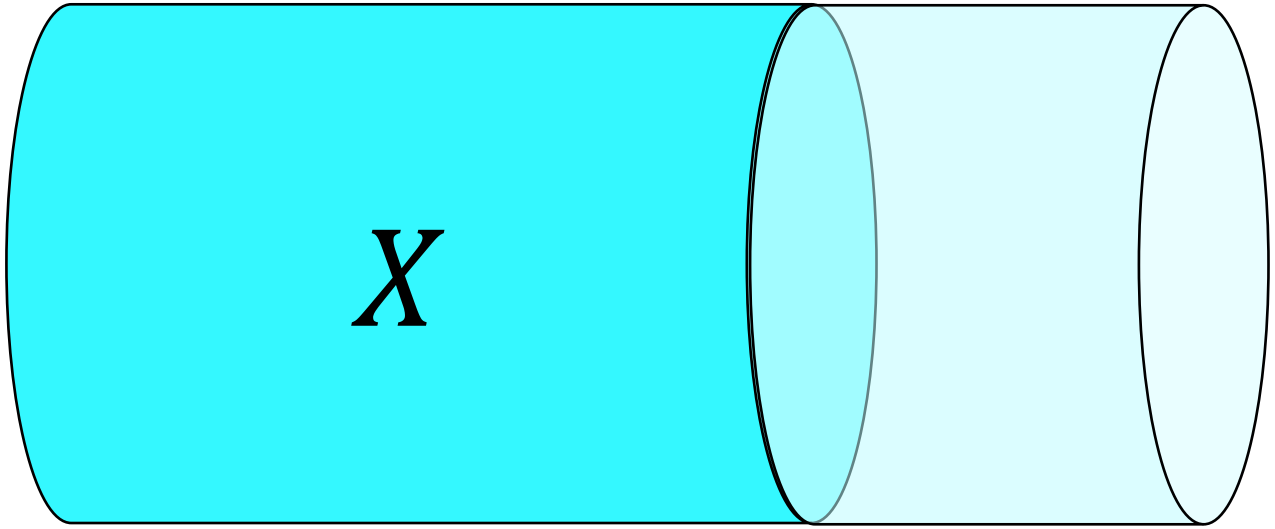
strong subsystem SPT => spurious TEE **Converse?**

Boundary MPS

The entanglement entropy $S(X)$ is invariant under any unitary within X and X^c





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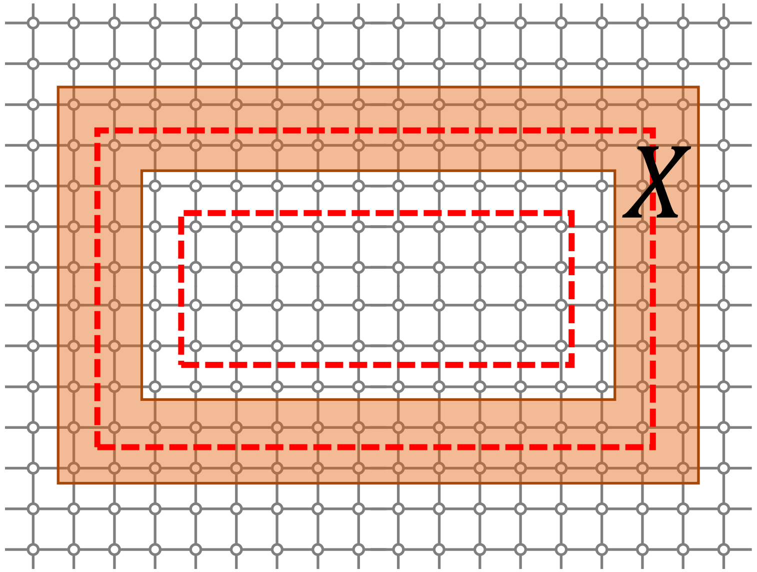

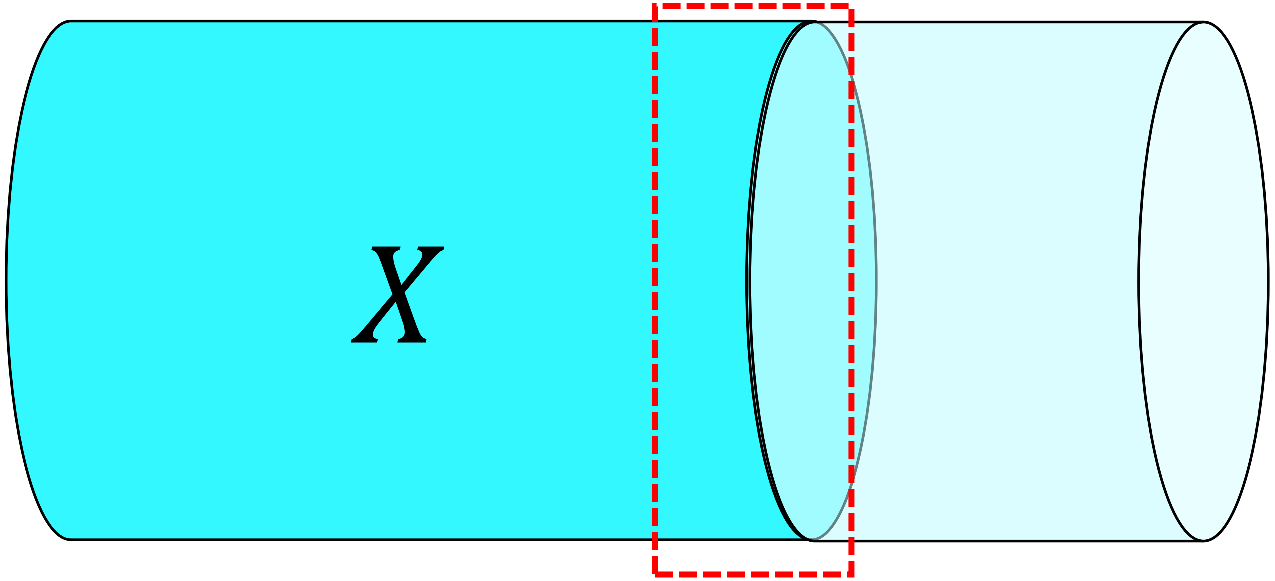

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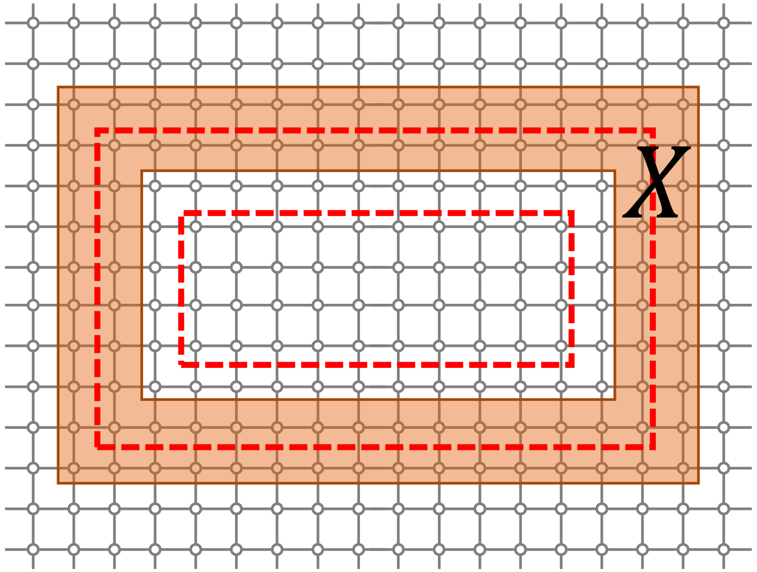

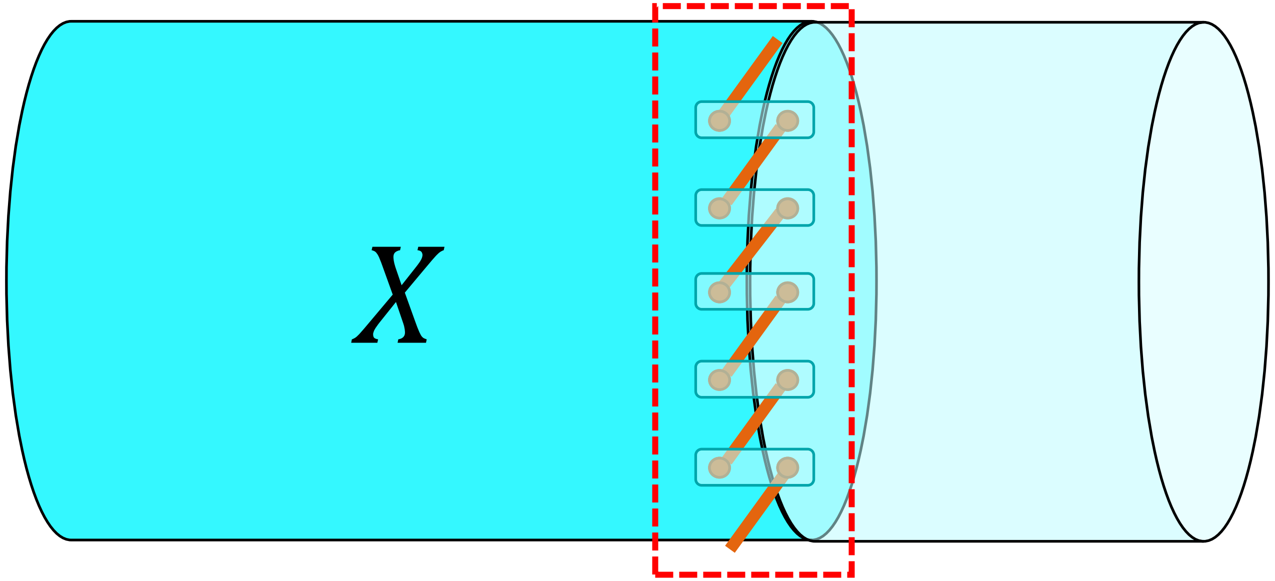

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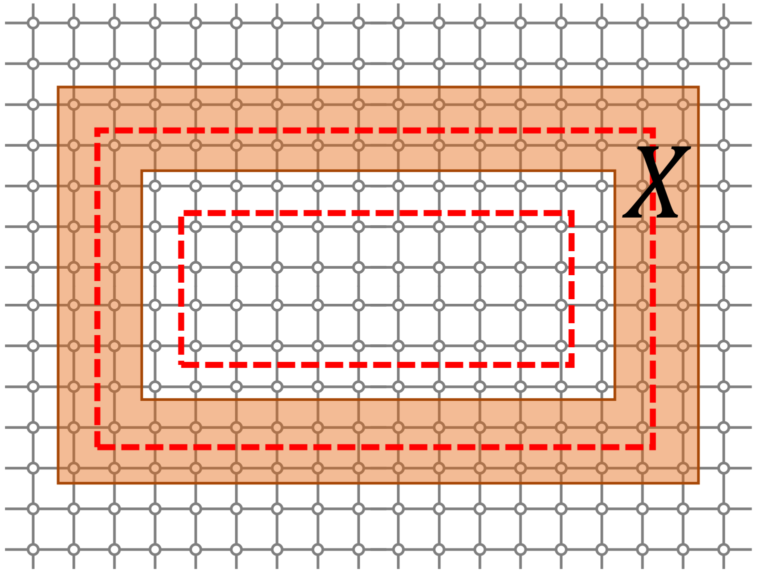
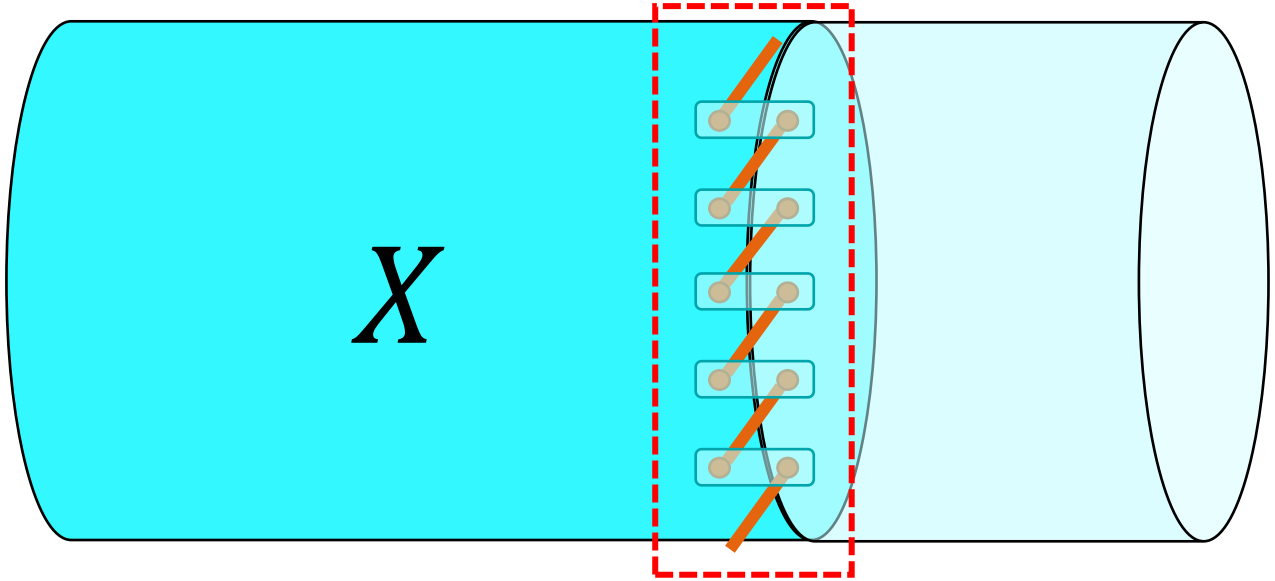
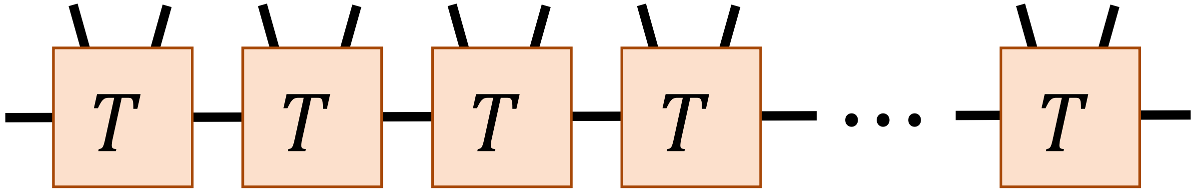




The boundary state around ∂X can be modeled by a MPS.



