3/2023

UCL

Holographic Duality between Local Hamiltonians from Random Tensor Networks

Harriet Apel

Joint with Toby Cubitt & Tamara Kohler

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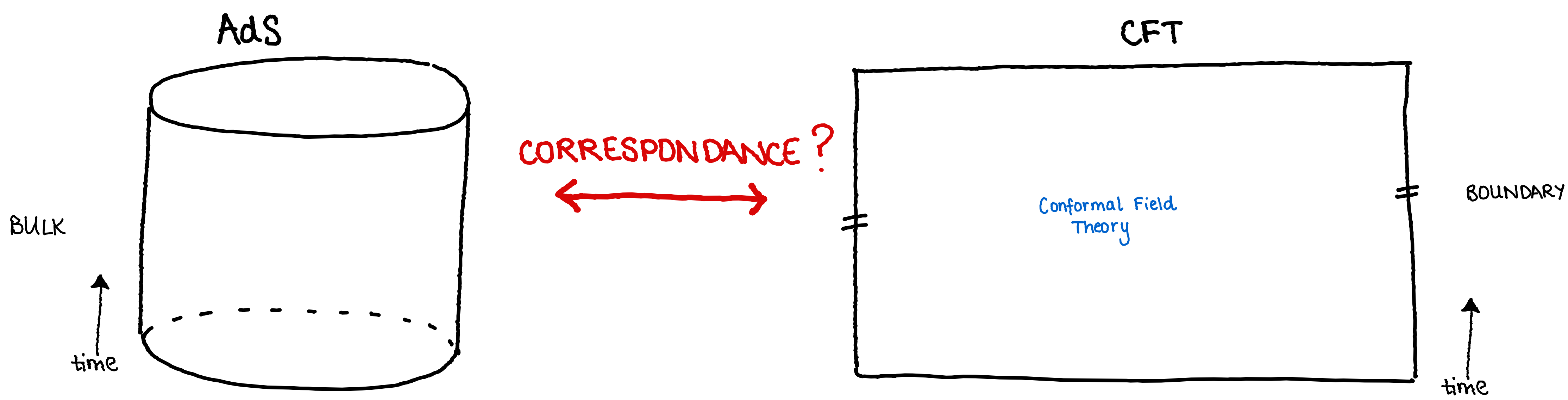
arXiv:2105.12067

MOTIVATION FROM HIGH ENERGY PHYSICS

Realisation of the holographic principle.



Weakly interacting semi-classical gravity

Strongly interacting CFT ?

Quantum gravity ??

'Just' a QFT ?

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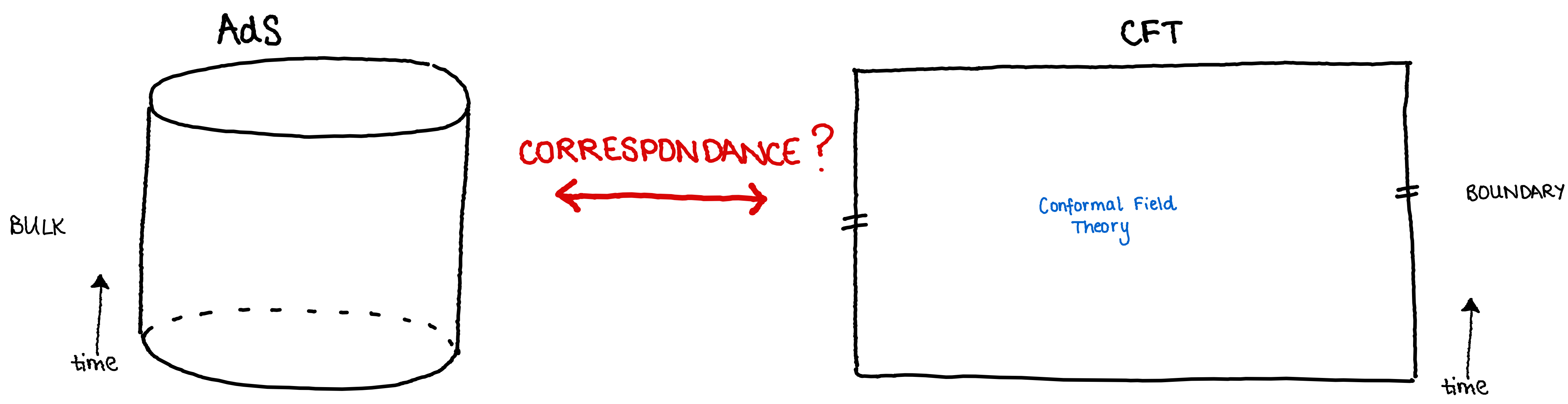

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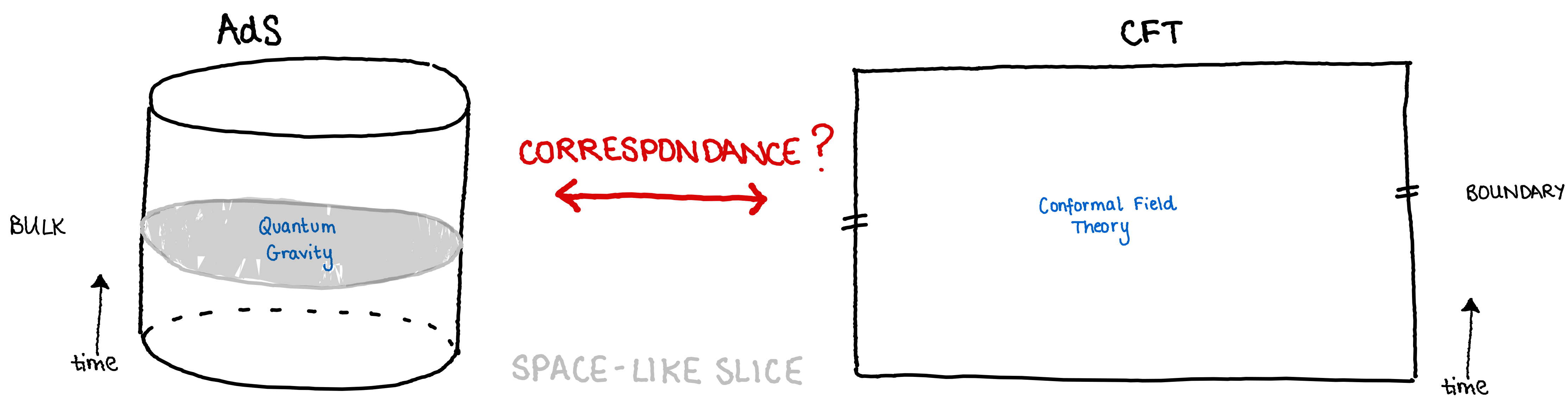

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TOY MODEL OF AdS/CFT

① ERROR CORRECTION

AdS/CFT behaves as an erasure code.

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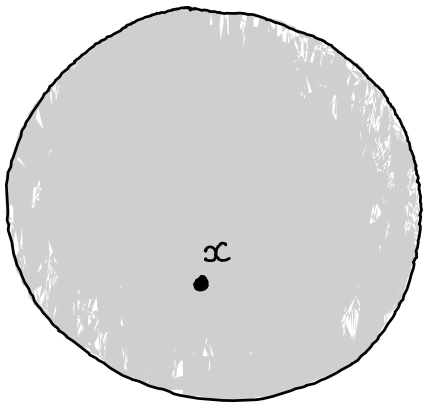

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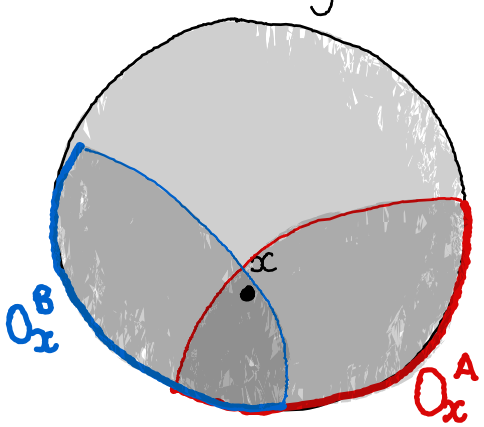

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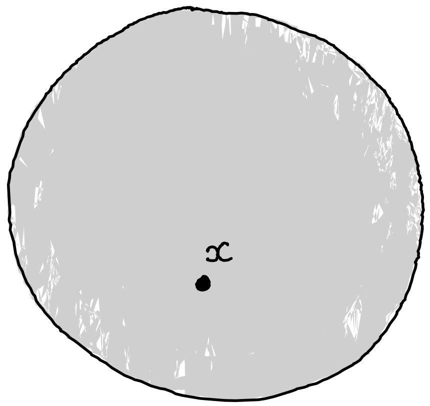

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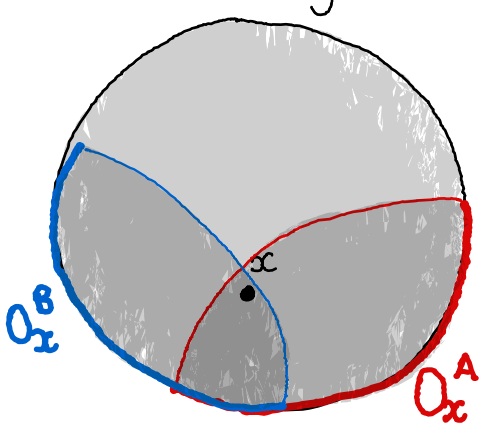

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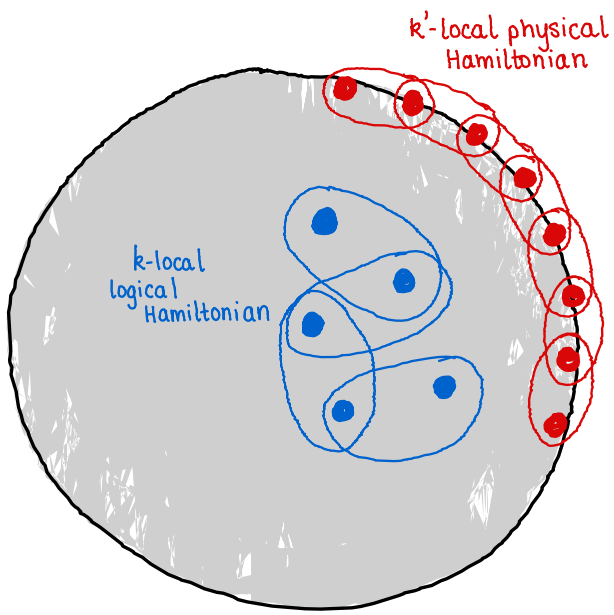
② LOCAL HAMILTONIANS

A local Hamiltonian in the bulk corresponds to a local boundary Hamiltonian.



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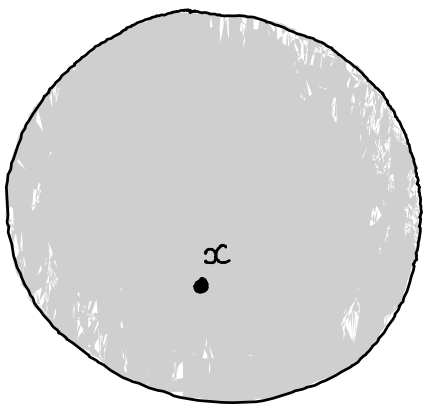

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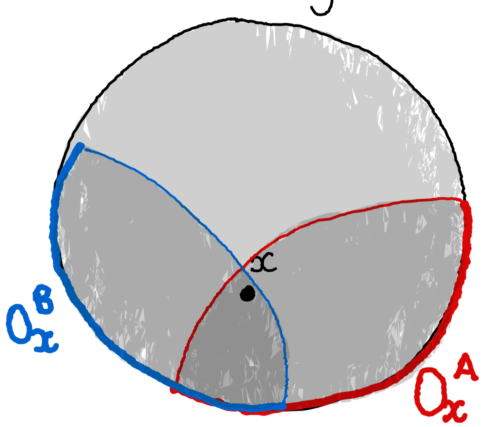

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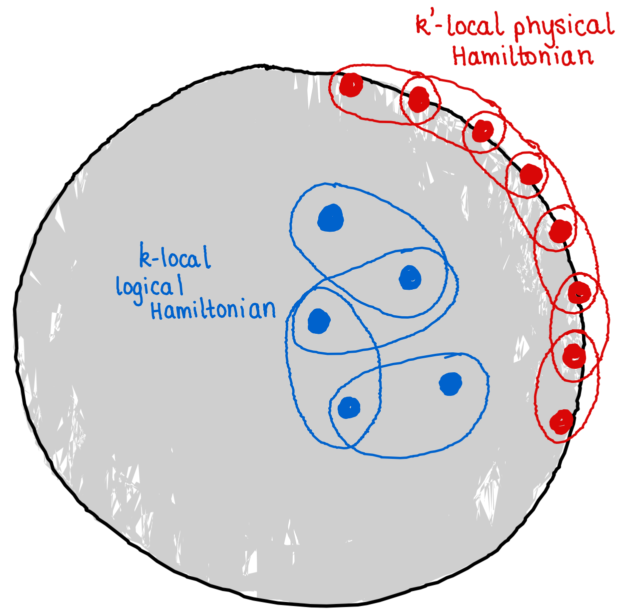

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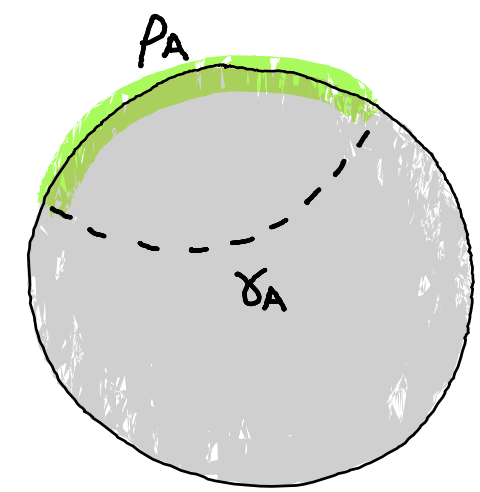
③ ENTANGLEMENT

Entropy of a codeword restricted to a subregion of the boundary, ρ_A



is related to an area in the bulk

$$S(\rho_A) \propto |\gamma_A|$$

the Ryu-Takayanagi formula.