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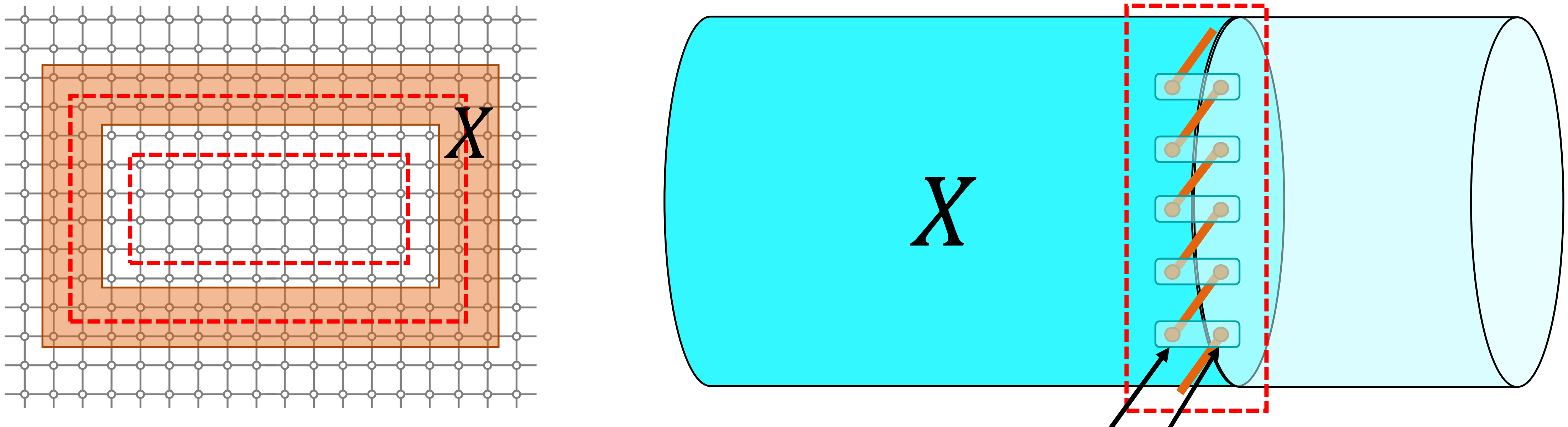

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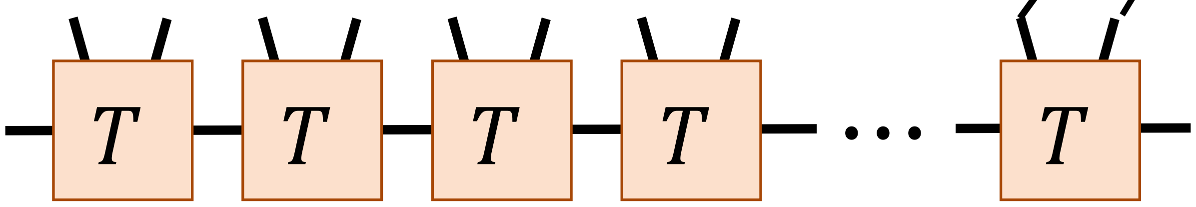
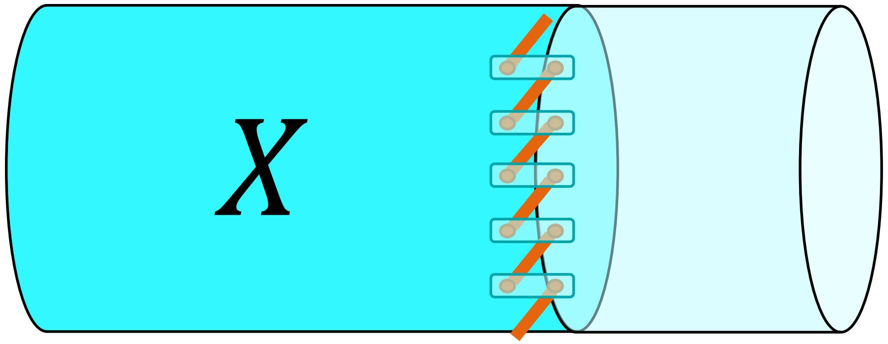


SSPT vs spurious TEE

2D subsystem SPT
(strong or weak along the cut)



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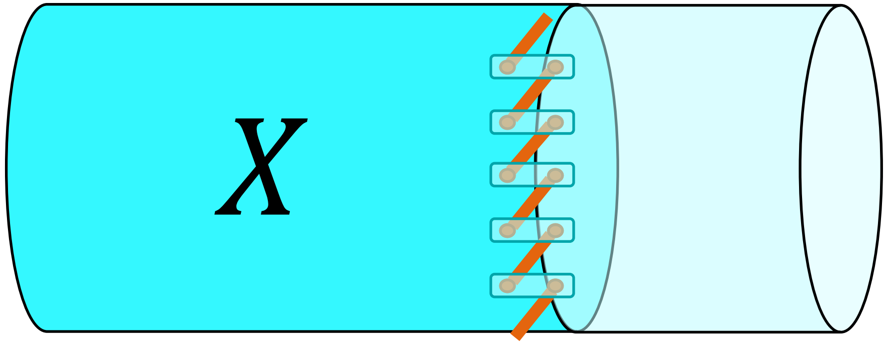

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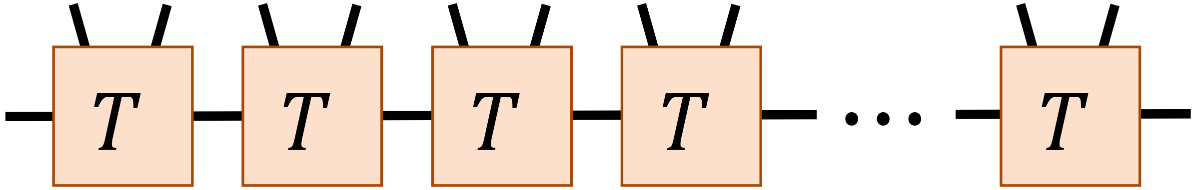

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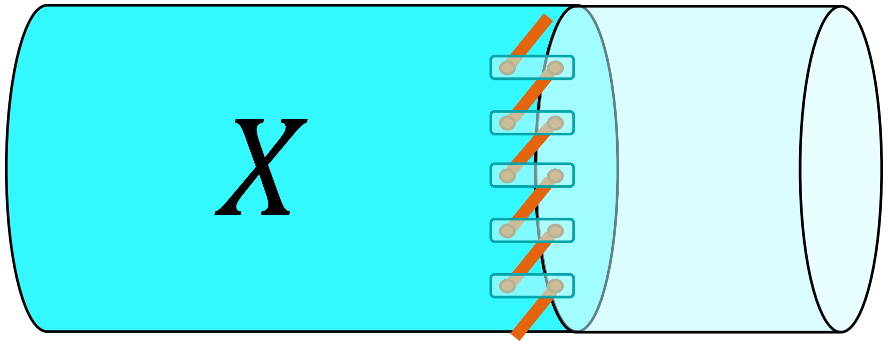

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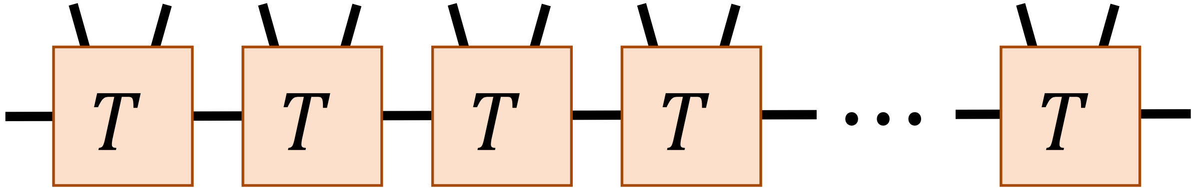
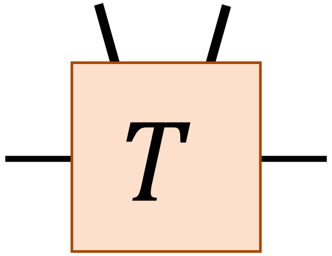


Spurious TEE occurs **[Zou, Haah '16]**

$$S(X)_\rho = \alpha|\partial X| + c + o(1)$$

Stabilizer boundary states

Consider an isometric MPS $|\psi_n\rangle$ at the boundary



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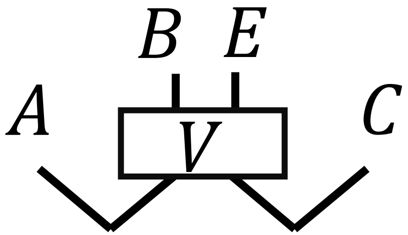

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$V: \mathrm{AC} \to BE$

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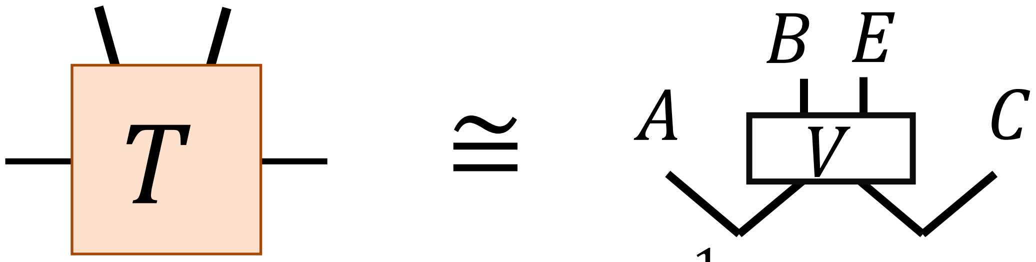

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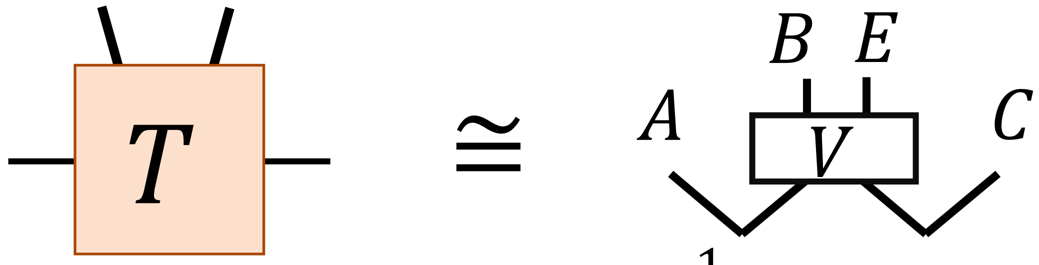

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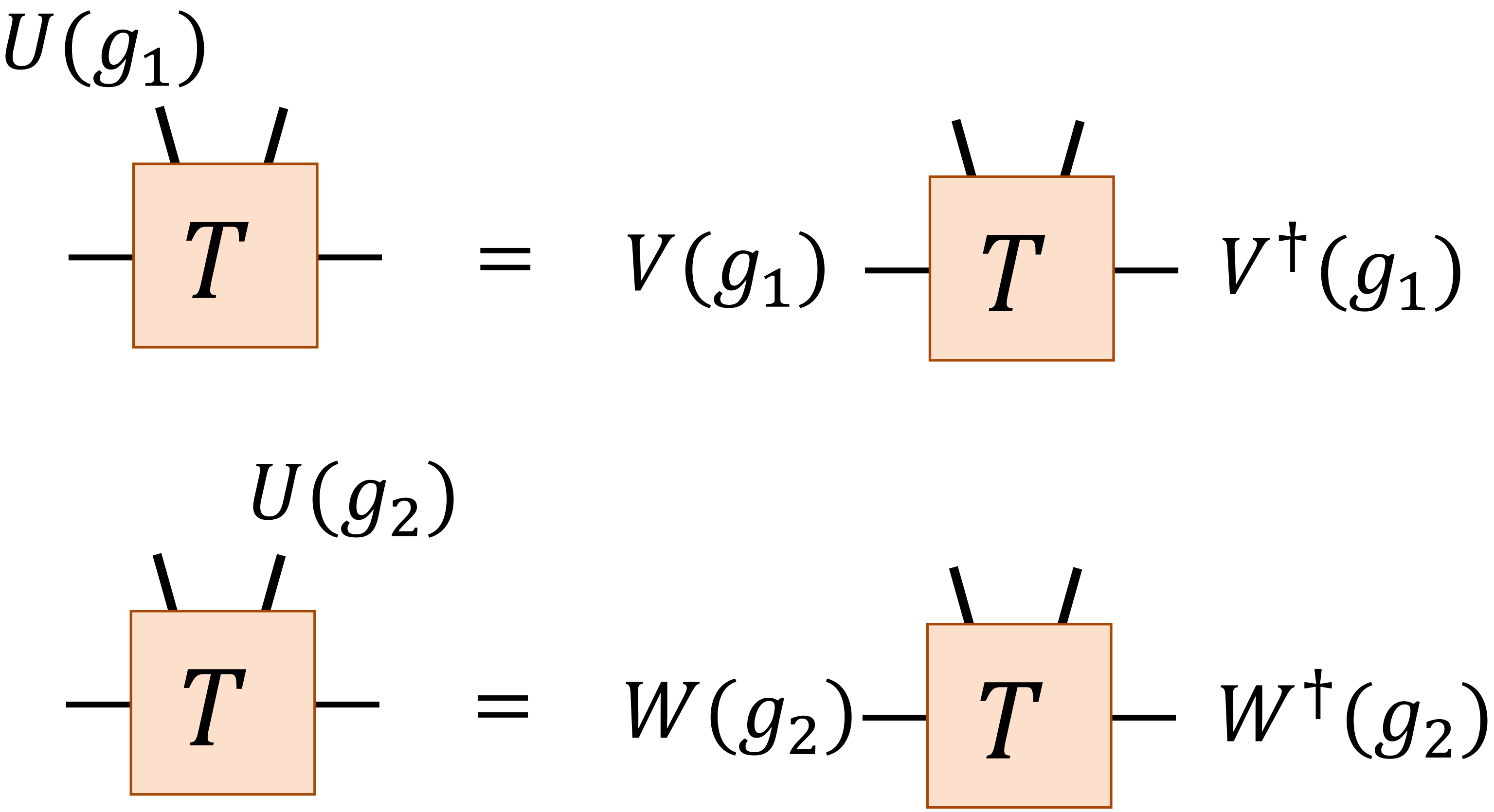
Non-trivial task: find the symmetry from given $|\psi^n\rangle$ with spurious TEE

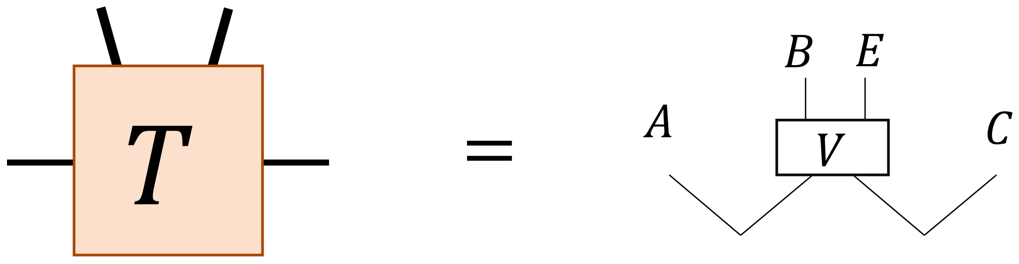
MPS in a 1D SPT phase

$G = G_1 \times G_2$



$$\exists (g_1, g_2) \in G, [V(g_1), W(g_2)] \neq 0.$$

Symmetry condition as logical operators



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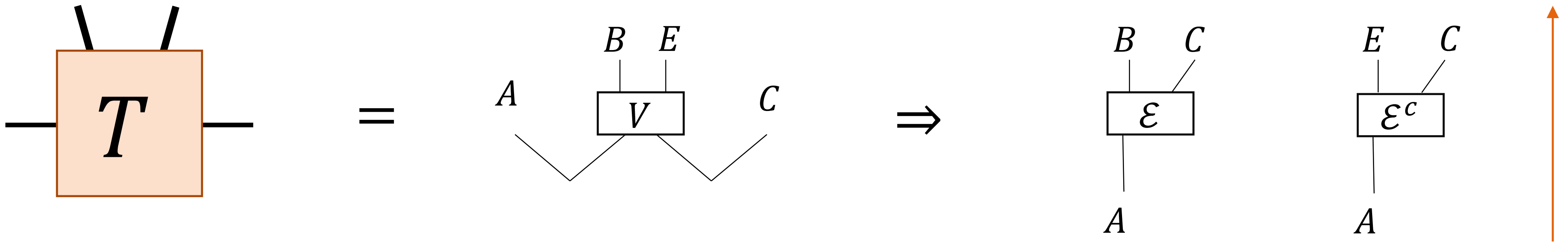

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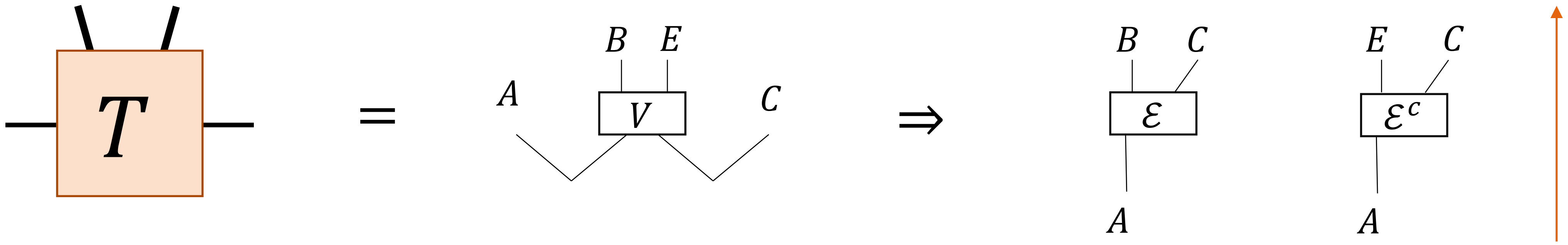

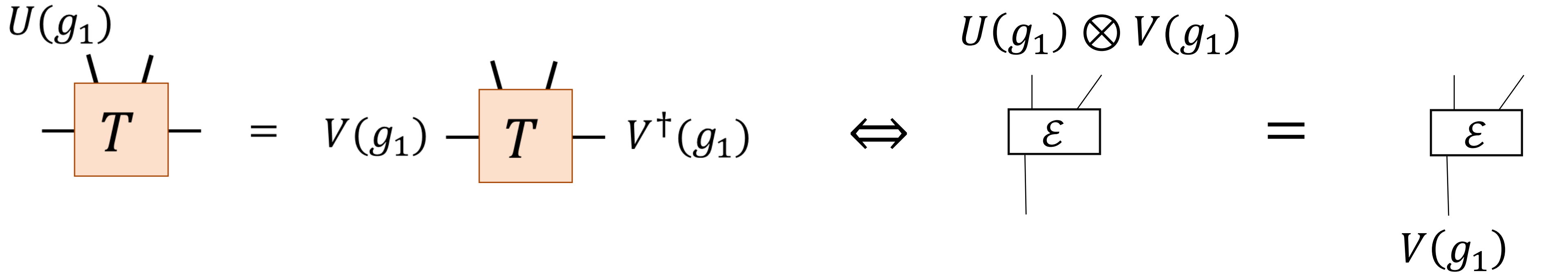

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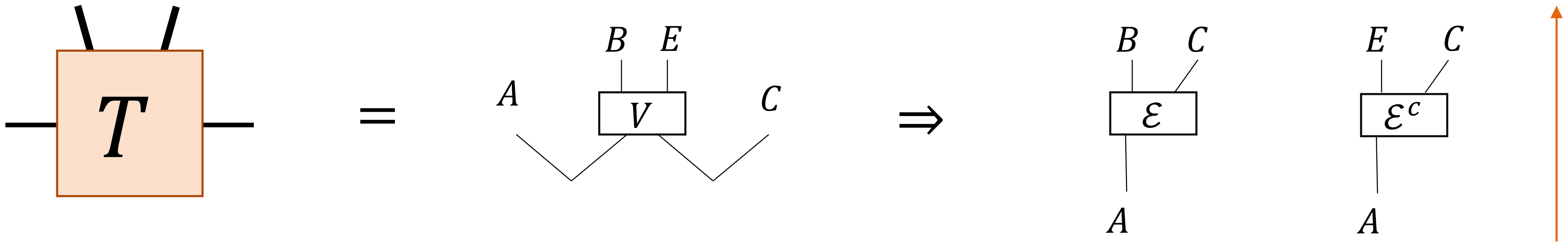

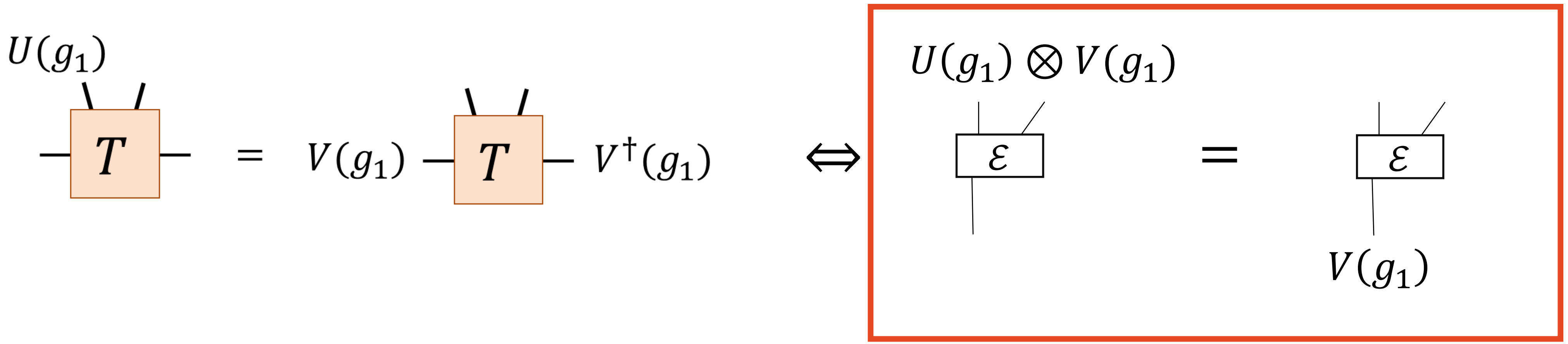

Point: $U(g_1) \otimes V(g_1)$ is a tensor-product ***logical operator*** of $V(g_1)$ acting on ***the correctable algebra*** of $\mathcal{E}$.