OUTLINE

MAIN RESULT [AKC22] : We construct a holographic approximate error correction code that is the first model to simultaneously achieve these properties of AdS/CFT

1. Holographic tensor network codes
   - isometric networks
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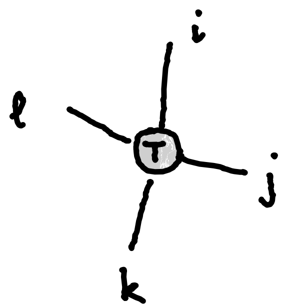
HOLOGRAPHIC TENSOR NETWORKS

↳ Quick tensor recap



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HOLOGRAPHIC TENSOR NETWORKS

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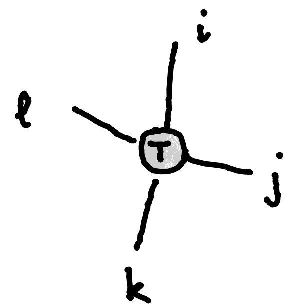

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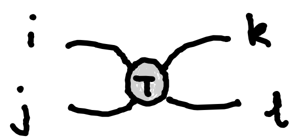

$$V\,|i\rangle|j\rangle = \sum_{kl} T_{ijkl} |k\rangle|\ell\rangle$$

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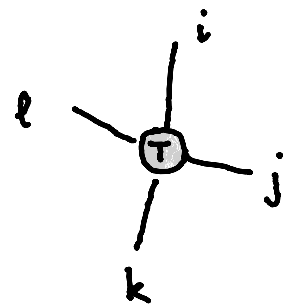

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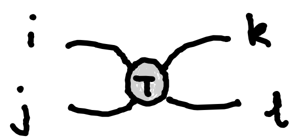

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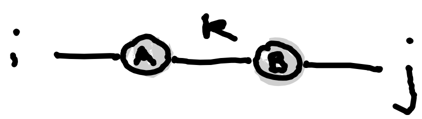
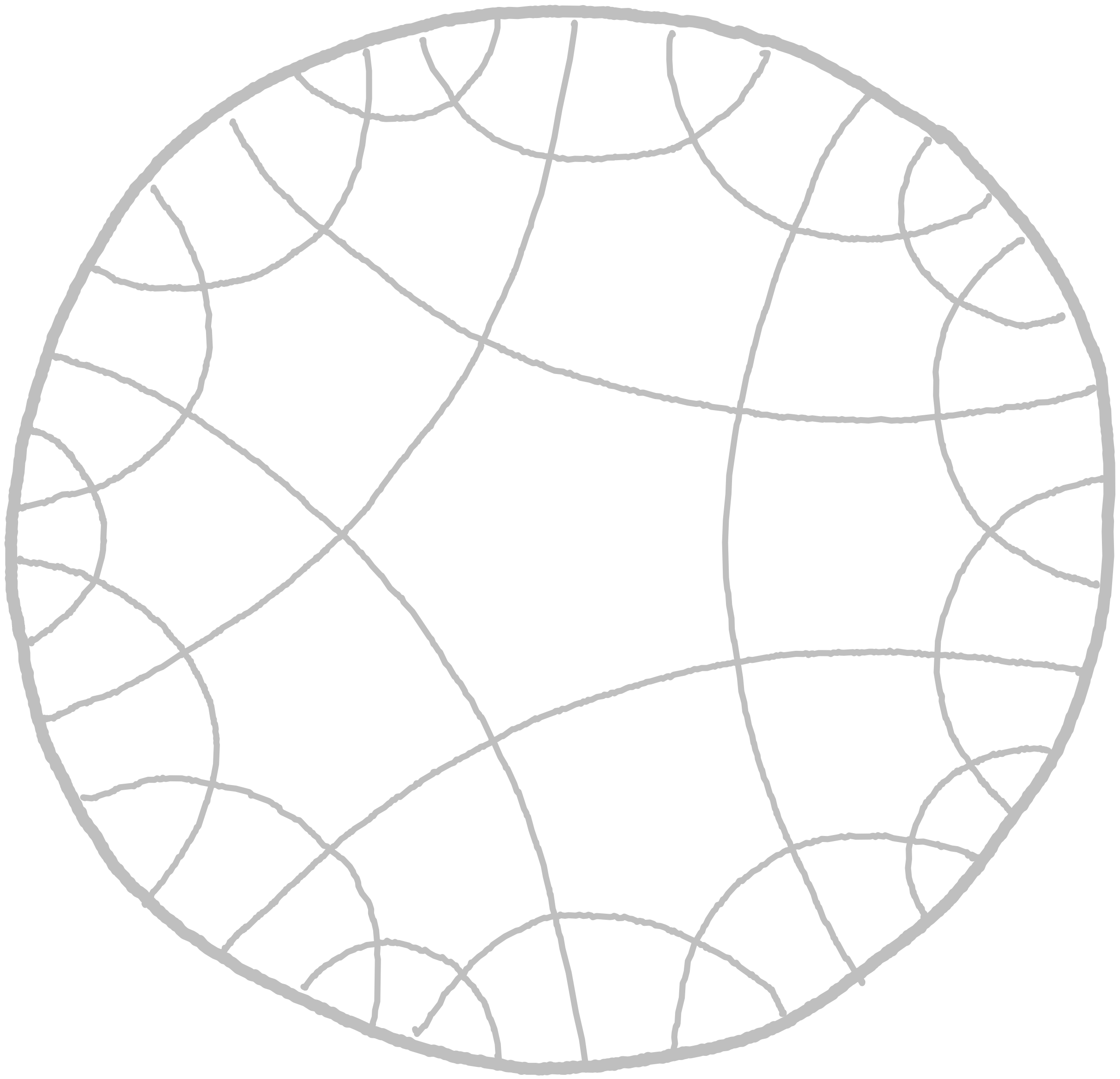


$$\sum_{k} A_{ik} B_{kj}$$

HOLOGRAPHIC TENSOR NETWORKS

[HaPPY 2015]



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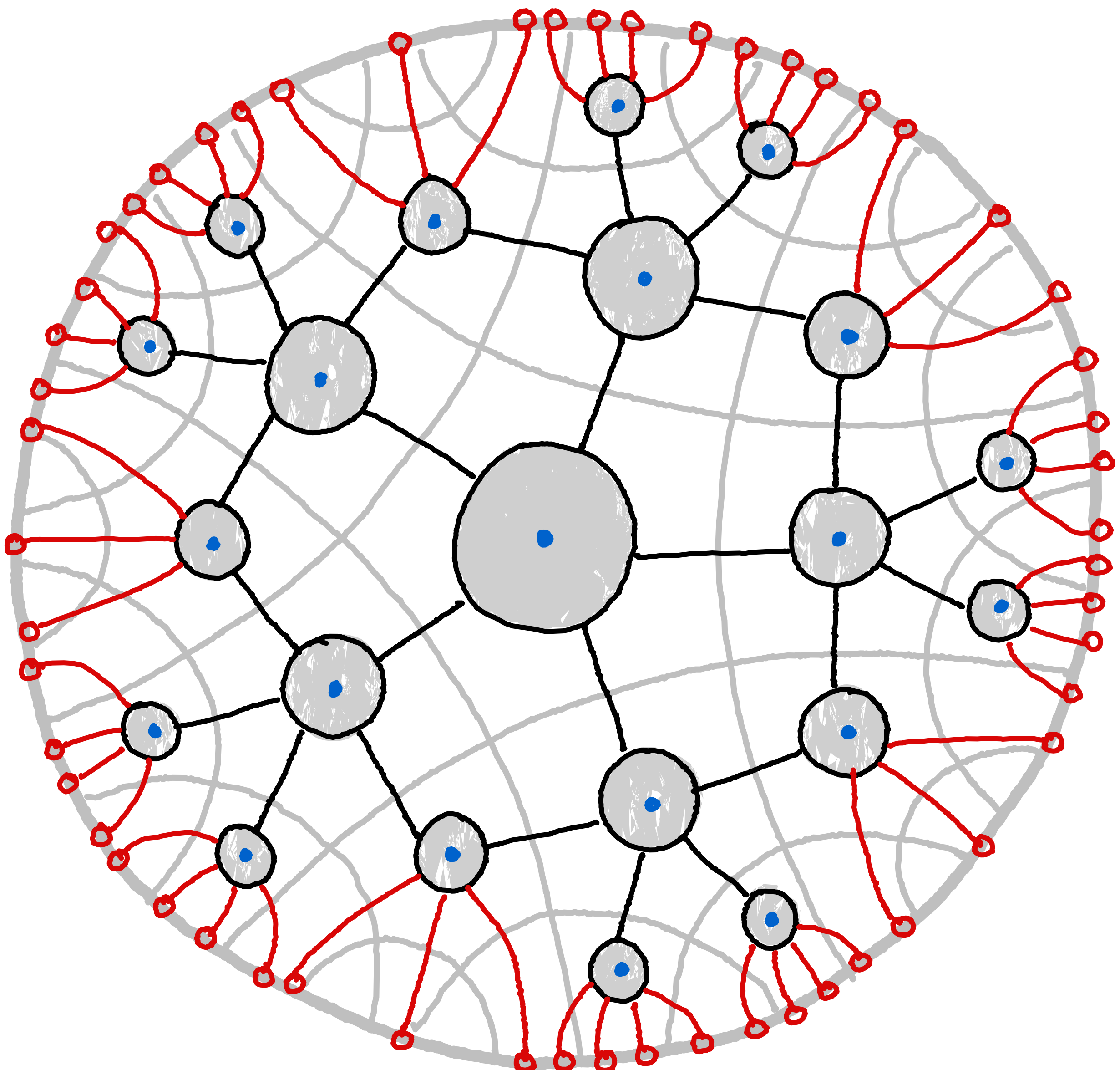
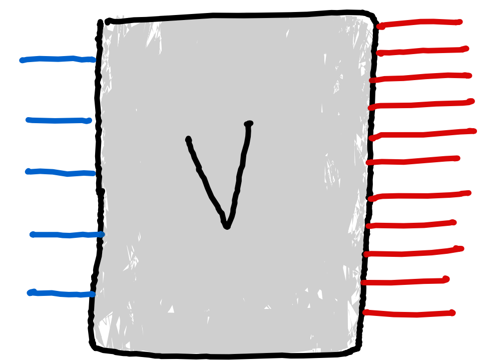


Tensor network describes a map

$\cong$



- Bulk degrees of freedom
  LOGICAL QUBITS

∘ Boundary degrees of freedom
  PHYSICAL QUBITS

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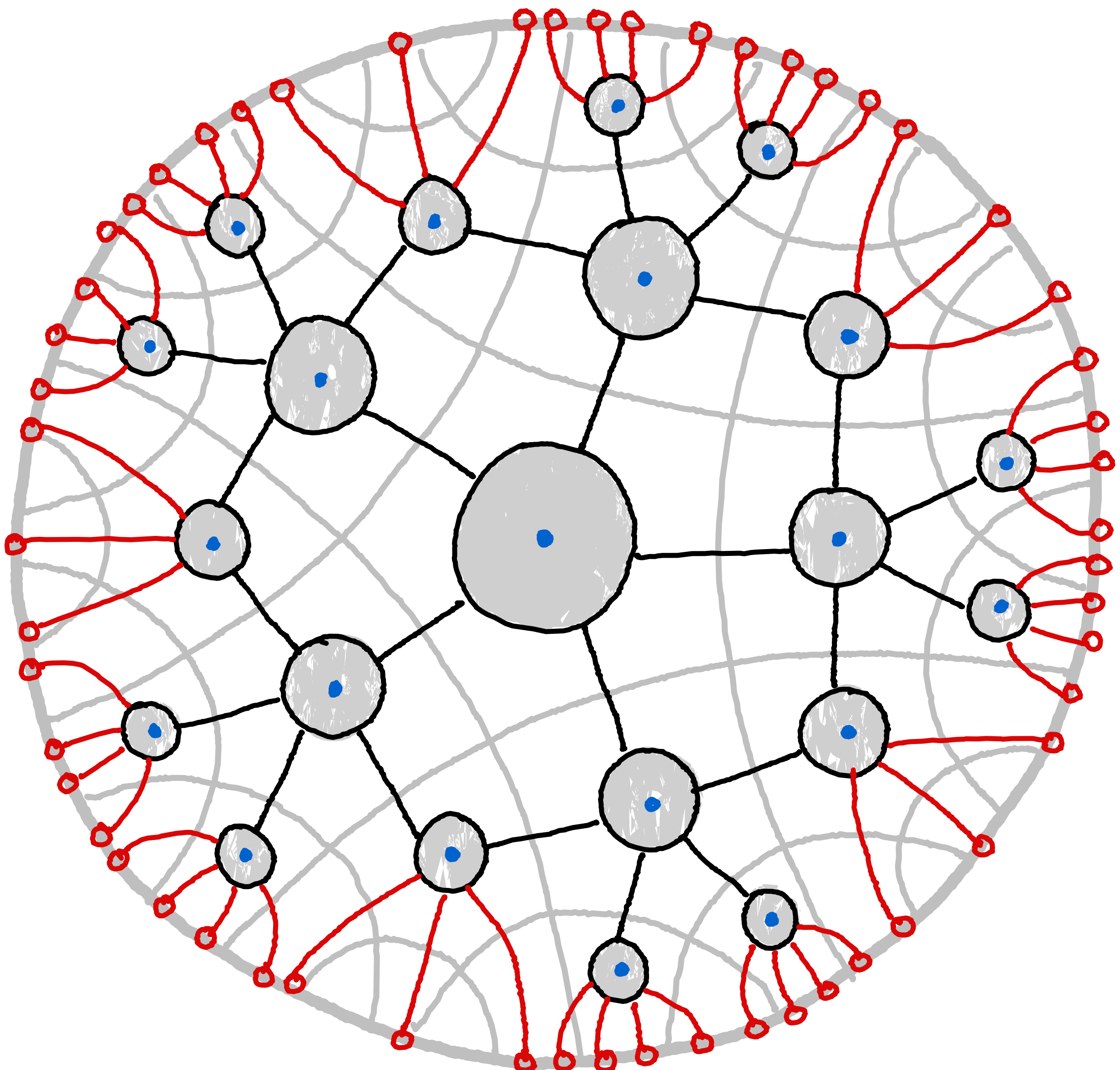

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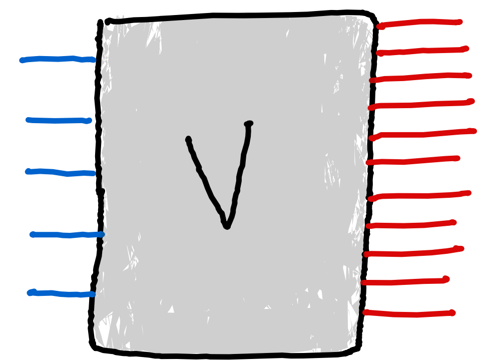
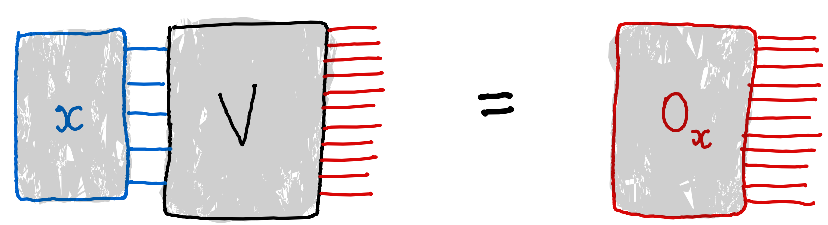


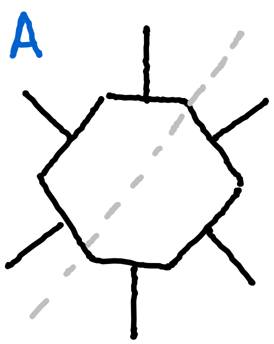
Operators are pushed through the network



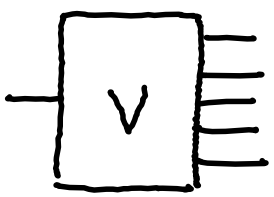
Bulk operator

Boundary operator

CHOICE OF TENSOR → isometric map



$$\rho_A = \frac{\mathbb{1}_A}{d_A}$$



Perfect tensors := special class of tensors that are maximally entangled across all partitions

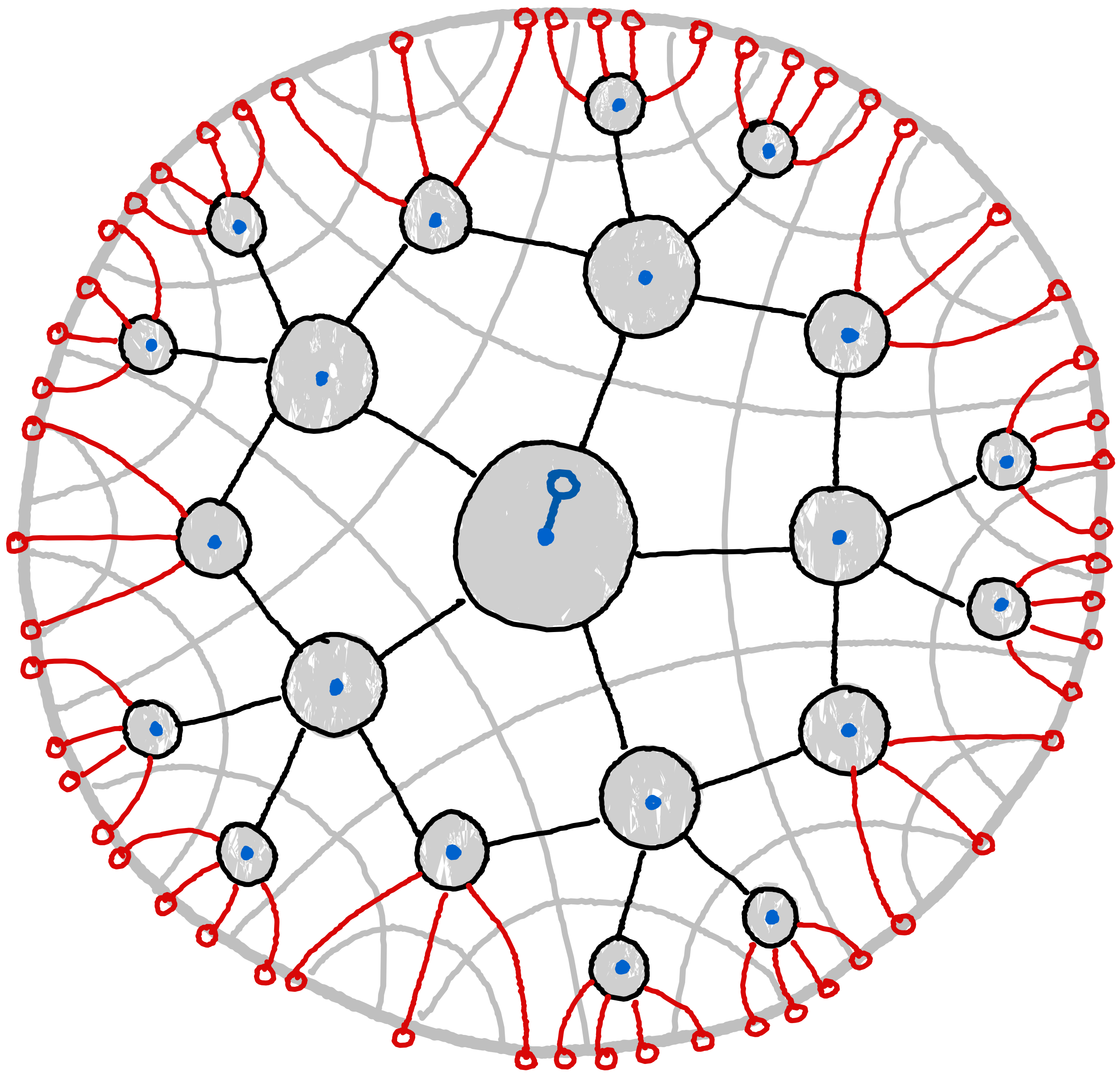
→ describe encoding isometries when
input legs ≤ # output legs

e.g. 6-leg perfect tensor describes the 5-qubit code
[[5, 1, 3]]

distance = 3 so can recover all information after erasure of any 2 output indices

→ perfect tensors saturate the Quantum Singleton bound => have good error correcting properties

EC WITH PERFECT TENSORS



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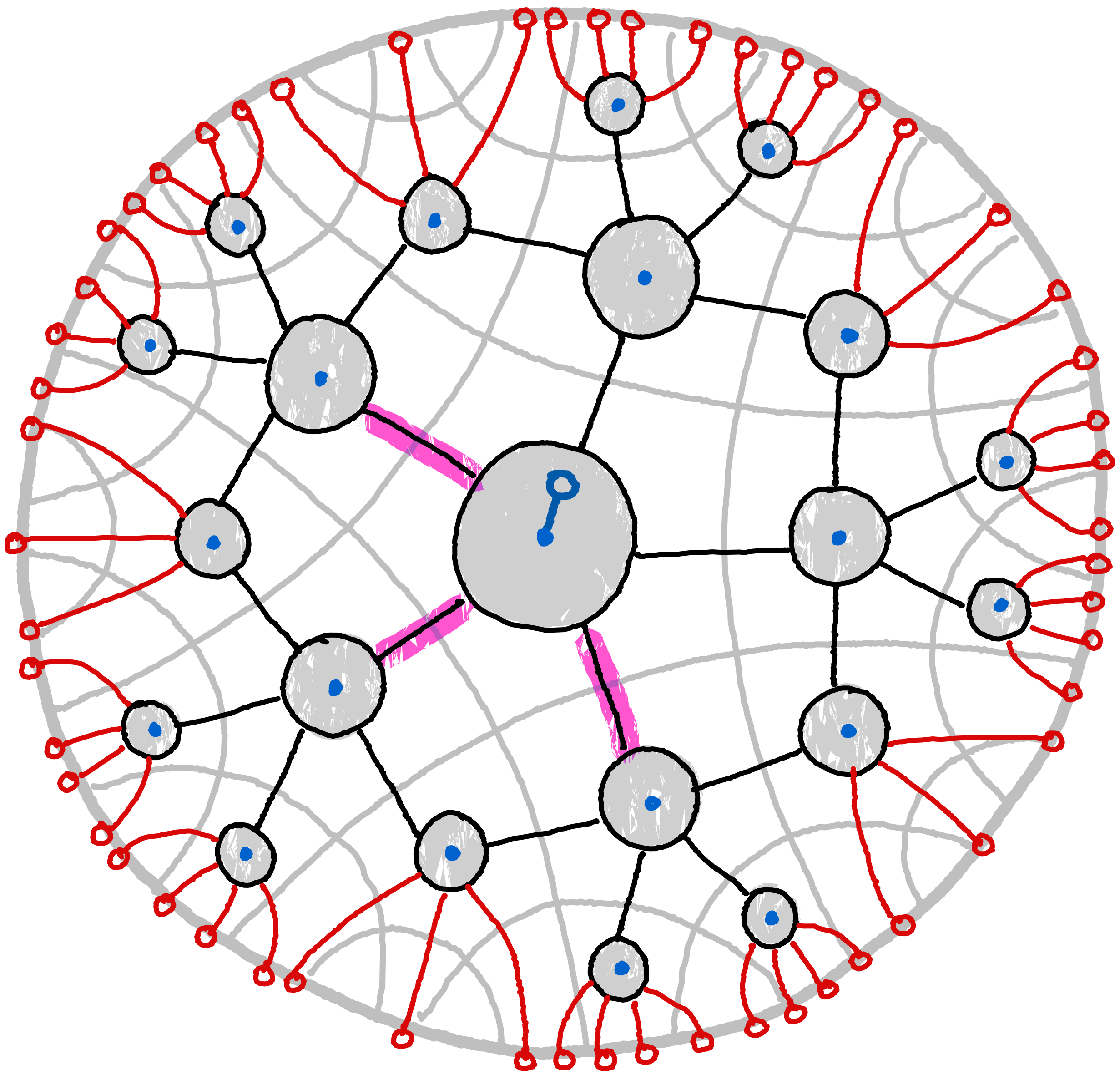

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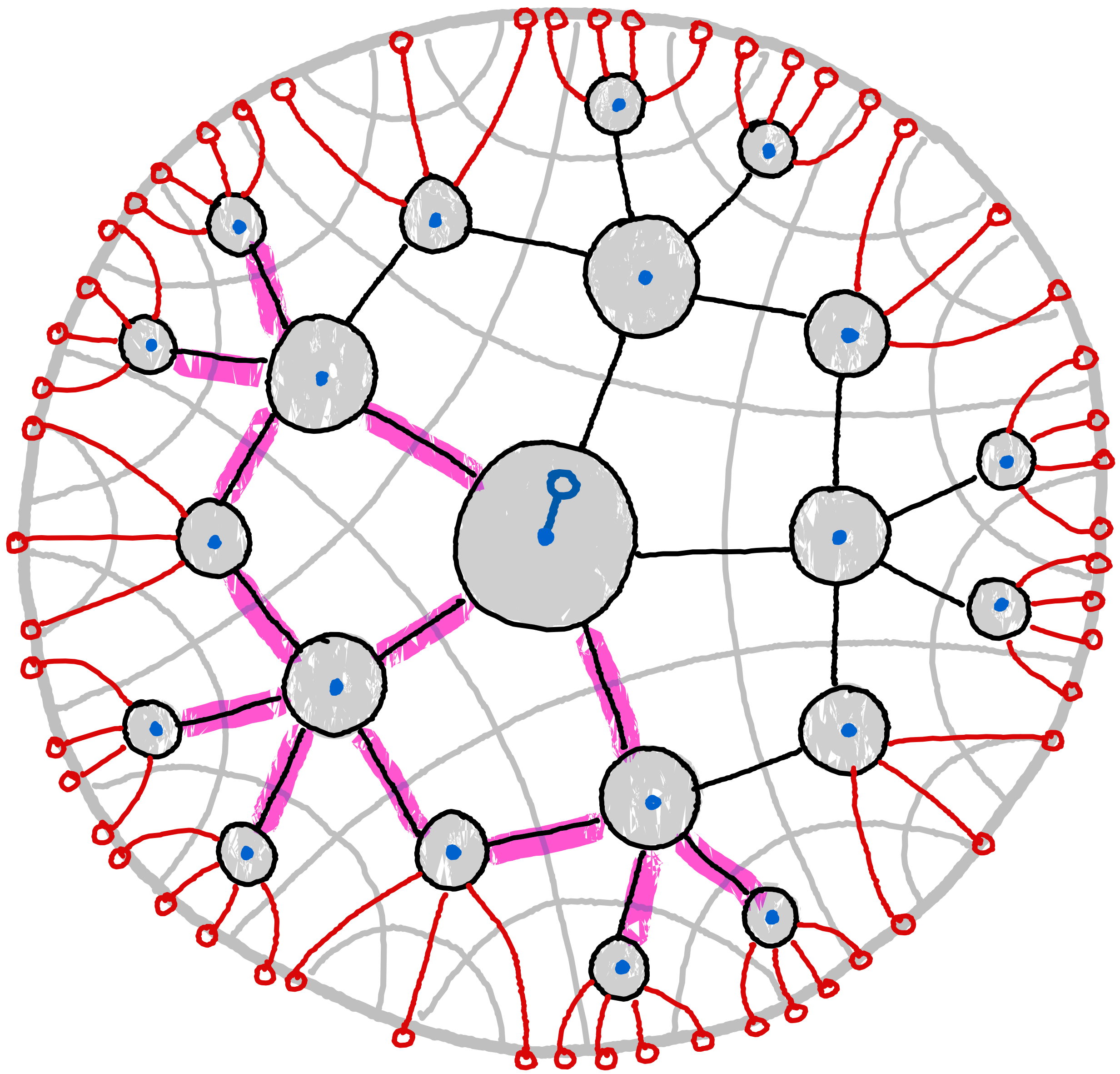

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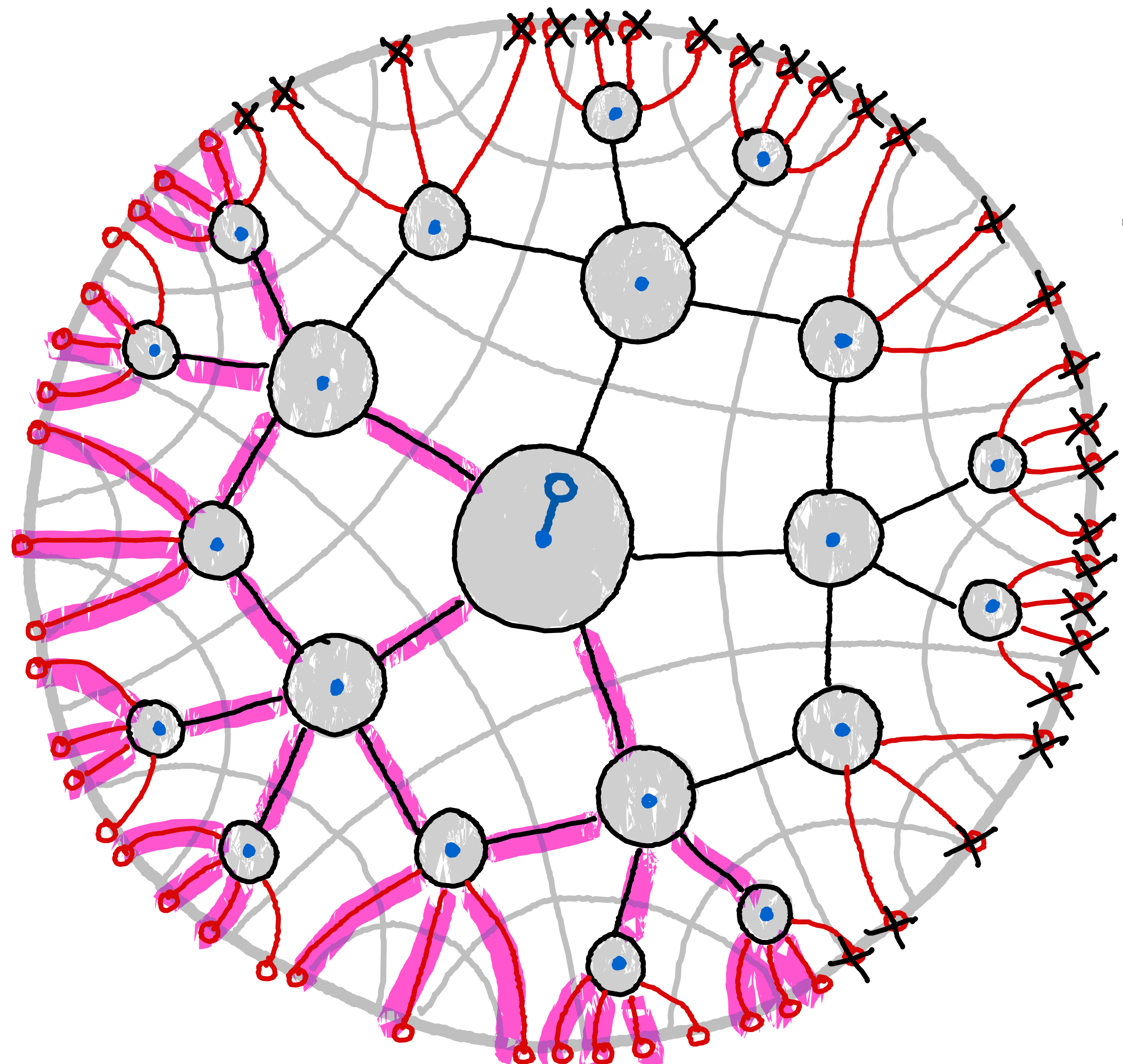
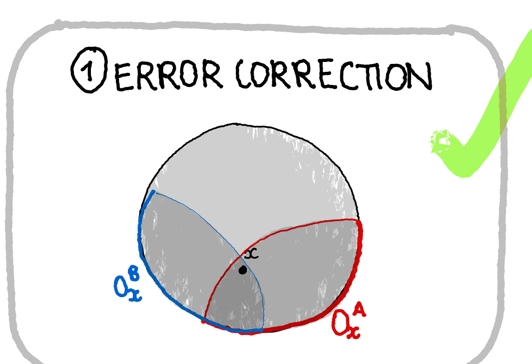


Information protected against erasure of the rest of the boundary qubits.