Correctable algebra

Operator-Algebra Quantum Error Correction [Beny, Kempf, Kribs, '07]

For a C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, a CPTP-map $\mathcal{E}$ is *correctable* if there exists a CPTP-map $\mathcal{R}$ s.t.,

$$X = (\mathcal{R} \circ \mathcal{E})^\dagger(X), \forall X \in \mathcal{A}, \forall \rho.$$

*$\mathcal{E}$ preserves all information in **subalgebra** $\mathcal{A}$*

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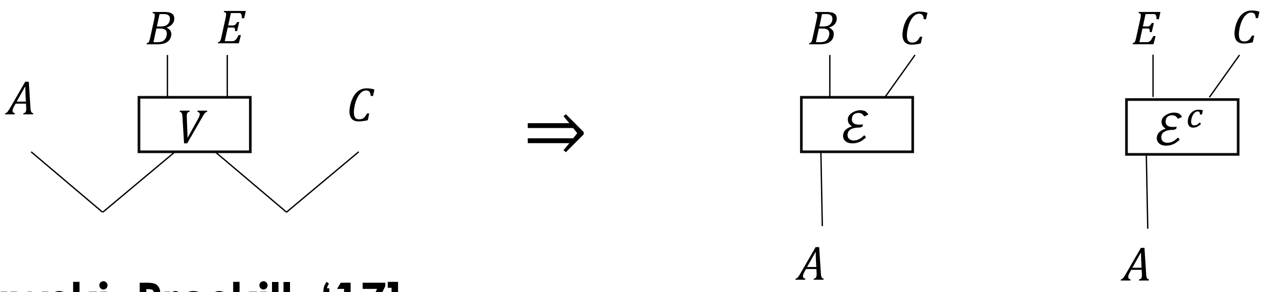
Correctable algebra $\mathcal{A}_\mathcal{E}$: the maximal subalgebra such that $\mathcal{E}$ is correctable.

$$\mathcal{A}_\mathcal{E} = \left(Alg\{E_b^\dagger E_a\}\right)', \qquad \mathcal{E}(\cdot) = \sum_a E_a \cdot E_a^\dagger.$$

"Maximum information" exactly preserved by $\mathcal{E}$

Spurious TEE>0 => non-trivial OAQEC



Define 4 correctable algebras

$$\mathcal{A}^{BC}, \mathcal{A}^{B}, \mathcal{B}^{EC}, \mathcal{B}^{E} \subset \mathcal{B}(\mathcal{H}_A)$$

[Pastawski, Preskill, '17]

Complementarity recovery condition: $\mathcal{A}_{\mathcal{E}^c} = \mathcal{A}'_{\mathcal{E}}$ (satisfied for stabilizer states)

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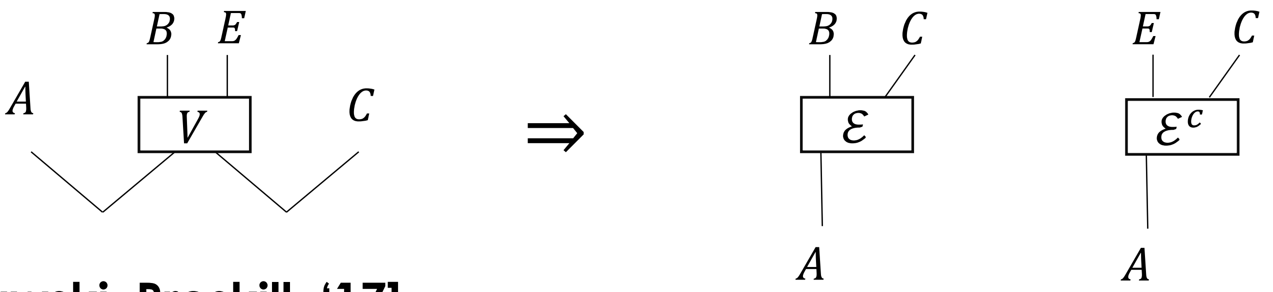

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$$\rightarrow \mathcal{A}^{B} = (\mathcal{B}^{EC})', \qquad \mathcal{B}^{E} = (\mathcal{A}^{BC})'$$

Simply denote $(\mathcal{A}^{BC}, \mathcal{B}^{EC})$ by $(\mathcal{A}, \mathcal{B})$

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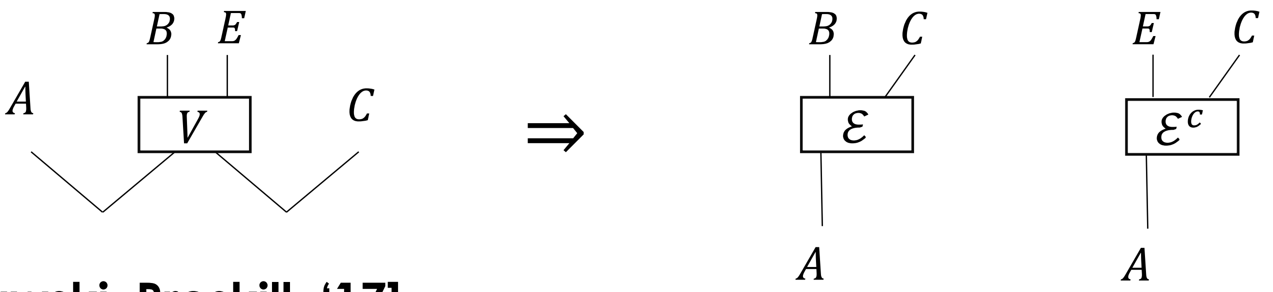

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Complementarity recovery condition + constant spurious TEE$>$ 0

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Proof sketch (1/3)

Theorem [Kato, Brandão, ‘19]

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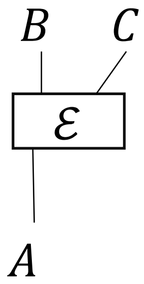

$\mathcal{G}_{BC}$: The set of tensor product logical unitary operators on BC

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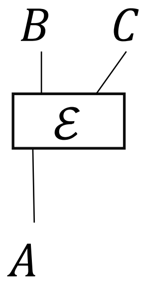

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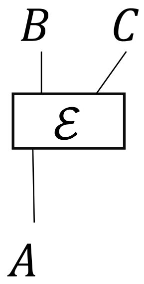

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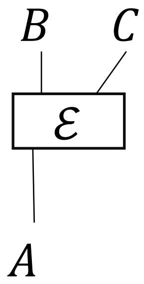

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$$G_1 \cong \mathcal{G}_{BC}/\mathcal{C}_B$$

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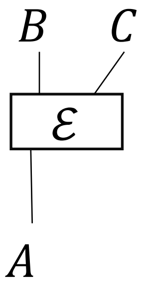

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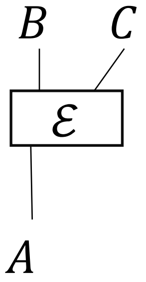

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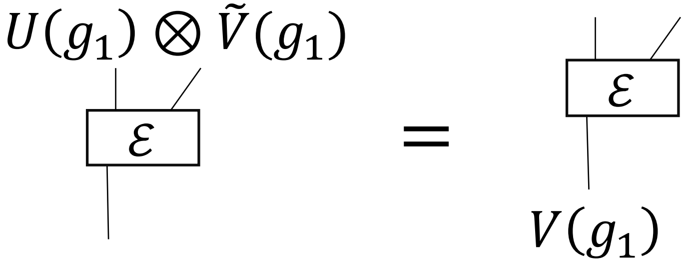

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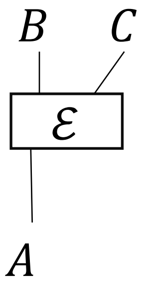

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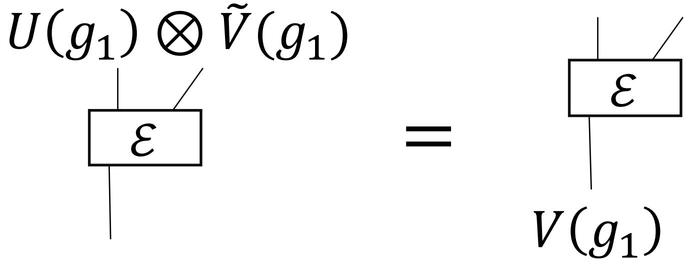

one can show $V(g_1) \mapsto \tilde{V}(g_1)$ is a permutation in G_1.

Proof sketch (2/3)

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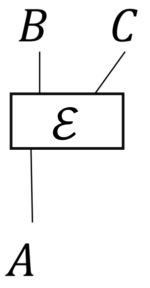

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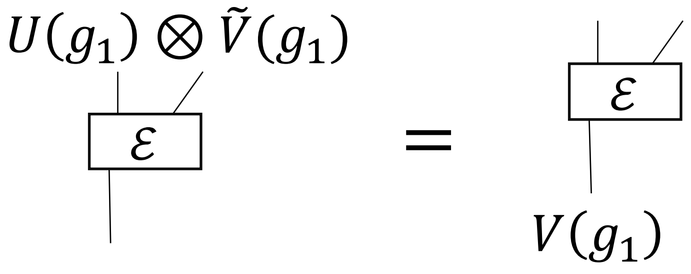

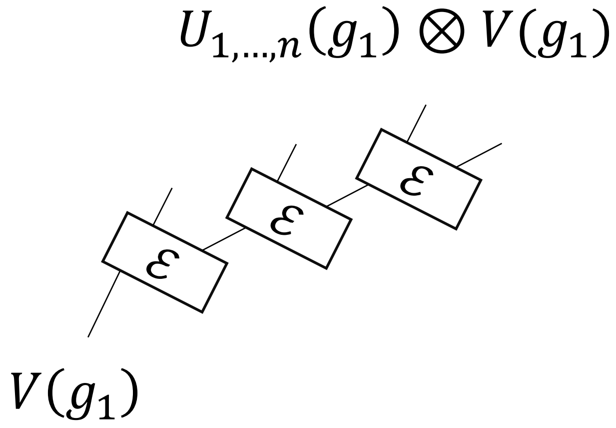

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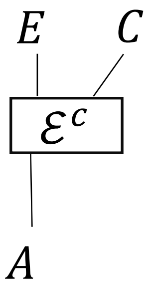

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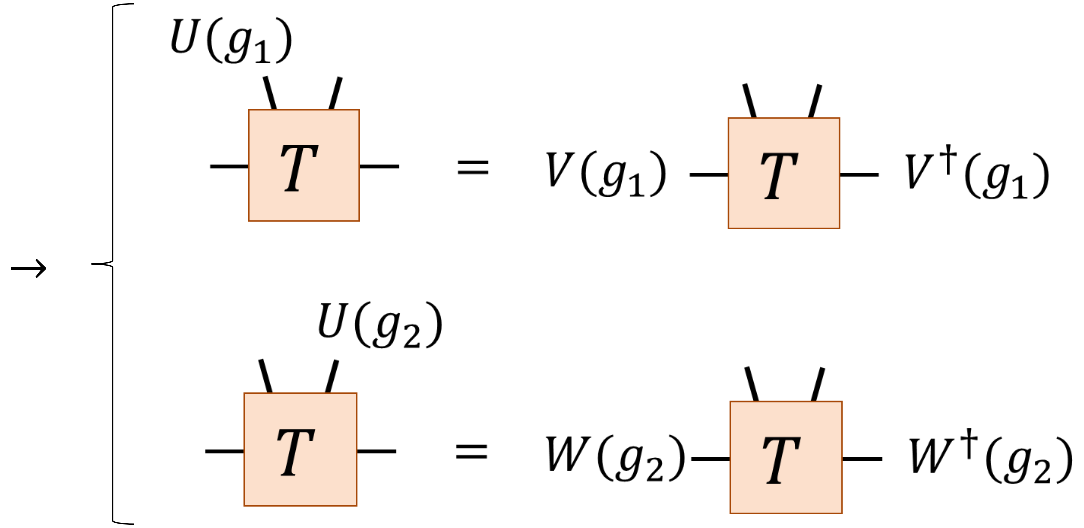

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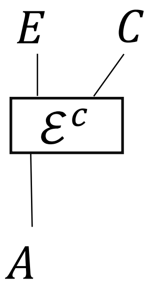

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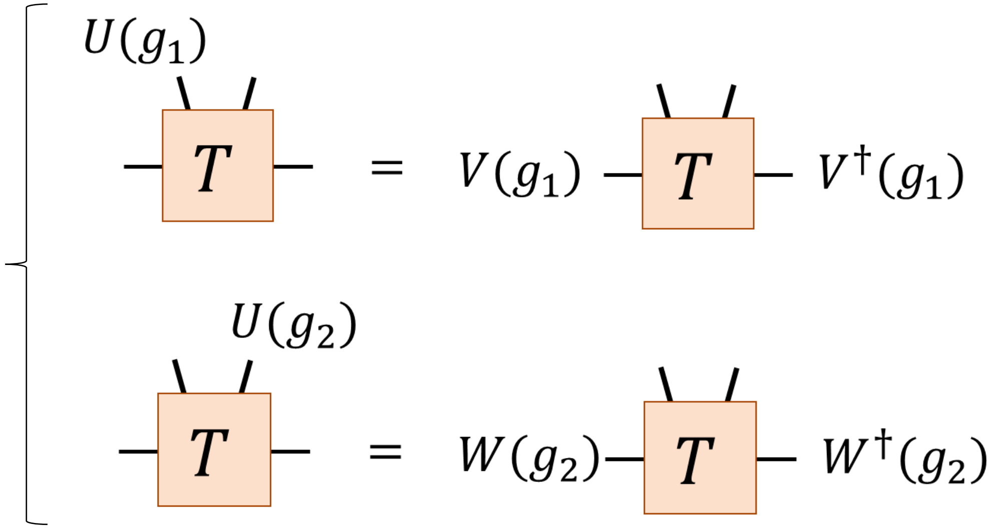
$\rightarrow$



Proof sketch(3/3)

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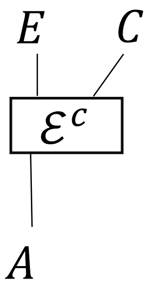