ENTANGLEMENT OF PERFECT NETWORKS

GOAL:

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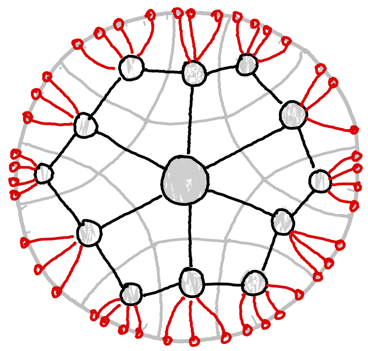

$\longrightarrow$ RT Formula holds

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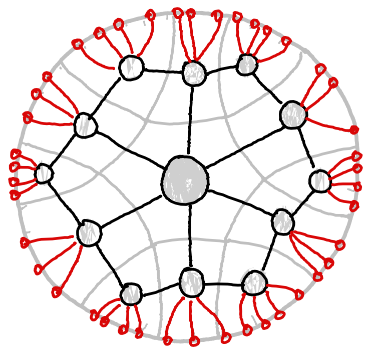

→ RT Formula holds

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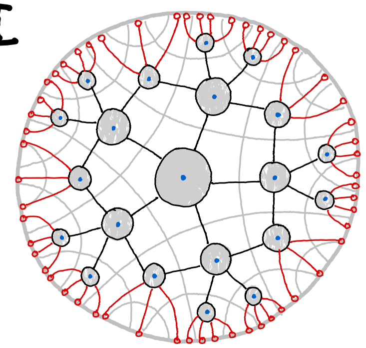

→ Provable entropy bounds

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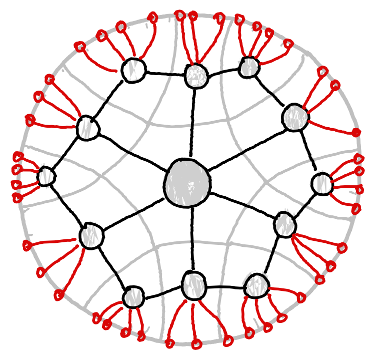

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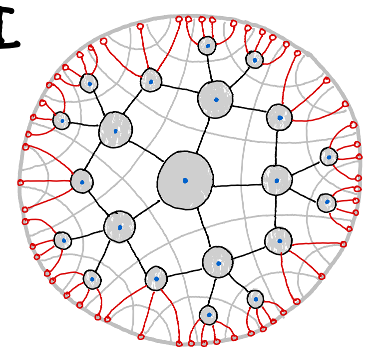

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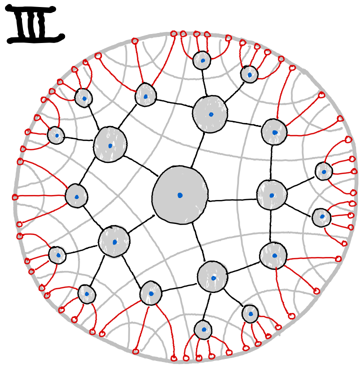

→ Proof techniques fail

BETTER PROOF TECHNIQUES FOR RANDOM NETWORKS

Random tensor ~ random state, generated by applying a Haar random unitary to a reference state.

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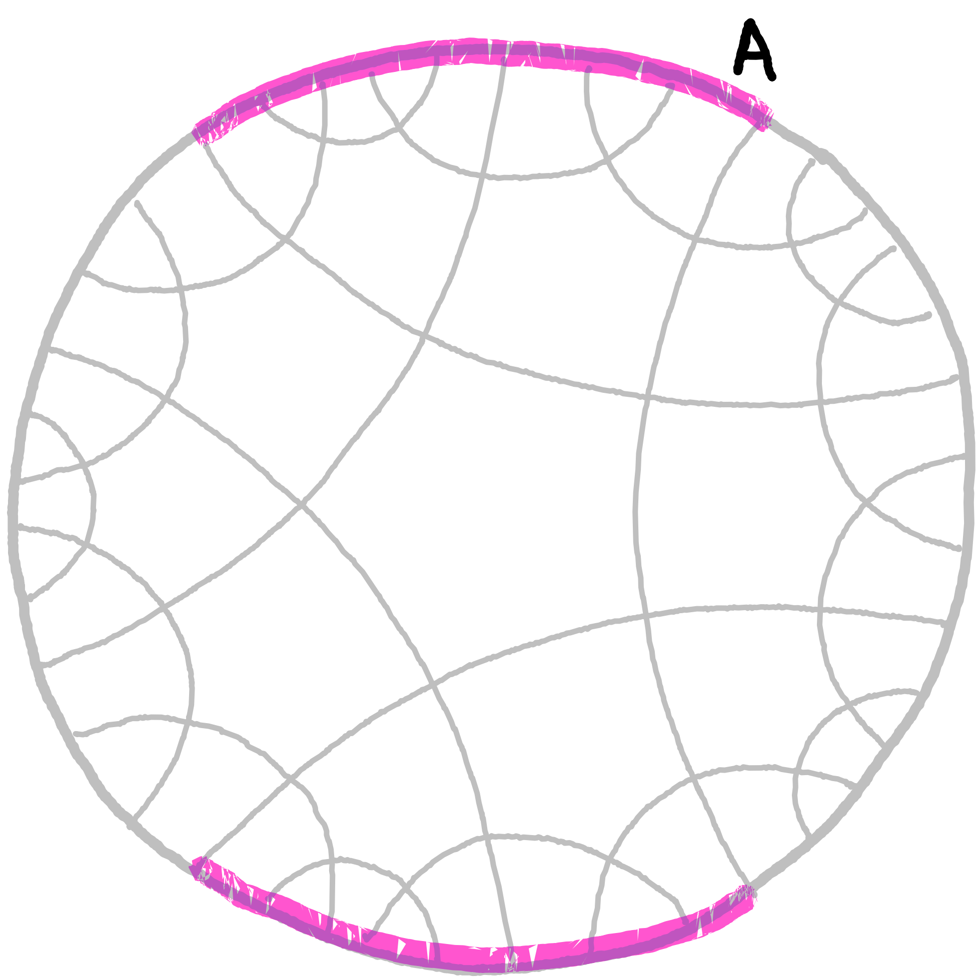

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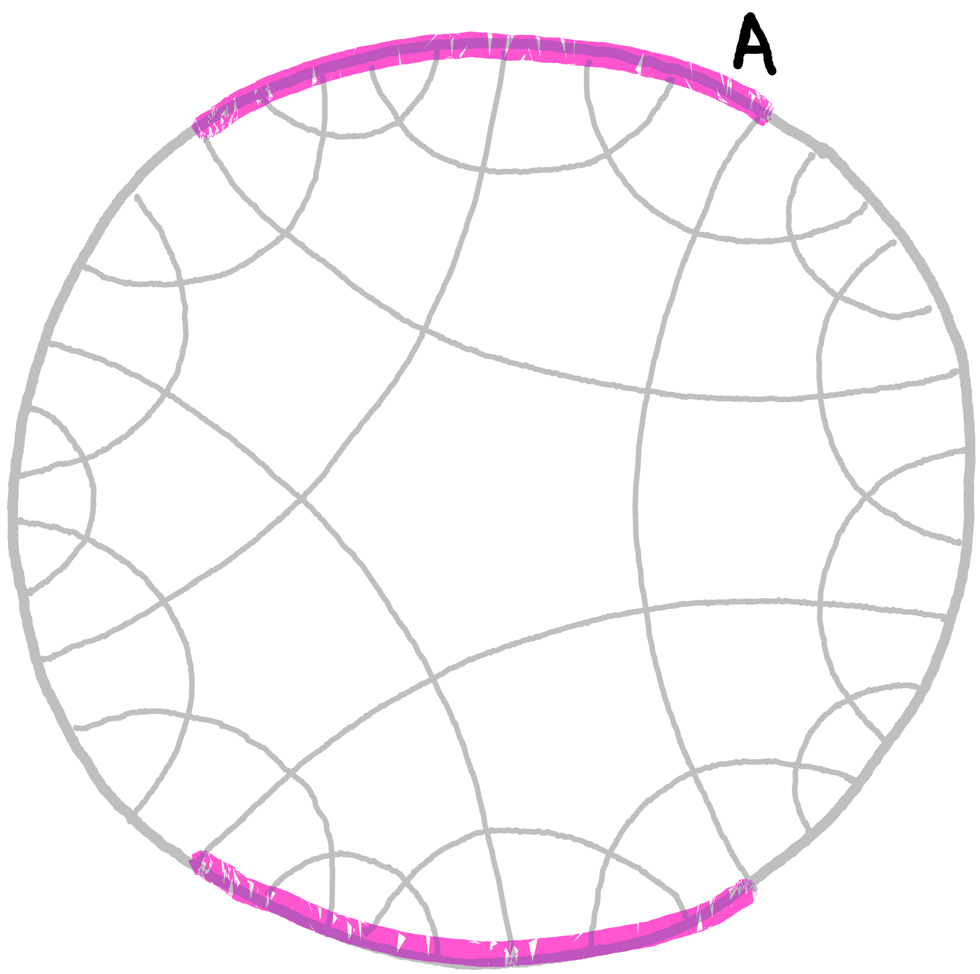

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Ising Action

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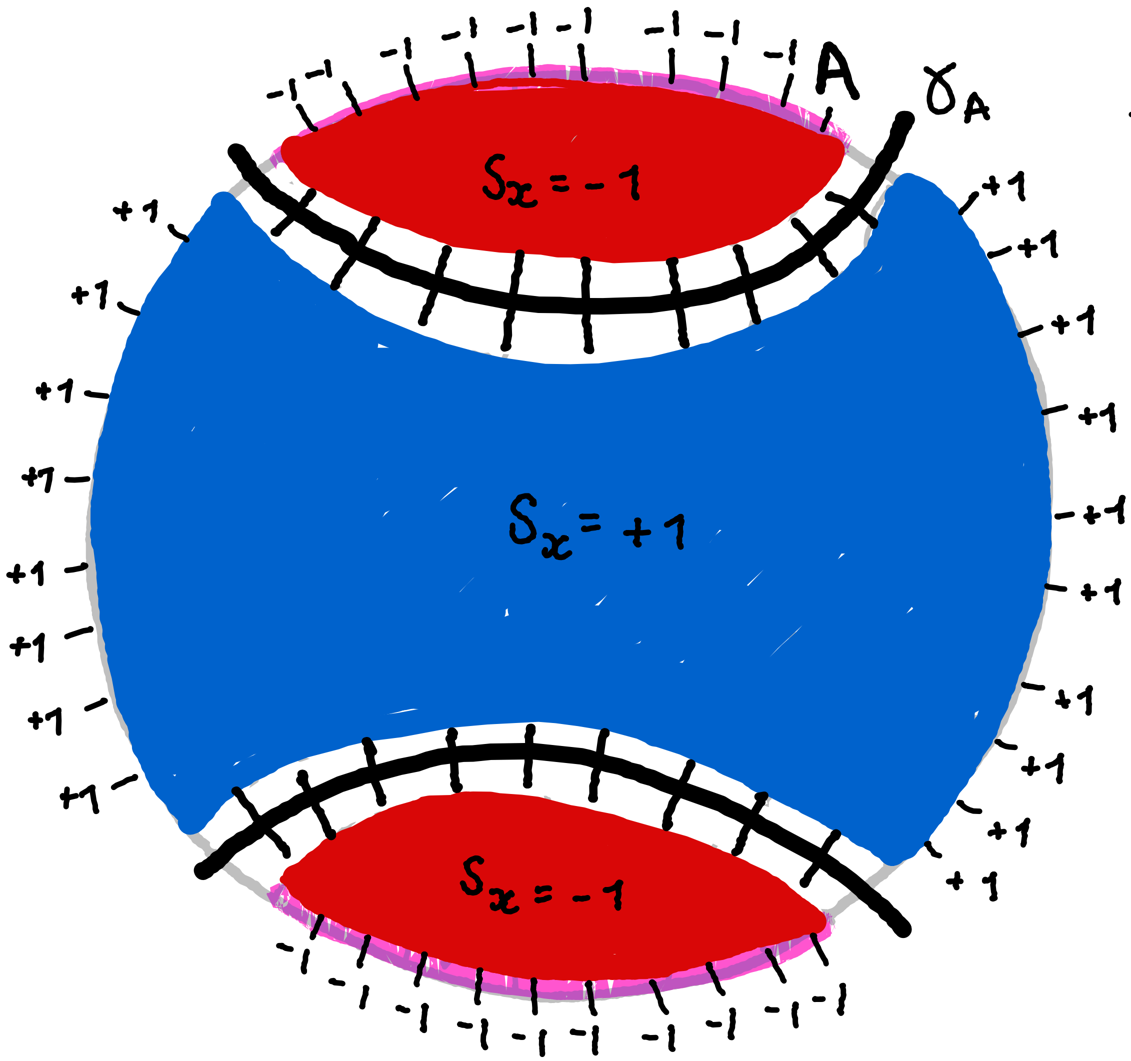

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Ising Action

→ High bond dimension regime ~ low temperature regime

∴ RT surfaces manifest as spin domain walls

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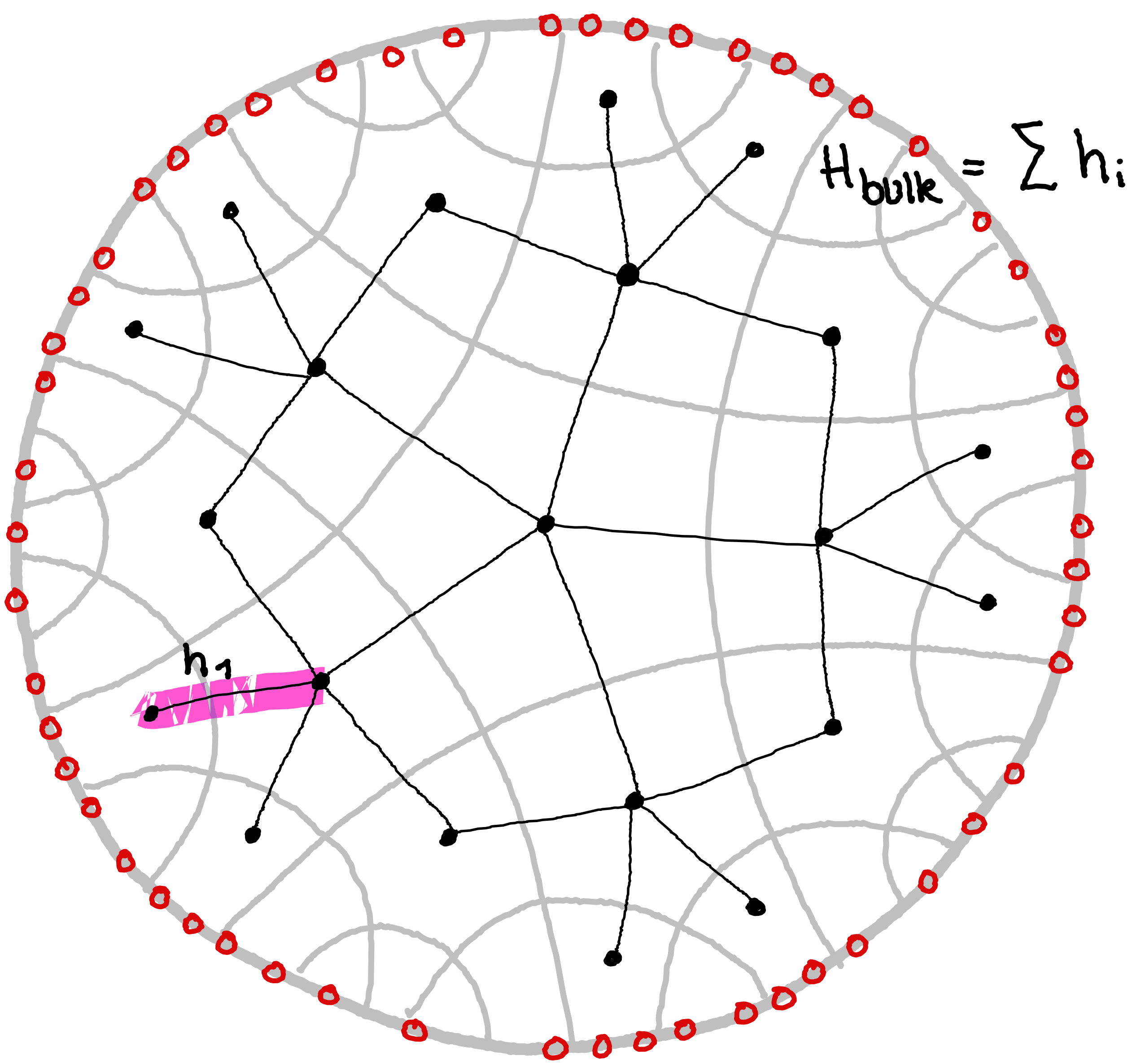

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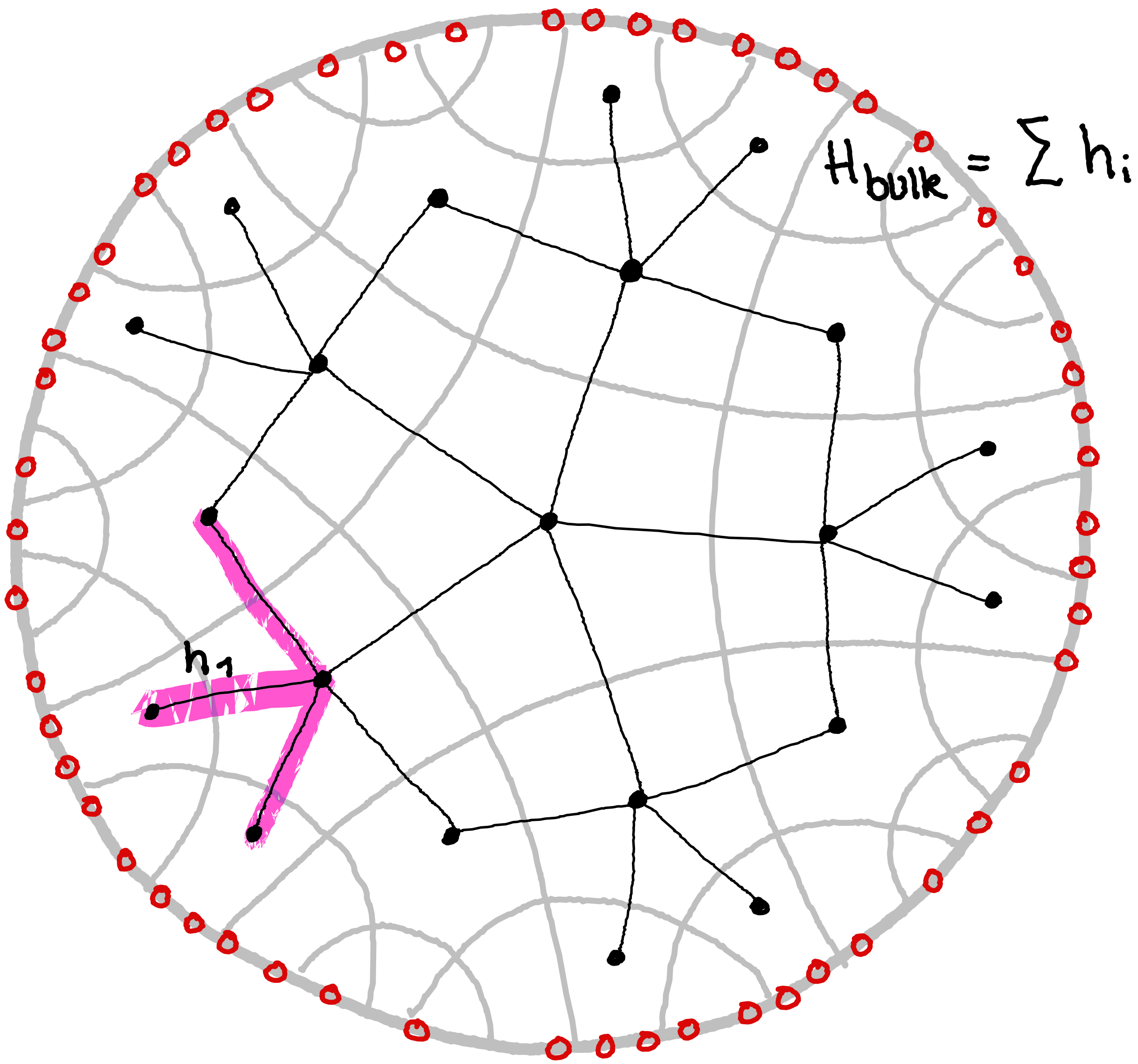

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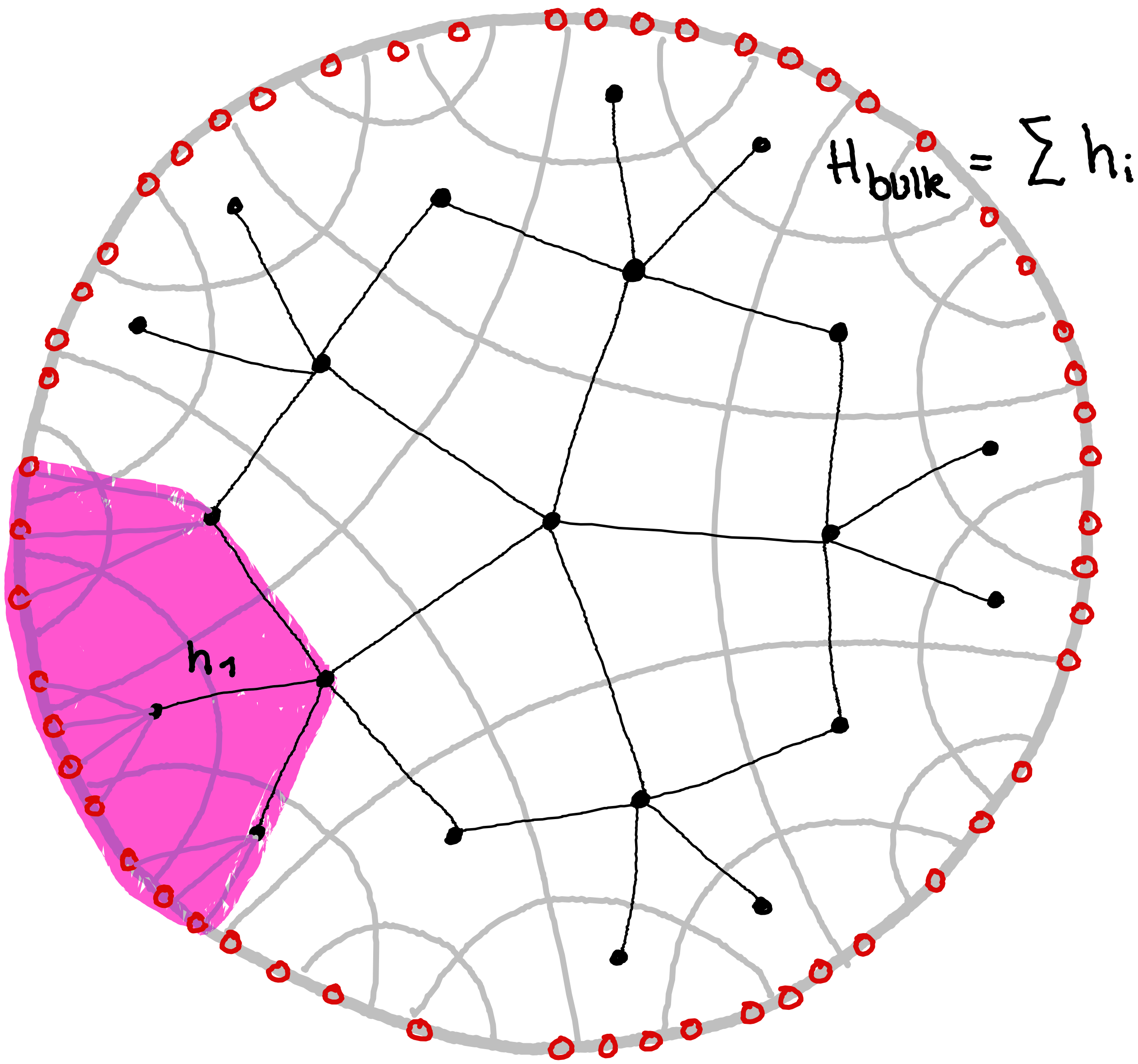

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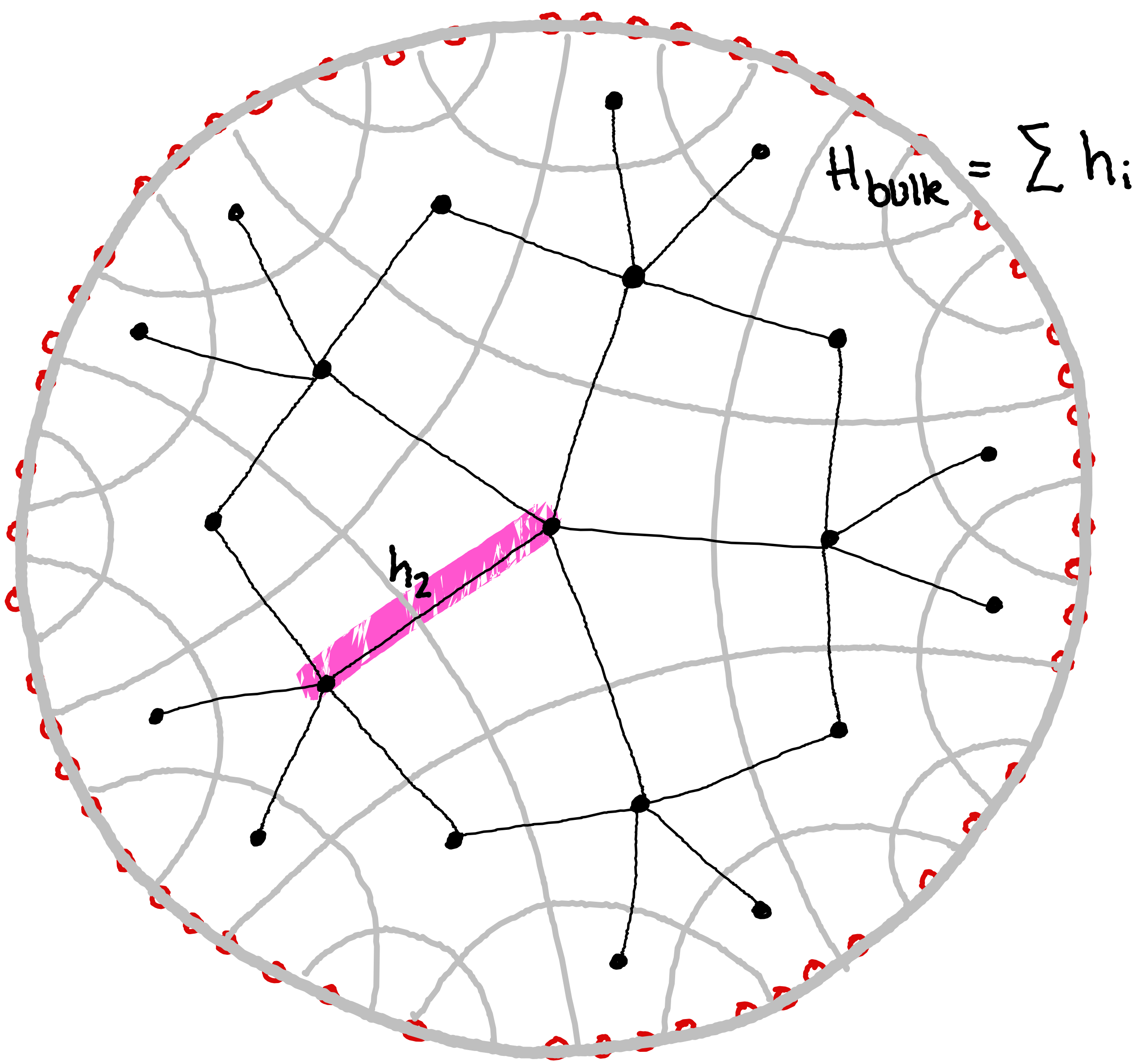

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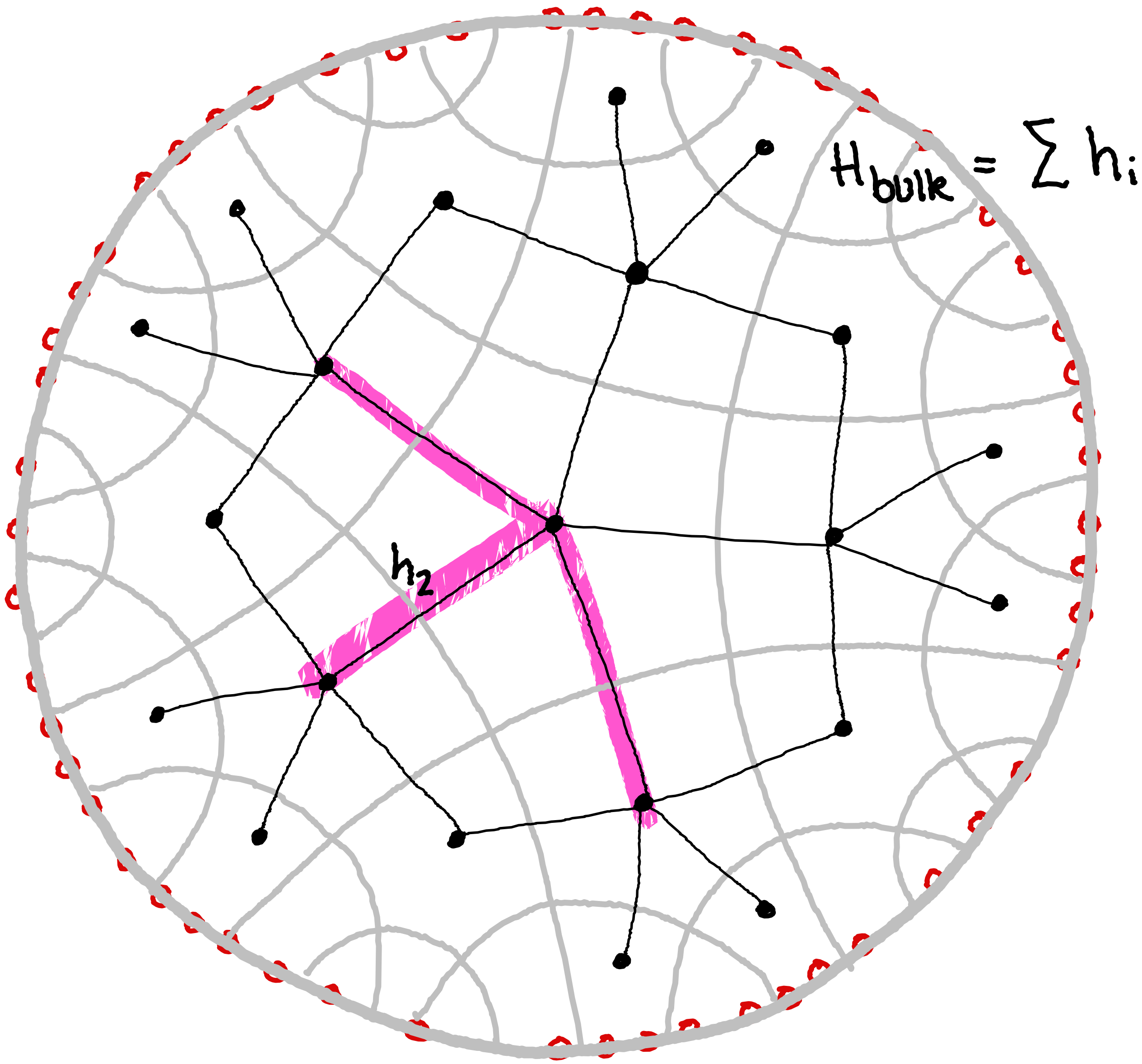