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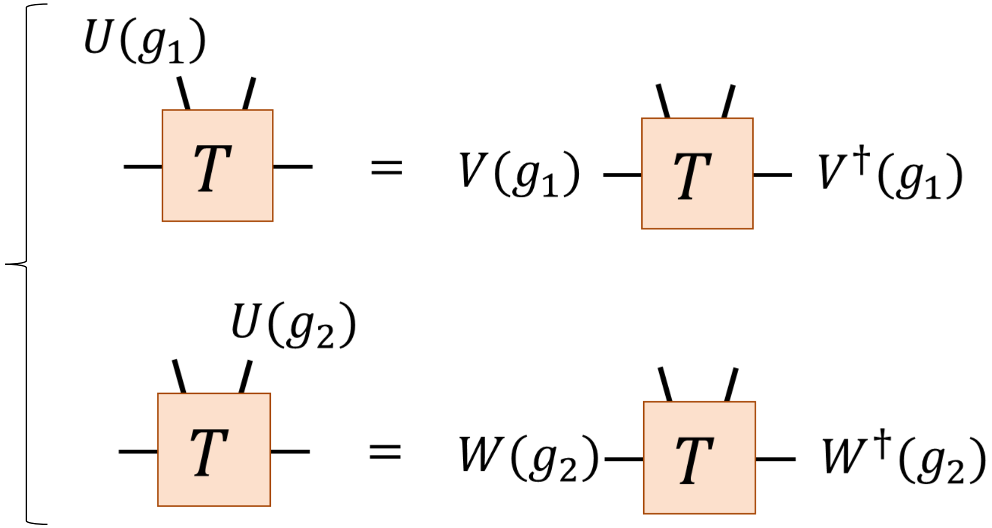

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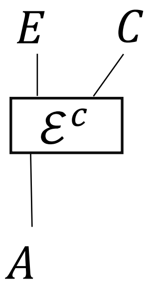

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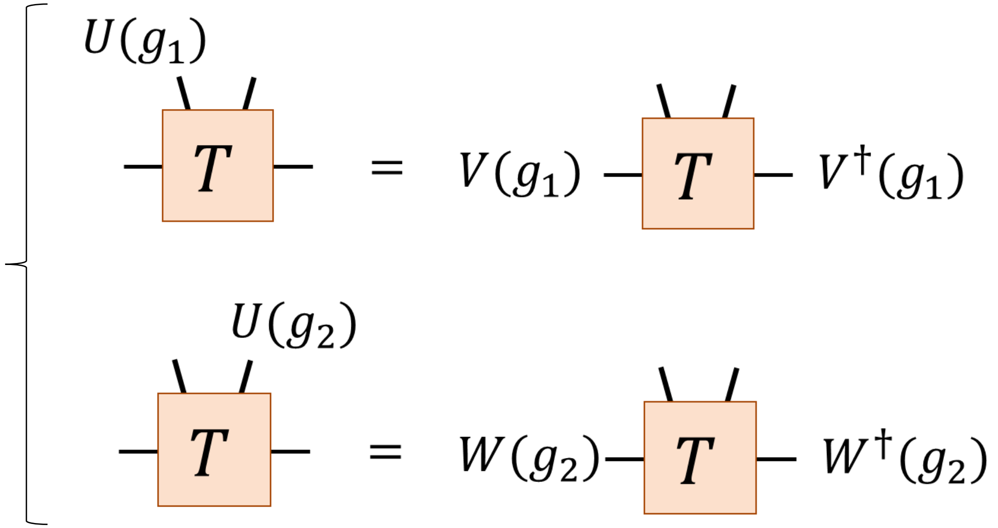

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$V(g_1) \in \mathcal{A}, \qquad W(g_2) \in \mathcal{B}$

$$\exists (g_1, g_2) \in G, [V(g_1), W(g_2)] \neq 0.$$

non-trivial $G_1 \times G_2$ 1D SPT ■

Beyond stabilizers

Q. Is there any 1D MPS which is **not** in any $G_1 \times G_2$ SPT but has non-zero spurious TEE?

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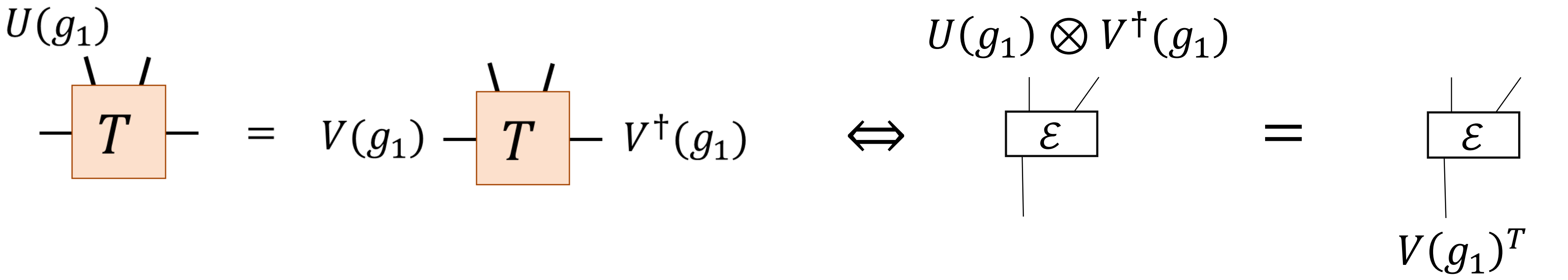

Beyond stabilizers

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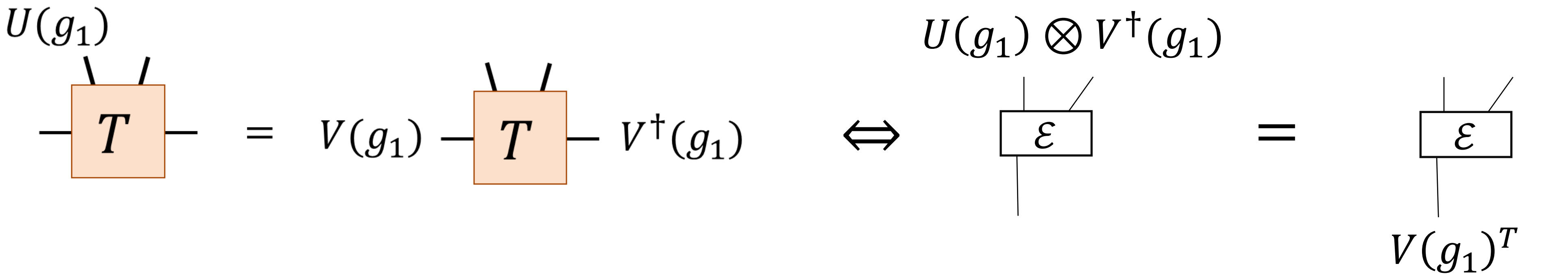
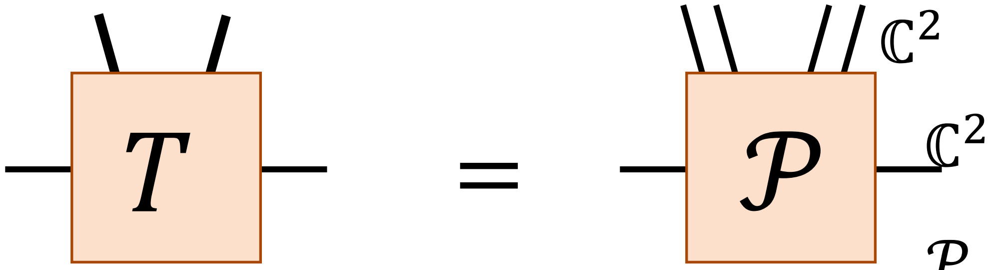


Find non-trivial OAQEC **without any tensor-product logical operator**

Candidate: perfect tensor state



$\mathcal{P}$ = 5-qubit code[[5,1,3]]

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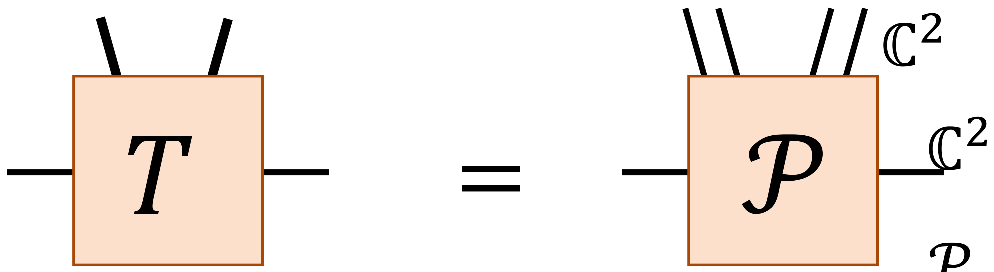

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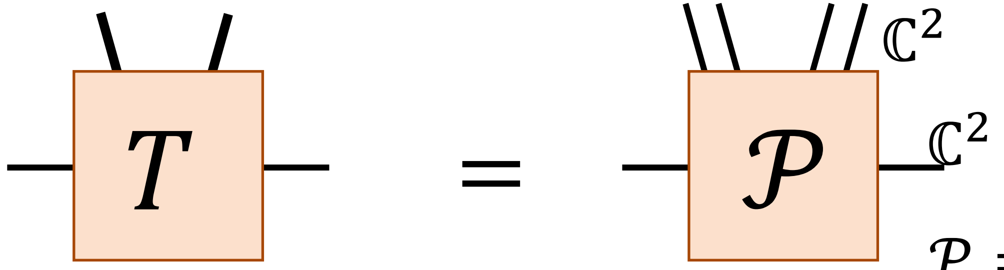

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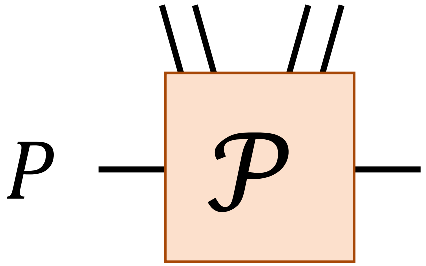

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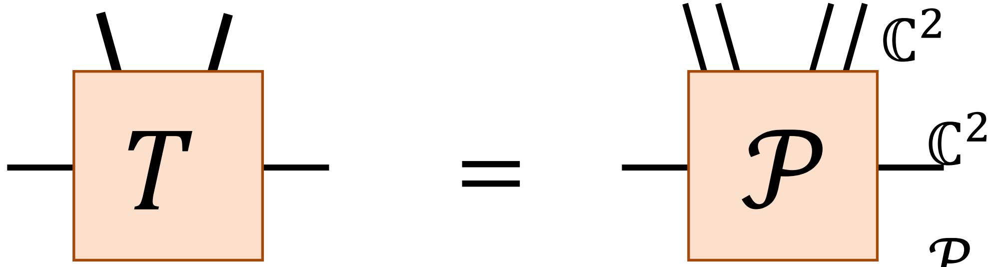

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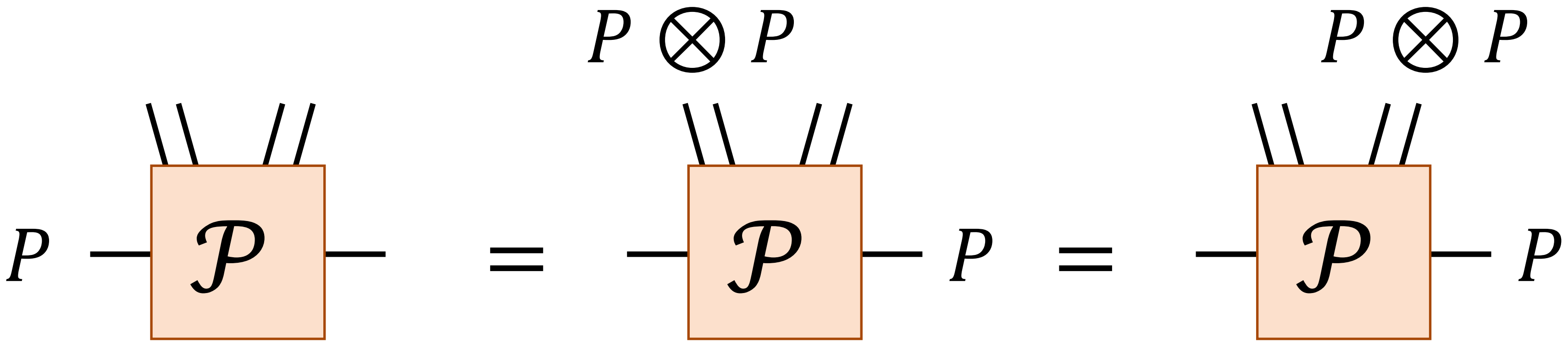

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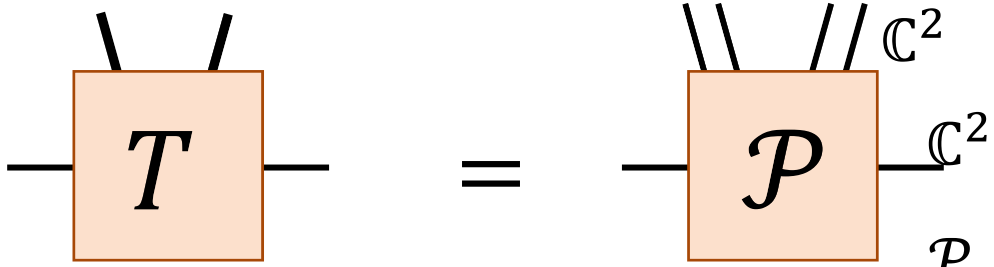