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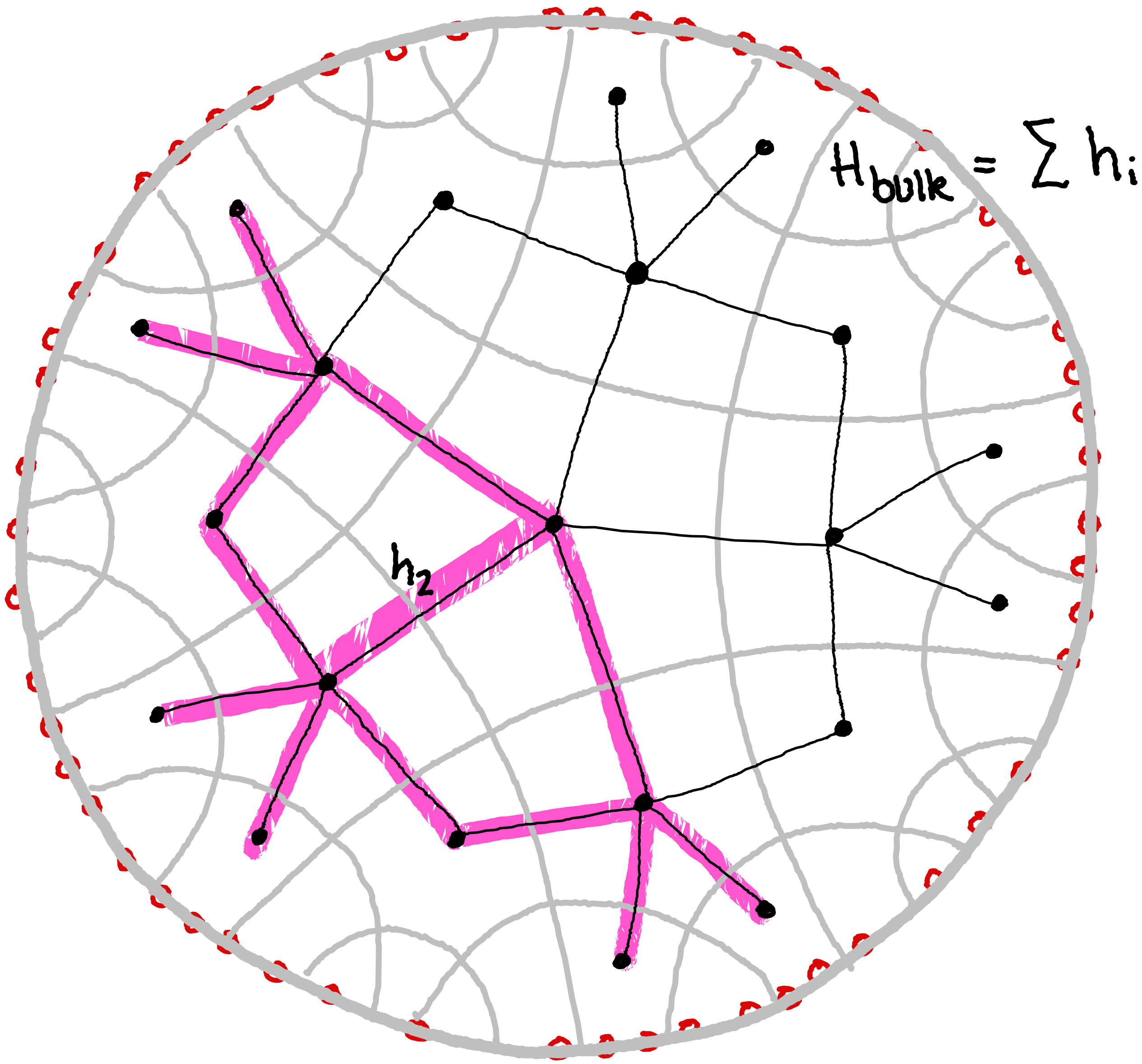

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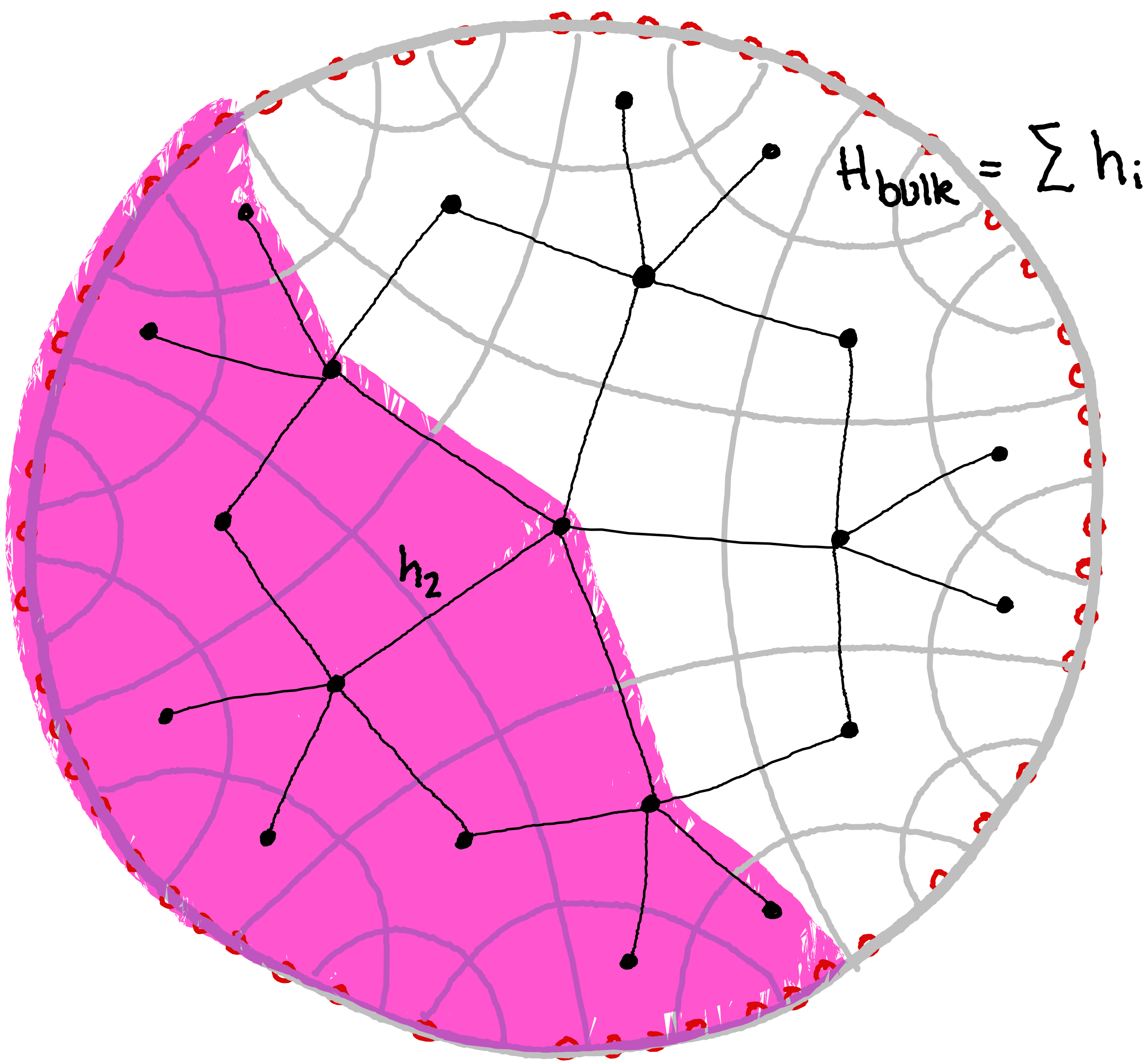
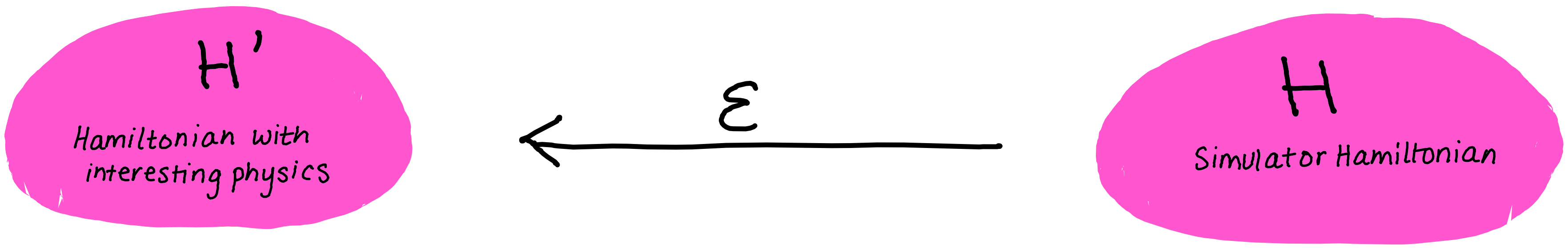


- Mapping a local bulk Hamiltonian gives a non-local boundary Hamiltonian
- Bulk Hamiltonian terms from deeper inside the bulk correspond to higher weight terms on the boundary
- Tensor network codes alone give good toy models of states and observables but not for Hamiltonians and dynamics

HAMILTONIAN SIMULATION



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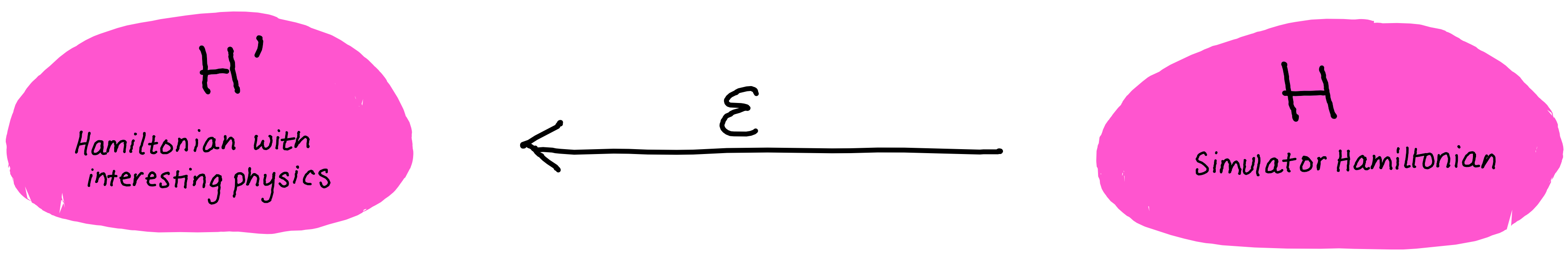

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$$\| \mathcal{E}(H)|_{\leq \Delta} - H' \|_\infty = 0$$

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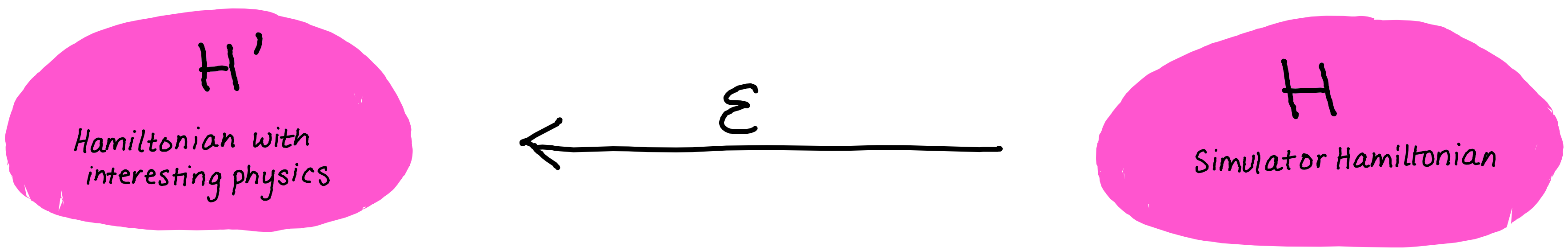

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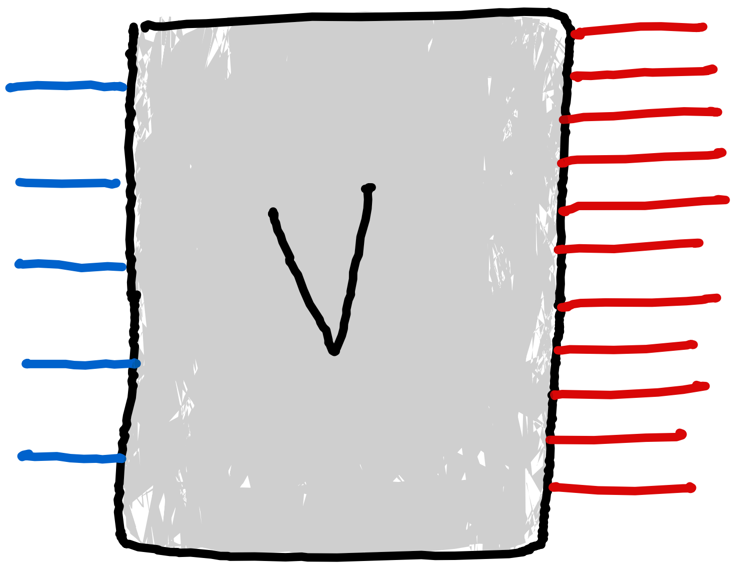
- Approximate simulation

$$\| \mathcal{E}(H)|_{\leq \Delta} - H' \|_\infty \leq \delta$$

QEC AS A SIMULATION

The tensor network encoding is a perfect simulation into a subspace



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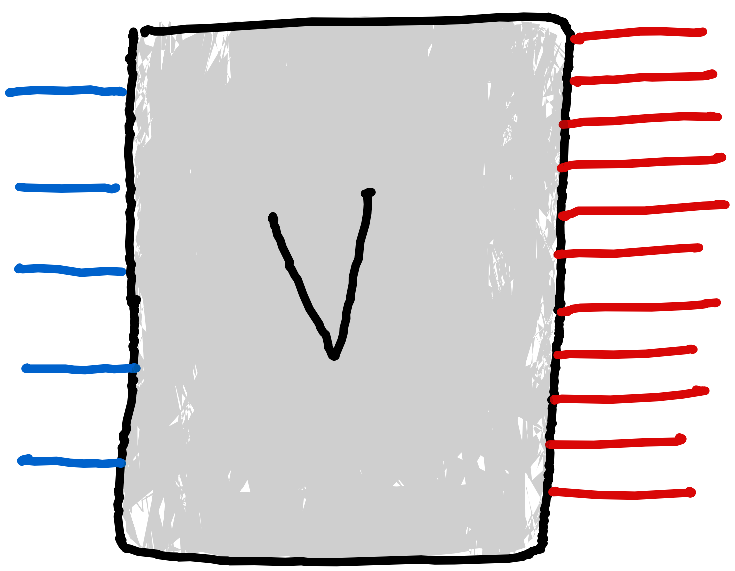

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support on the complement of the code space : adds an energy penalty to each stabilizer violated.

$$\left\| H_{bo}\big|_{\leq \Delta_s} - \mathcal{E}(H_{bu}) \right\| = 0$$

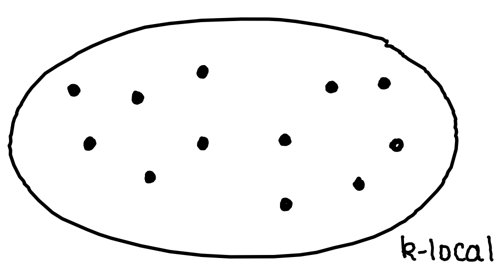
APPROXIMATE SIMULATION

Hamiltonian simulation techniques morph Hamiltonian interaction graphs while retaining all physical properties of the Hamiltonian up to a controllable error that can be made arbitrarily small.

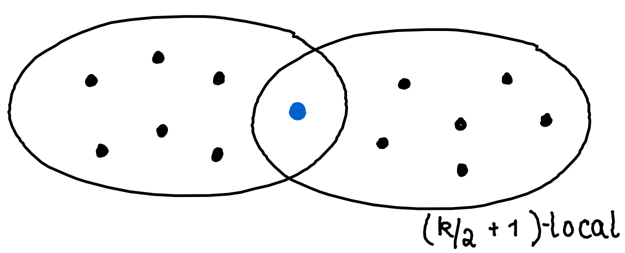
EG. PERTURBATION GADGETS

SUBDIVISION

→ Reduce the weight of interactions

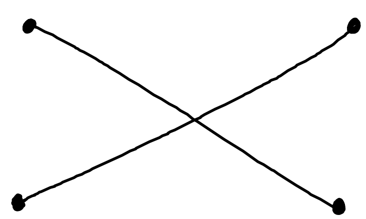


simulated by...



CROSSING

→ Remove crossings to make geometrically local



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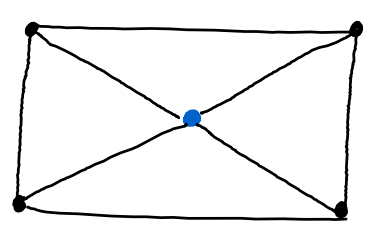
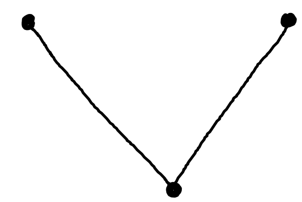


FORK

→ Reduce degree of vertices



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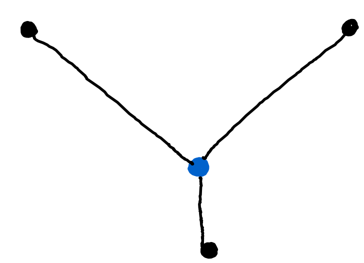

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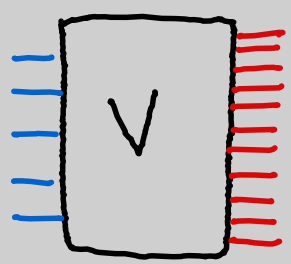
COMPOSE SIMULATIONS

LOCAL HAMILTONIAN ON BULK QUBITS

TENSOR NETWORK ENCODING

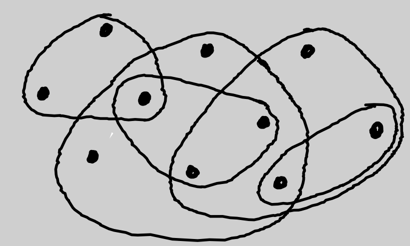


$H_{bo} = V H_b V^\dagger + \Delta_S \sum_s h_s$

is a perfect sim [KC19]

NON-LOCAL HAMILTONIAN ON BOUNDARY QUBITS

SIMULATION TECHNIQUES



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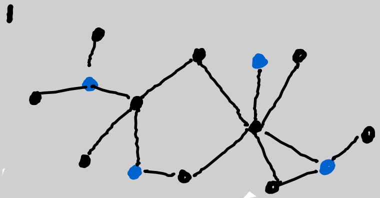

LOCAL HAMILTONIAN ON A NEW SET OF BOUNDARY QUBITS

COMPOSE SIMULATIONS

APPROXIMATE HOLOGRAPHIC QEC

LOCAL HAMILTONIAN ON BULK QUBITS

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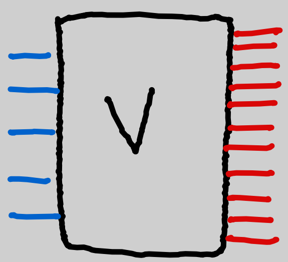

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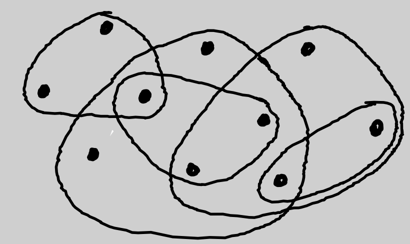

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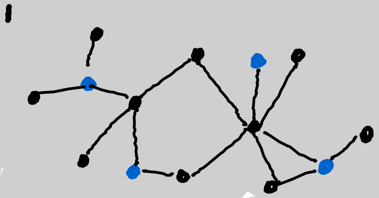
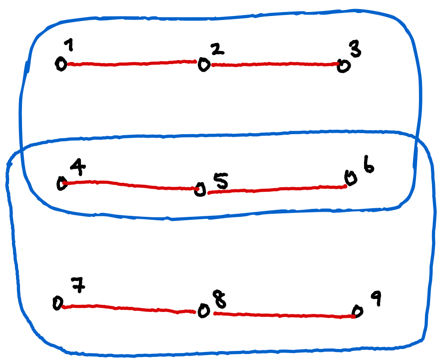


LOCAL HAMILTONIAN ON A NEW SET OF BOUNDARY QUBITS

- Error correcting properties preserved approximately
- Boundary Hamiltonian localised.

EXAMPLE : ROUNDS OF SIMULATION



Localising
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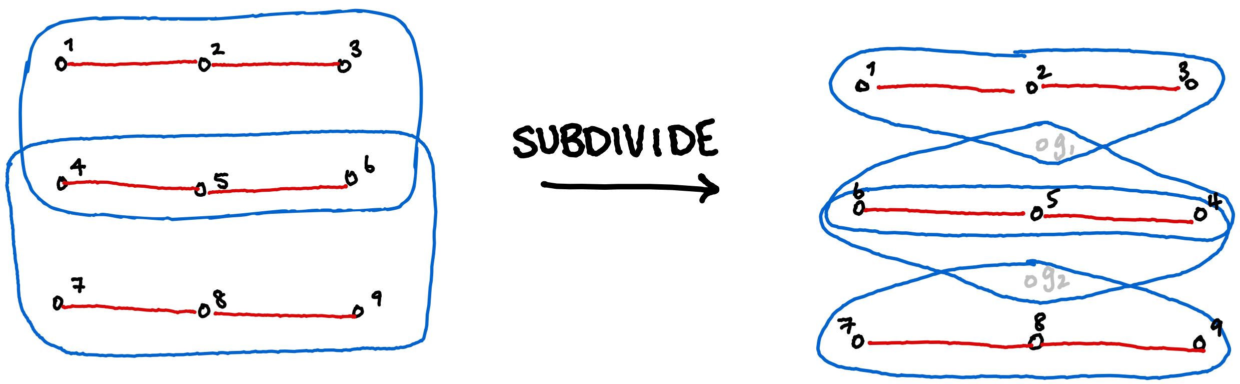

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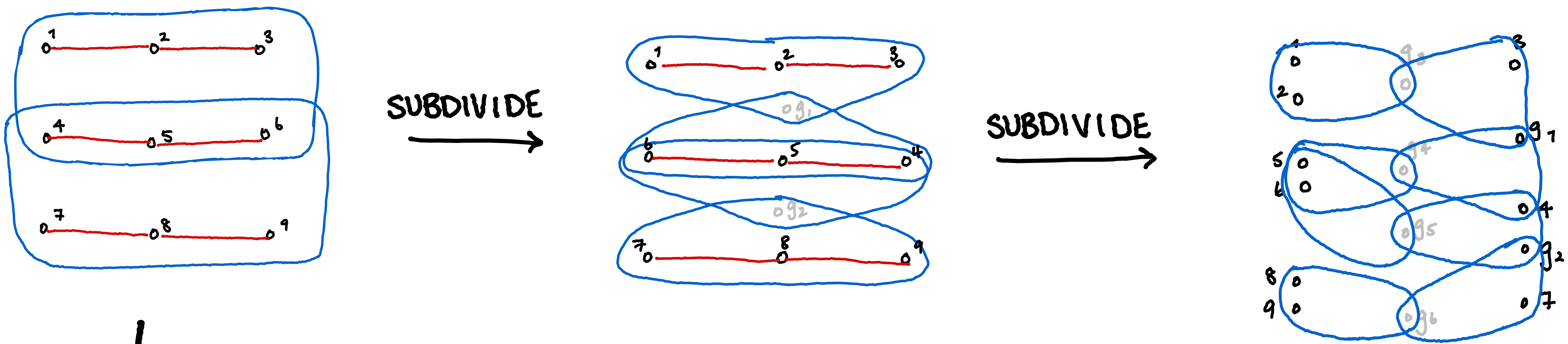

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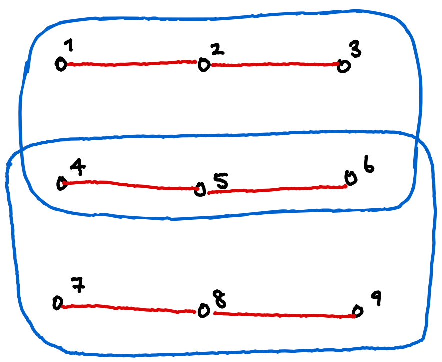
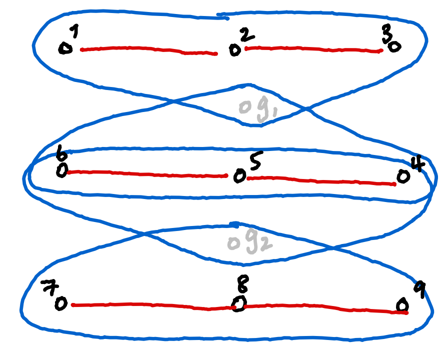


SUBDIVIDE



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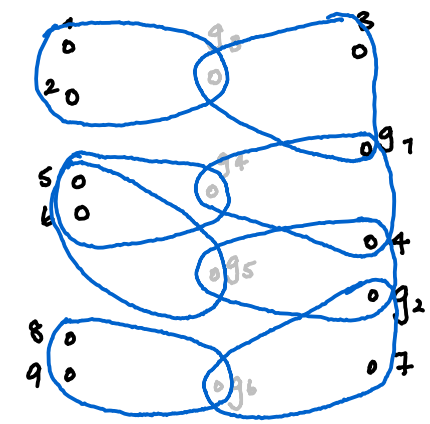
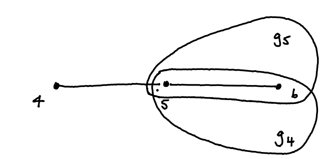
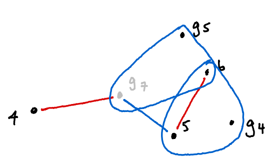


ZOOM IN + FORK





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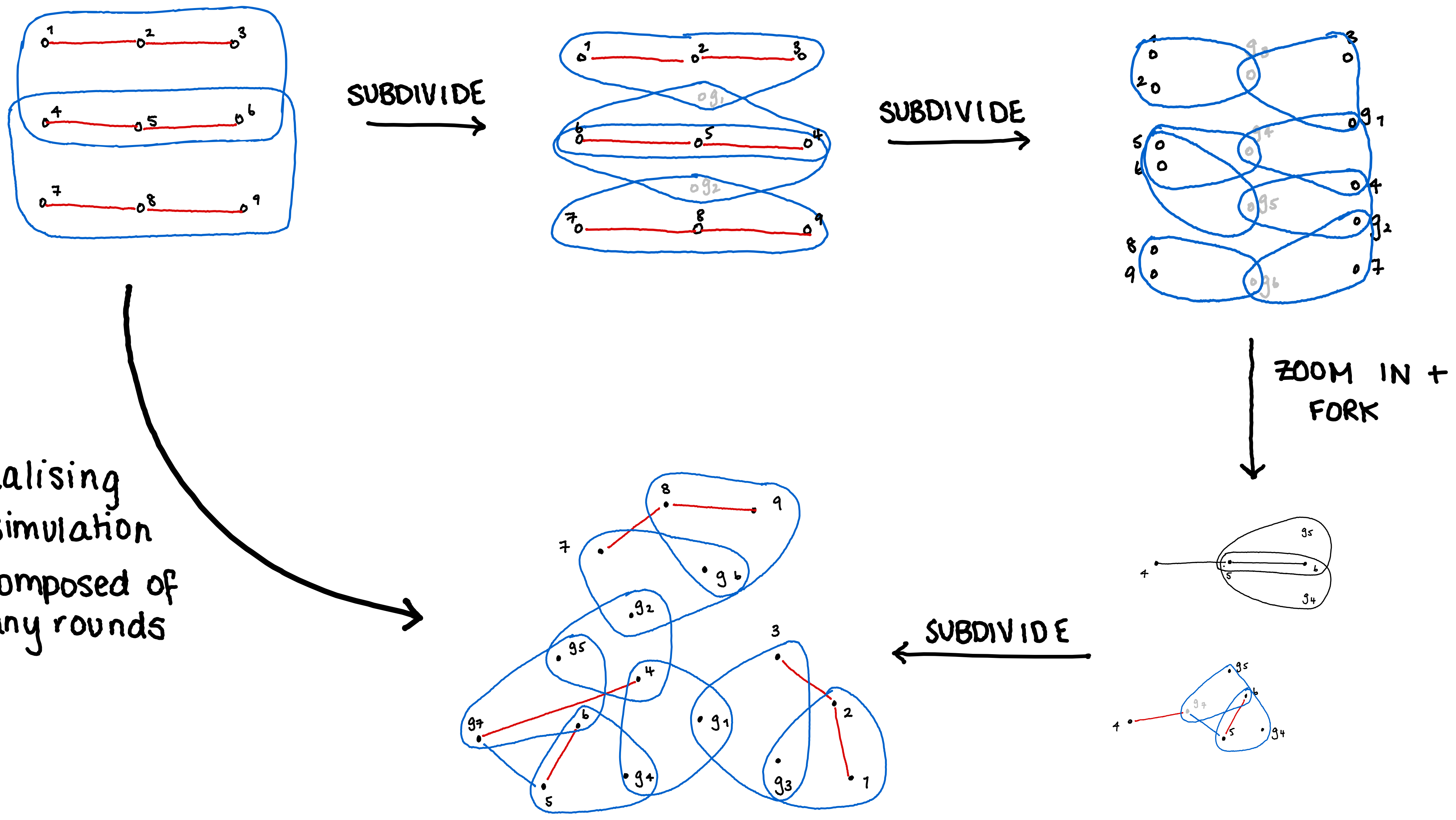

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COST OF LOCALISATION

Localising a Hamiltonian comes at the cost of

→ adding ancillary qubits

→ increased interaction strength

The size of the cost depends on the starting interaction graph (highest degree vertex, number of crossings, highest weight)

Require reasonable scalings for holographic codes

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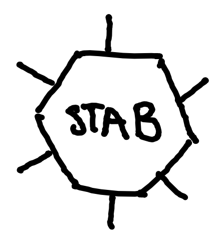
COSTS ARE CONTROLLED BY

I Chosing stabilizer tensors



- → Stabilizer tensors correspond to stabilizer states
- → They preserve the Pauli Rank of operators when mapped through

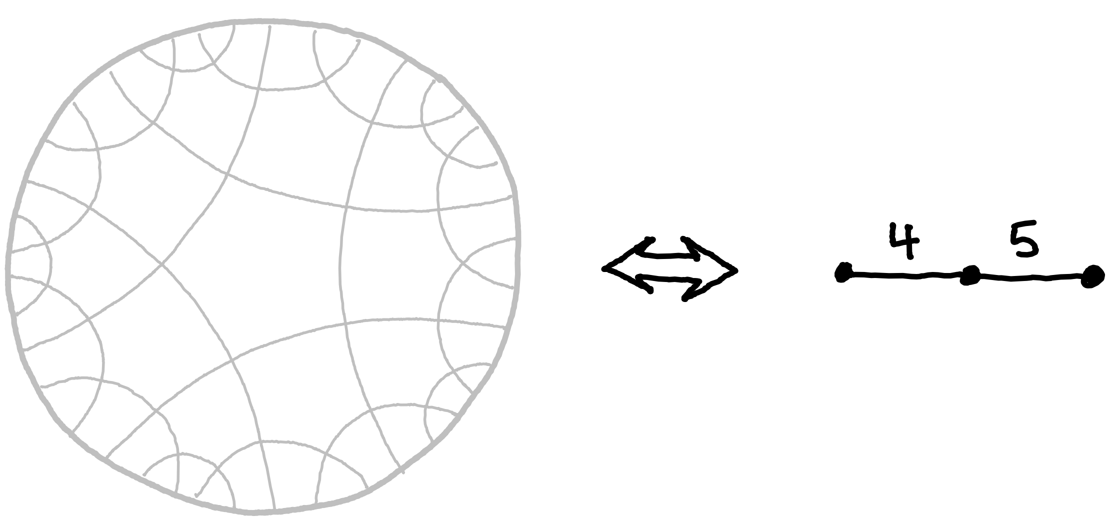
II Algebraic constraints on the geometry

Use coxeter group structure to ensure good properties of the network's geometry

$$W = \langle S \mid (s_i s_j)^{m_{ij}} = 1 \quad \forall \; i,j \in I \rangle$$

↳ Describes a tessellation of space with coxeter polytopes : a polytope with all dihedral angles integer submultiples of π