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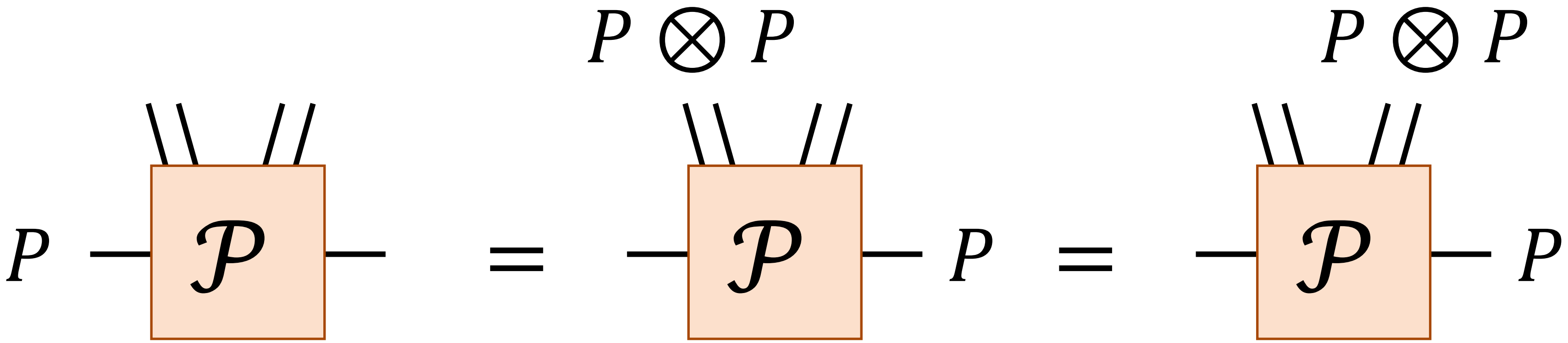

Corresponding MPS is a stabilizer SPT state

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U: non-Clifford, not diagonal in X or Z-basis

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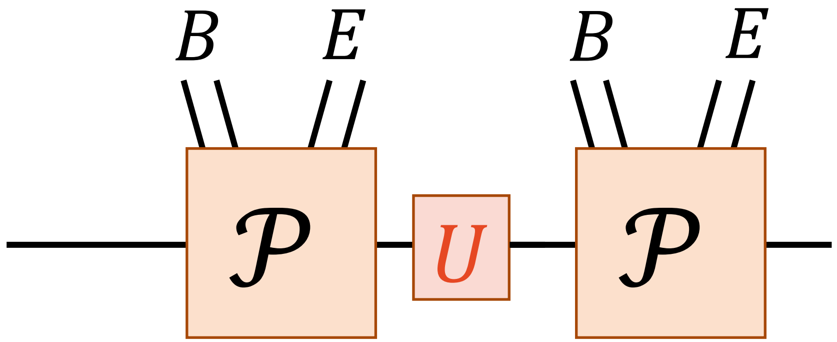

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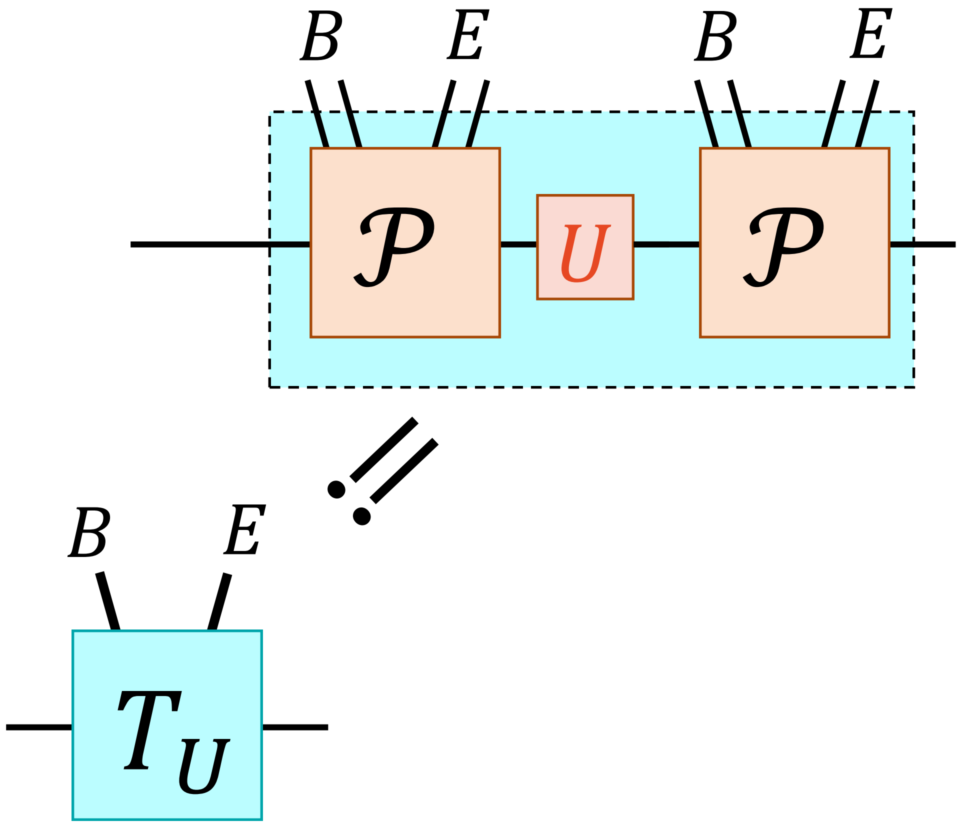

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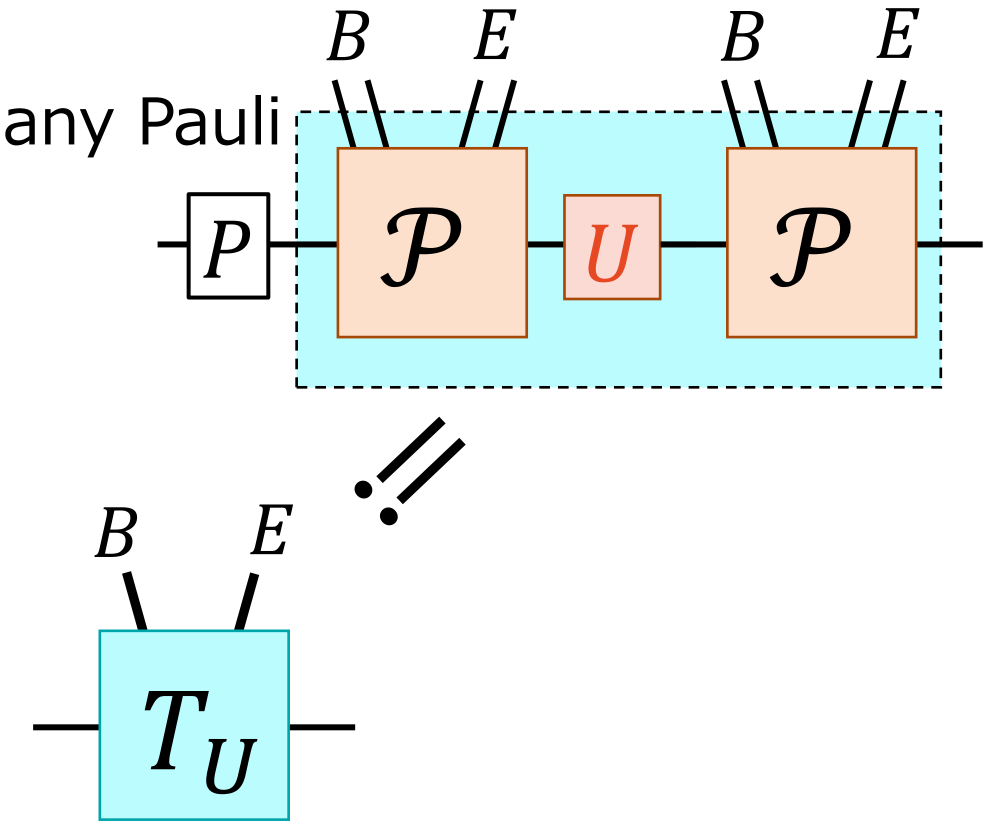

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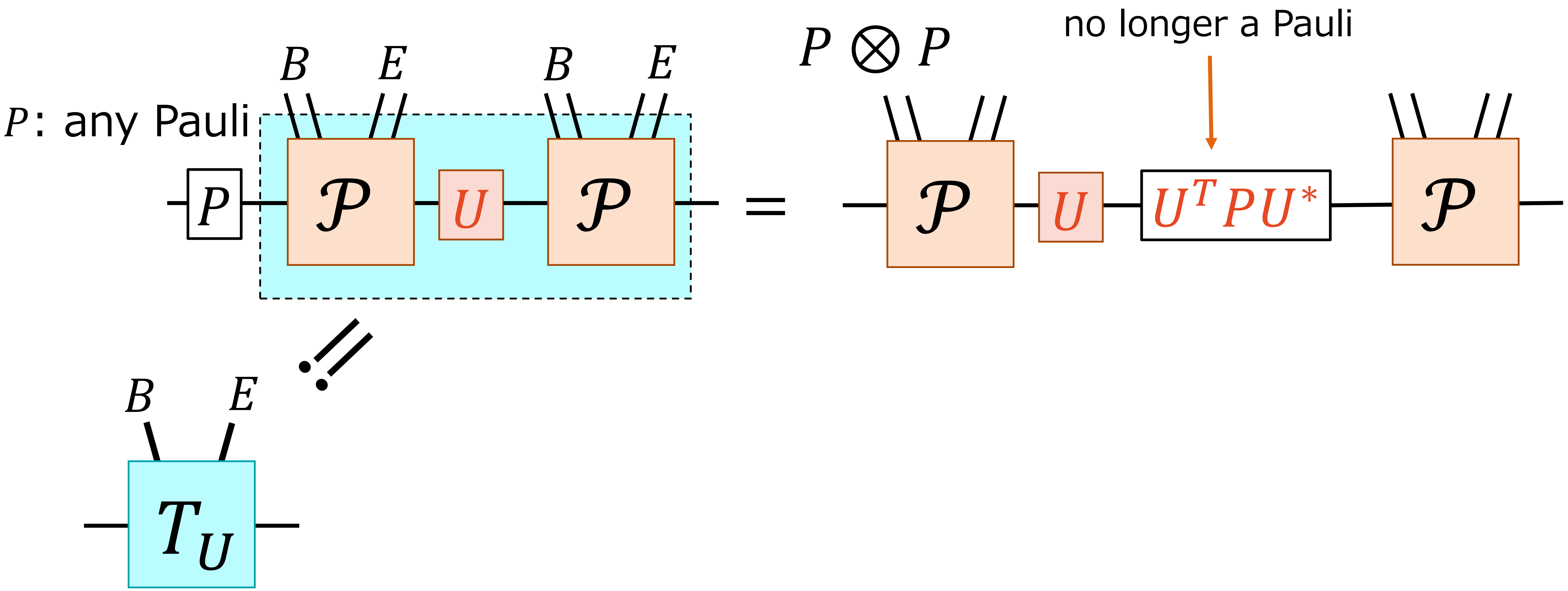

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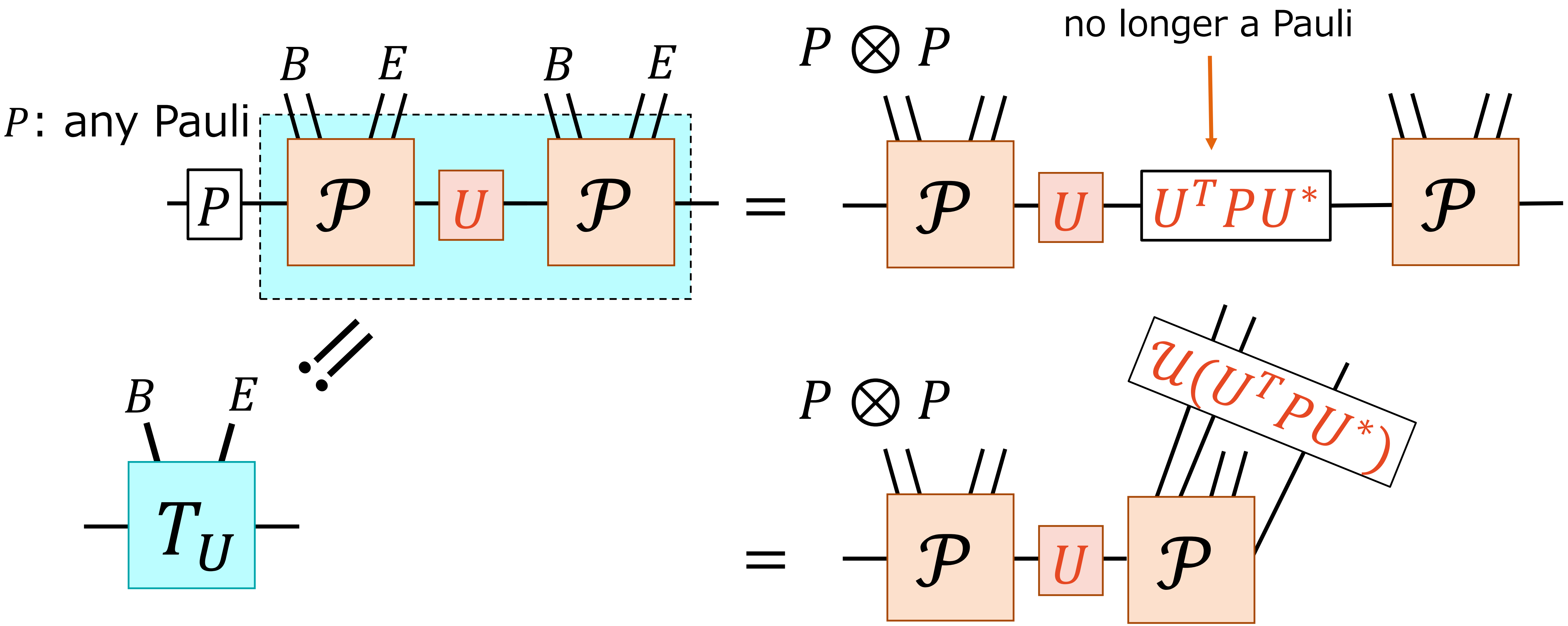

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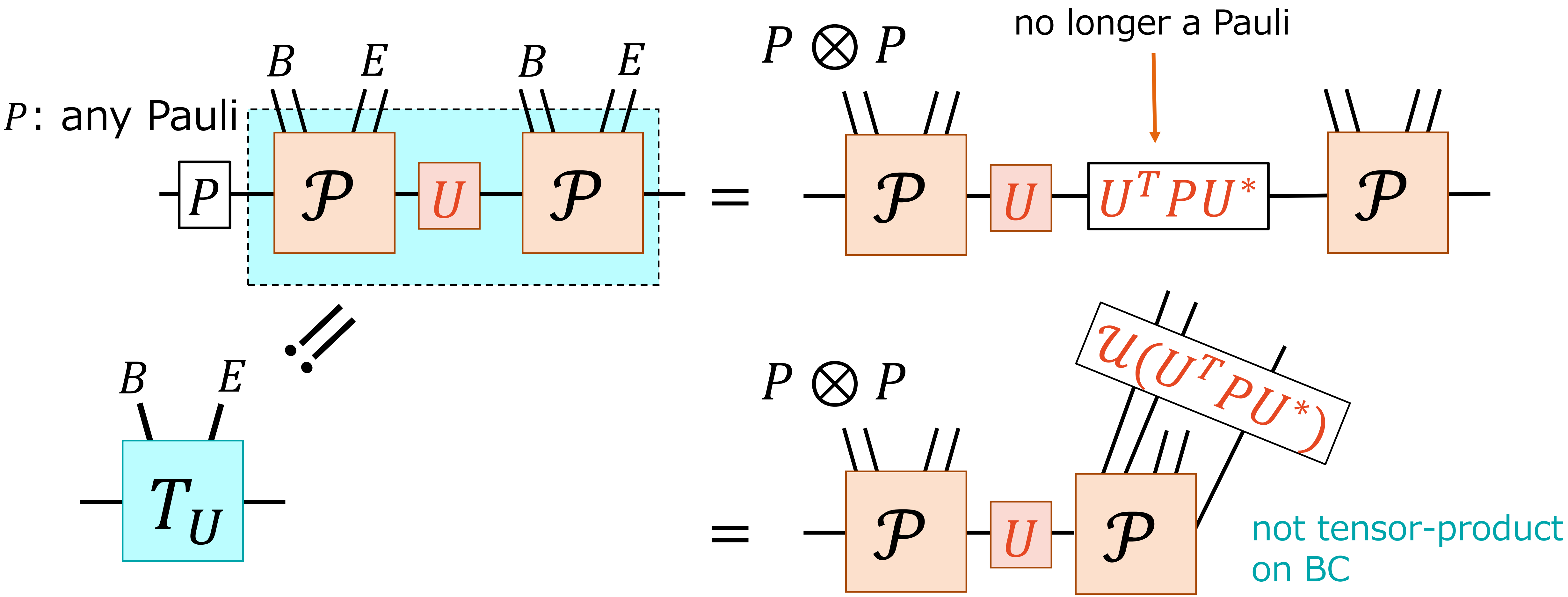
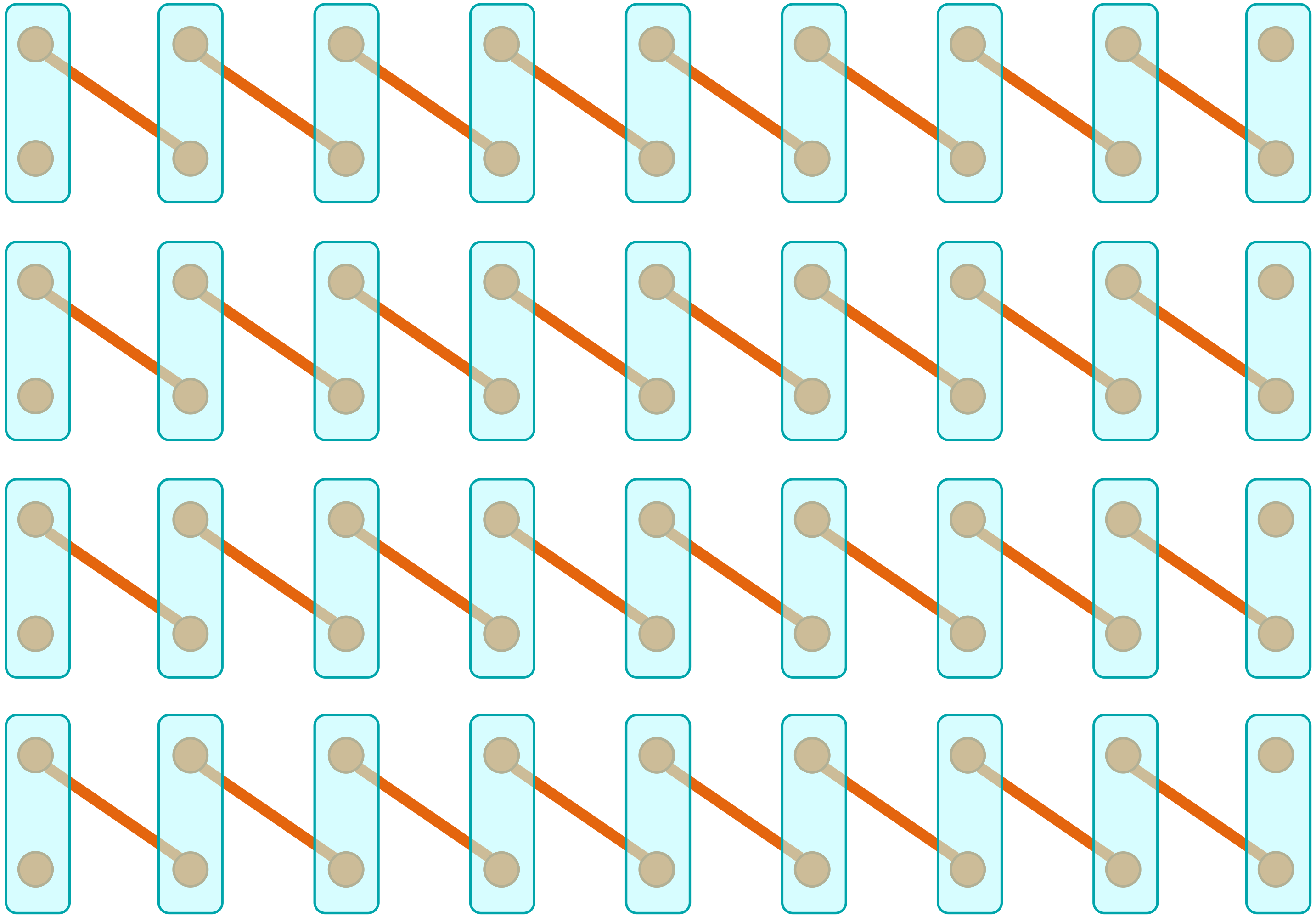


Toy model (no on-site subsystem symmetry)



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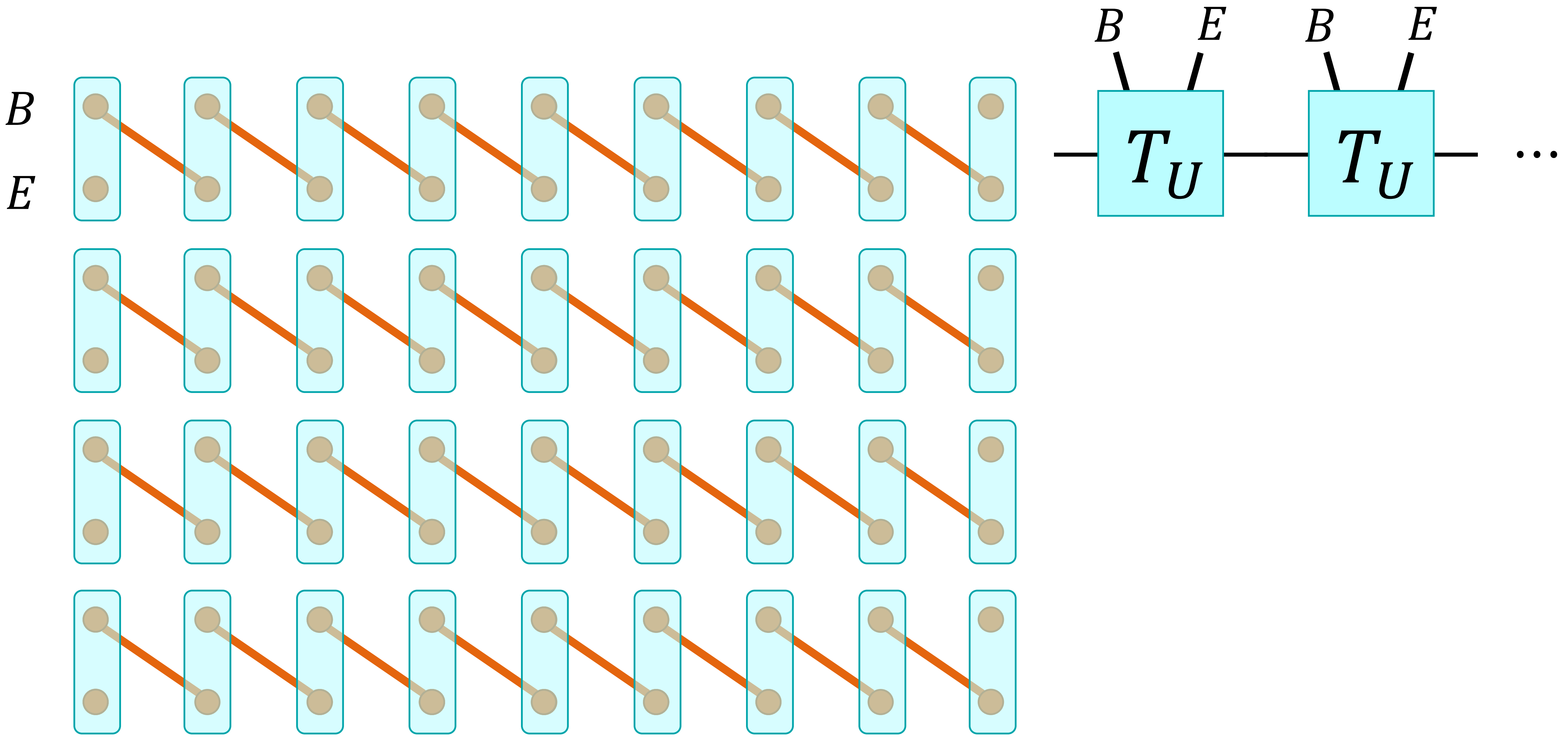

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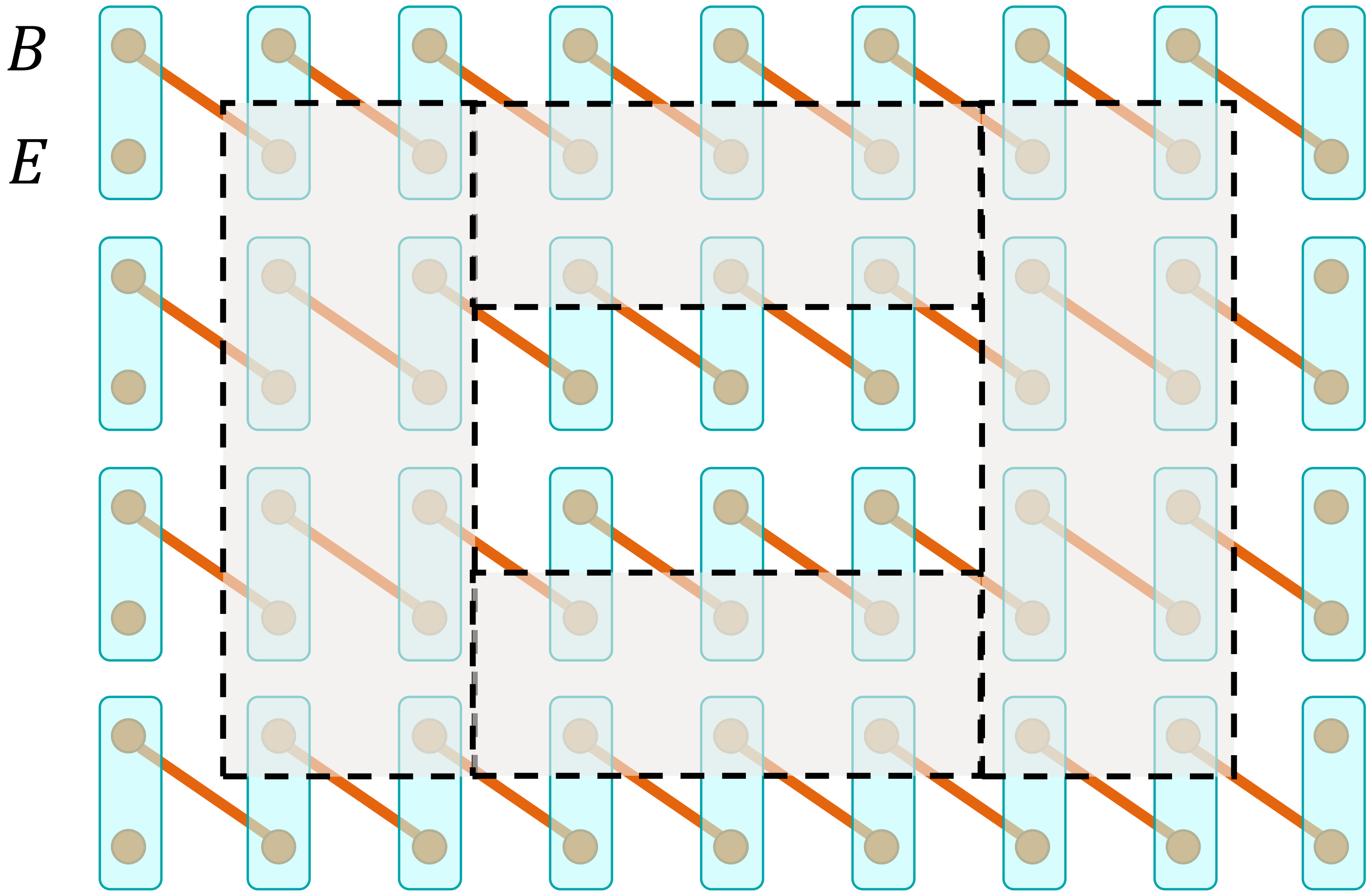

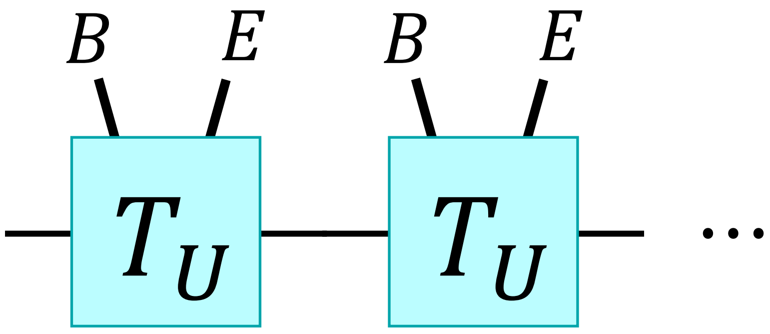

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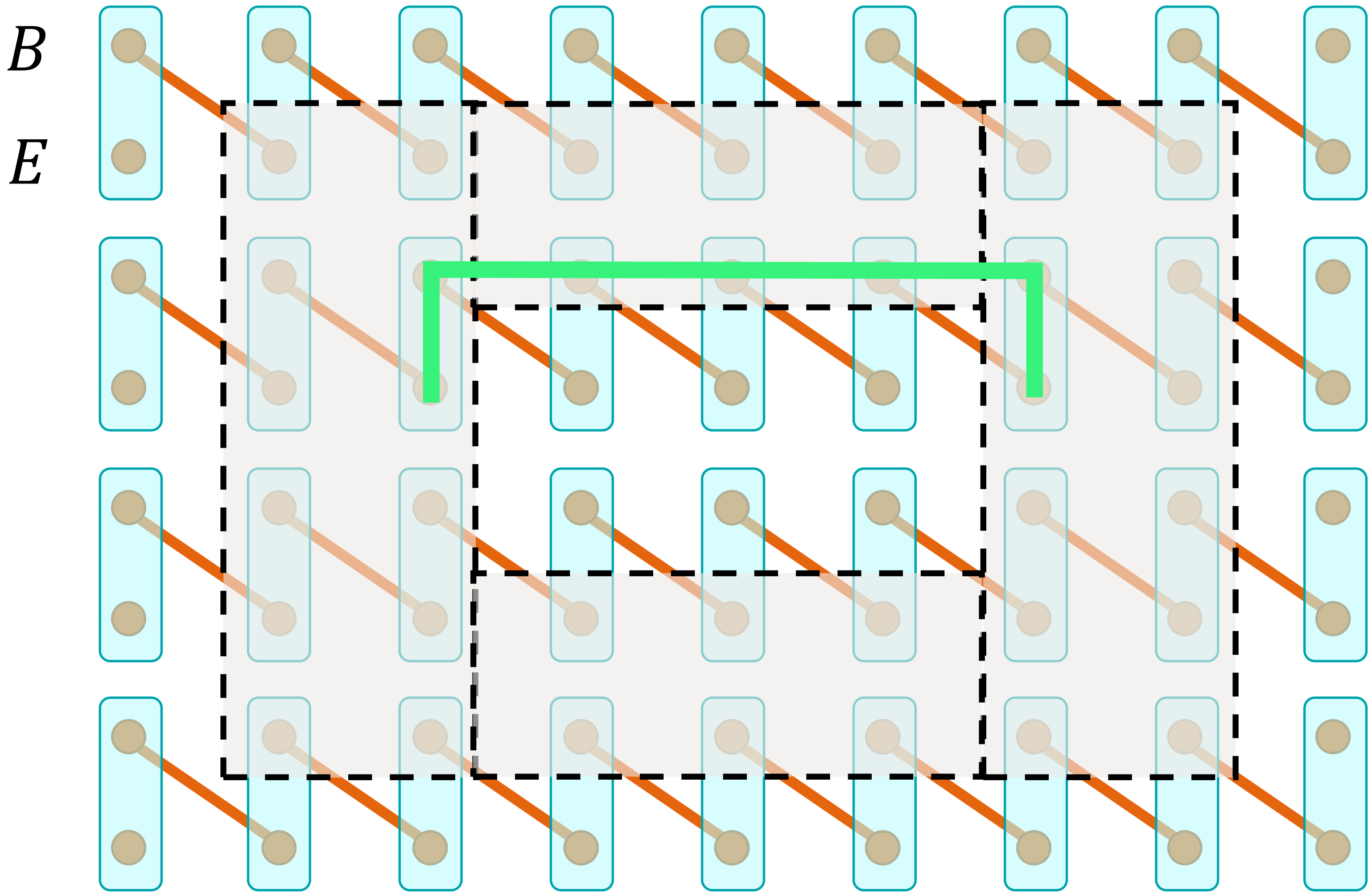

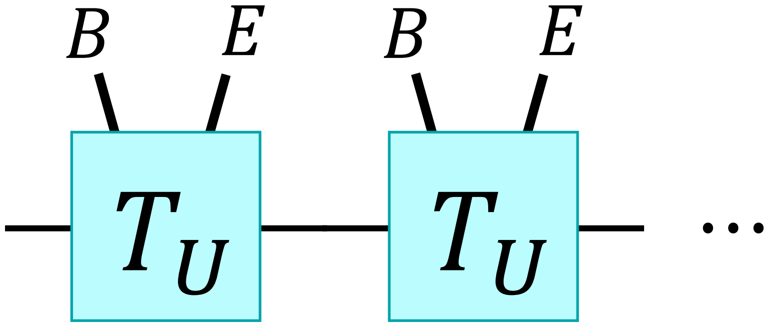

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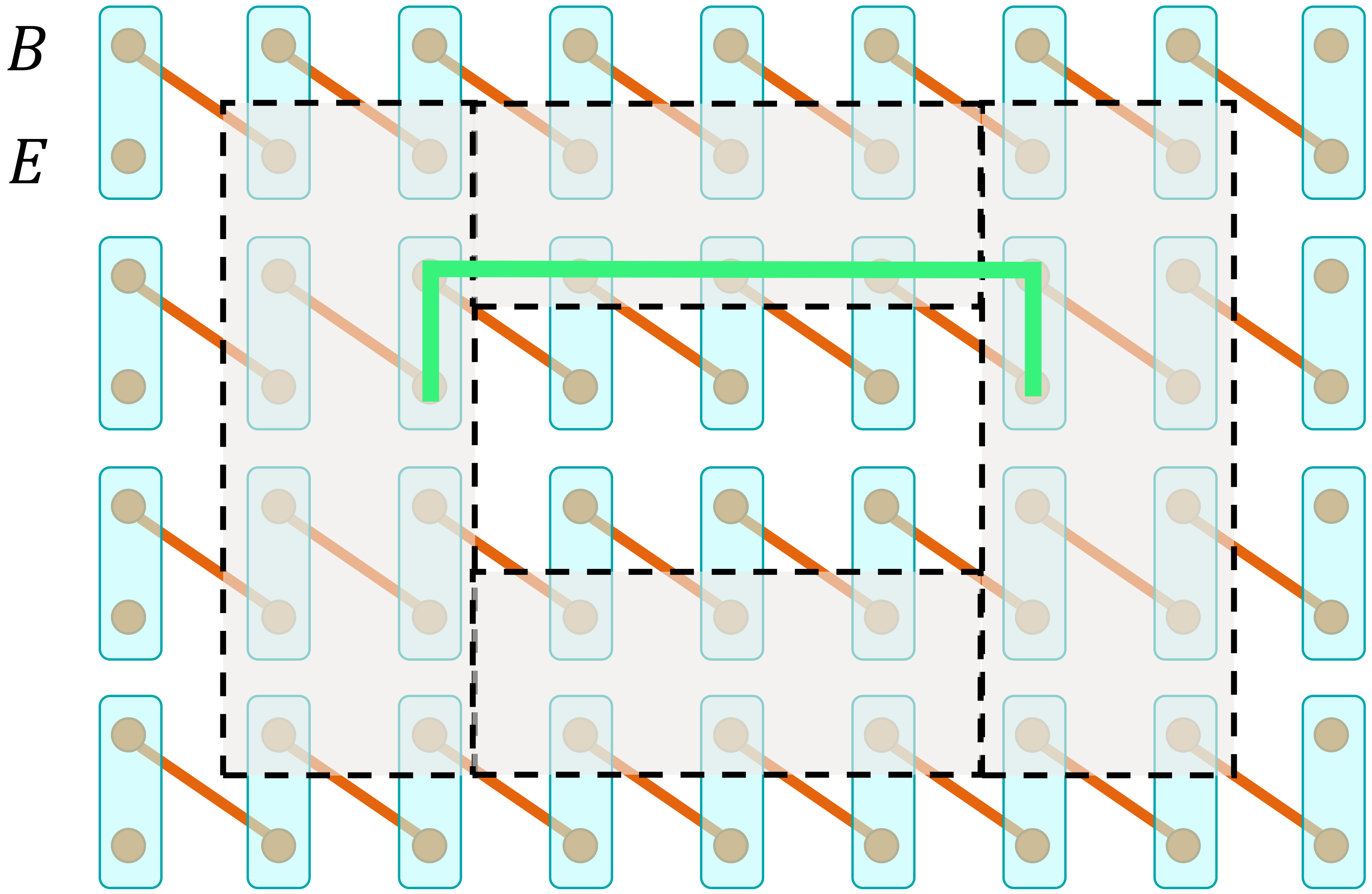

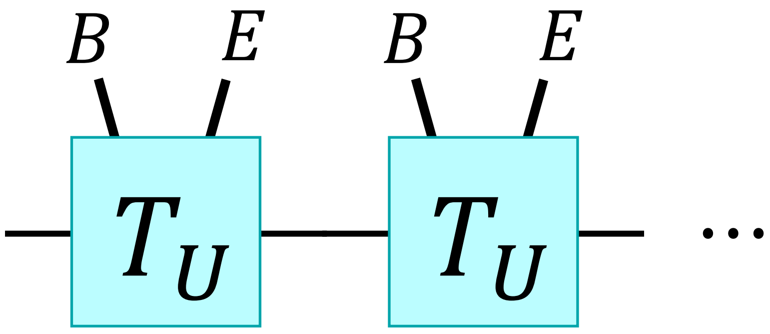

$$S_{topo} = 2 > 2\gamma = 0.$$

Open problems

Entanglement bootstrap

- What happen if the axioms are approximately satisfied?
- More general axiom replacing axiom A1?
- Higher-order central charge?

Spurious TEE

- What is the exact condition for the spurious TEE ?
- How can we distinguish different correlations?

References

Entanglement bootstrap
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