↳ Depicted by coxeter diagram



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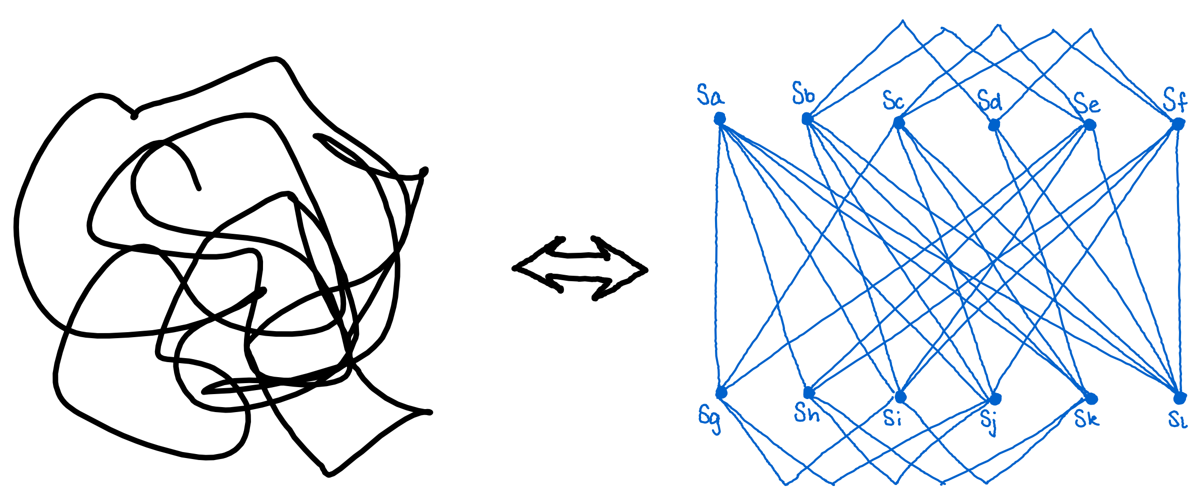


HYPERBOLIC COXETER GROUPS

- The number of cells in a ball of radius r goes as $O(\tau^r)$

  $\tau > 1$ gives hyperbolic coxeter groups
- Use coxeters groups to 1) prove EC properties for particular tessellations 2) track H boundary

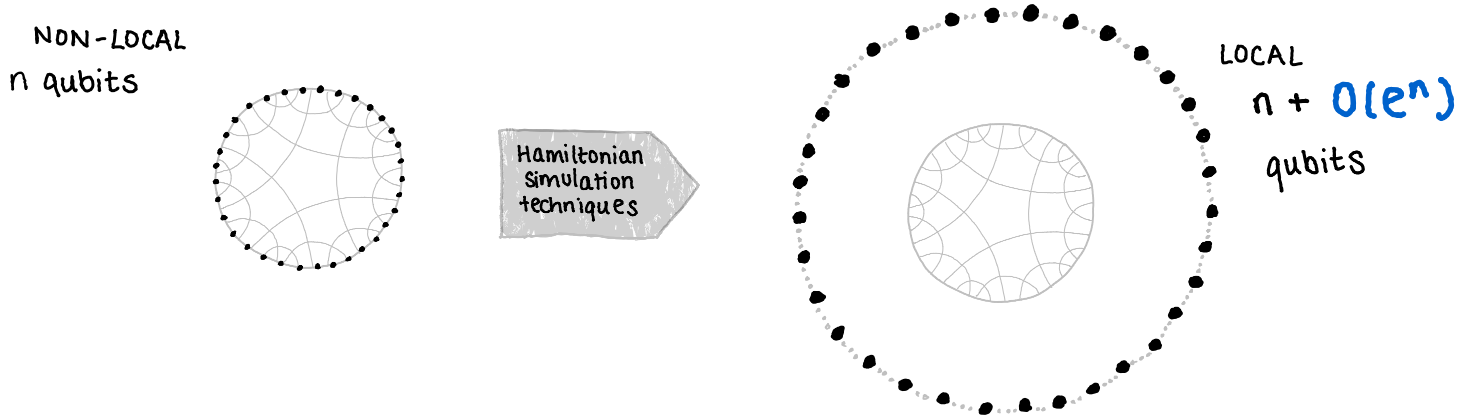
OUTLINE

MAIN RESULT [AKC22]: We construct a holographic approximate error correction code that is the first model to simultaneously achieve these properties of AdS/CFT

1. Holographic tensor network codes
   - isometric networks
   - random networks
2. Hamiltonian simulation techniques
3. Our construction

LOCALISING THE BOUNDARY HAMILTONIAN OF A RANDOM TN?

Applying simulation techniques to Haar random networks requires an unreasonable number of ancilla qubits as don't preserve Pauli rank.



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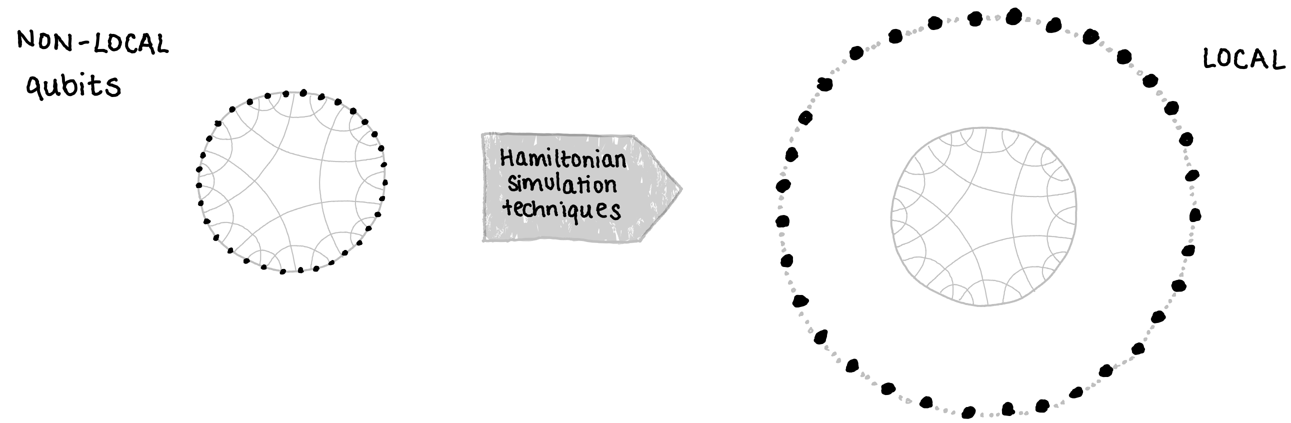
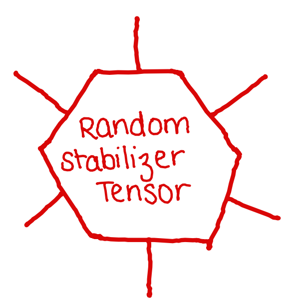


→ Our construction uses **random stabilizer tensors**



$\cong$ Random stabilizer state

[apply a random CLIFFORD unitary to a ref stabilizer state]

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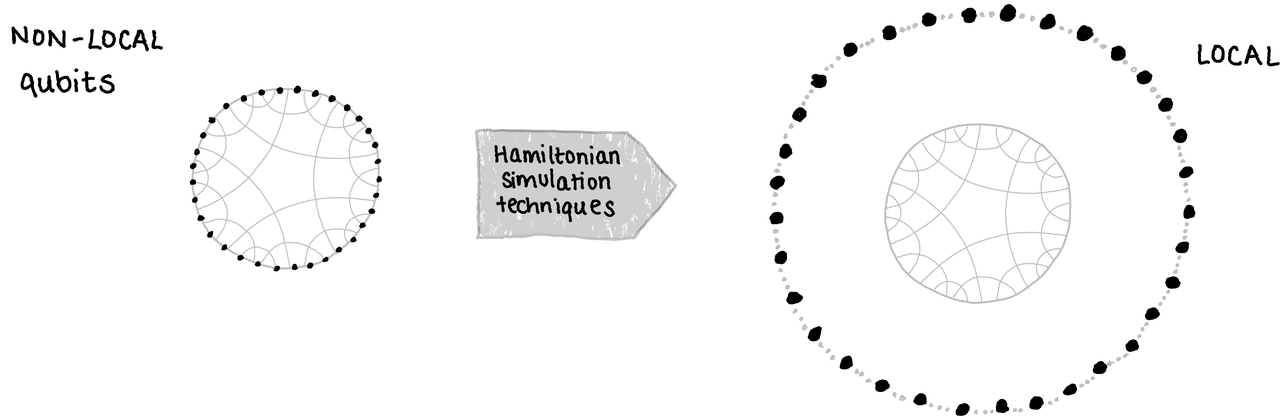

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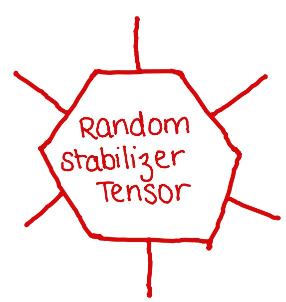

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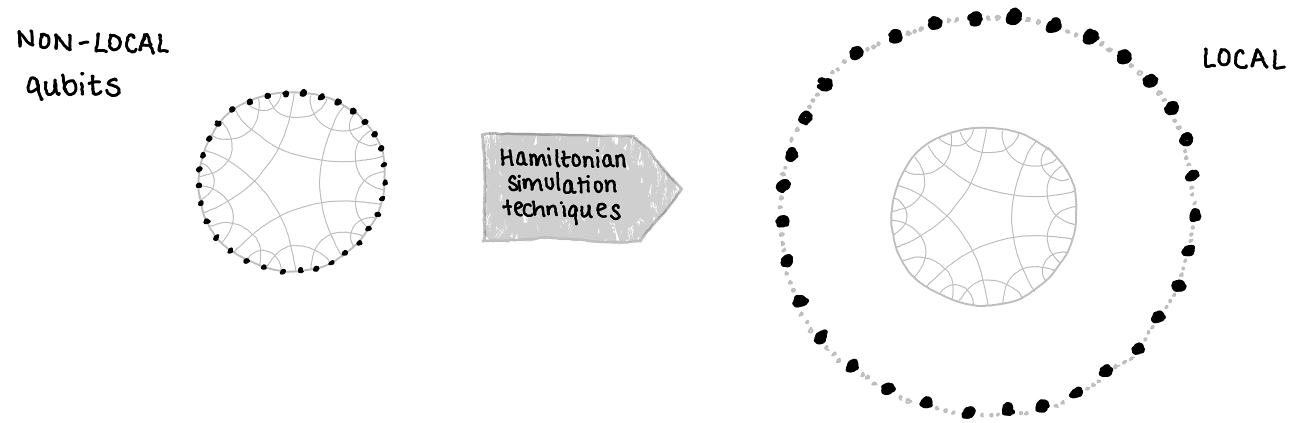

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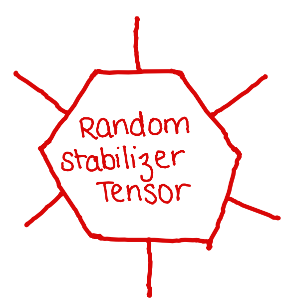

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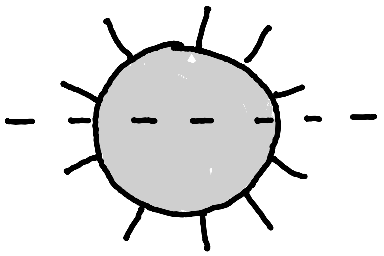
⇒ We were able to slightly modify the techniques from Haar random networks to prove exact 1st order RT for product bulk states.

RANDOM STABILIZER TENSOR NETWORKS DESCRIBE AN ISOMETRY

- Given a random stabilizer tensor with t legs each with dimension q

$$\mathrm{Prob}\left[S(\rho_A) \neq \log d_A\right] \lesssim \frac{1}{q^b}$$



where $0 \leq b < 1$ in the limit of large bond dimension*

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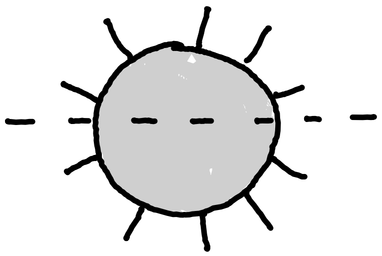

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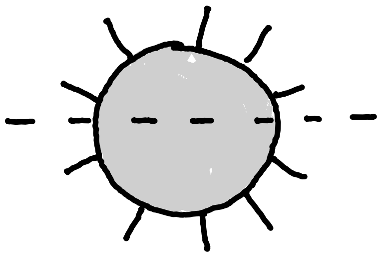

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$$P(A_1 \cap A_2 \cap \ldots \cap A_N) \geq \max\left\{0,\, P(A_1) + P(A_2) + \ldots + P(A_N) - (N-1)\right\}$$

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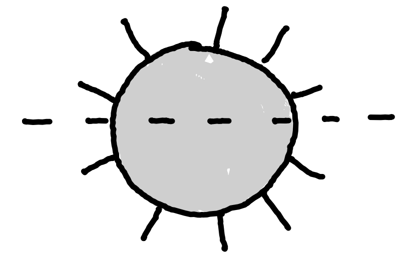

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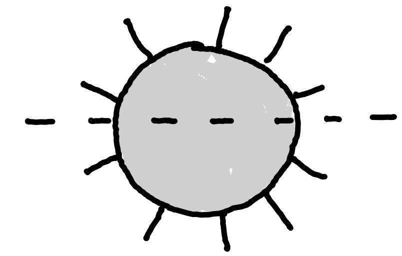

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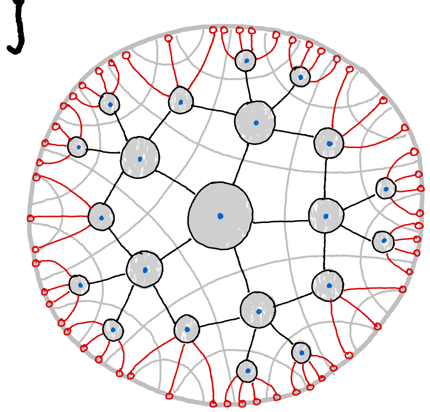
- For the network to be an isometry, this must be true for all n tensors, p^n



RETAIN TIGHT CONTROL

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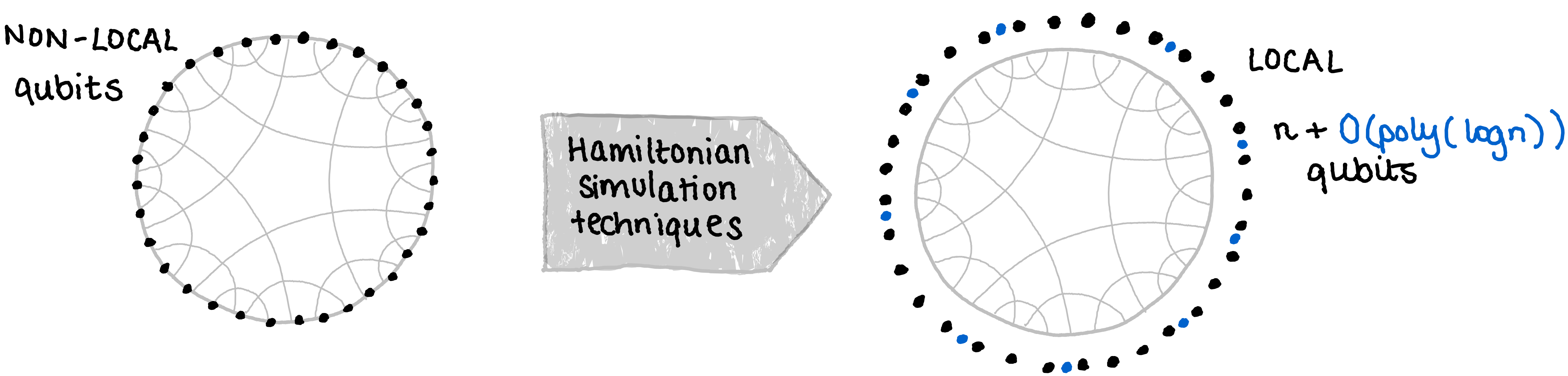
- Using this result we show that for a given sized network n, the is a choice of bond dimension such that

  1) With probability close to 1 the network is exactly an isometry exhibiting the expected error correction properties

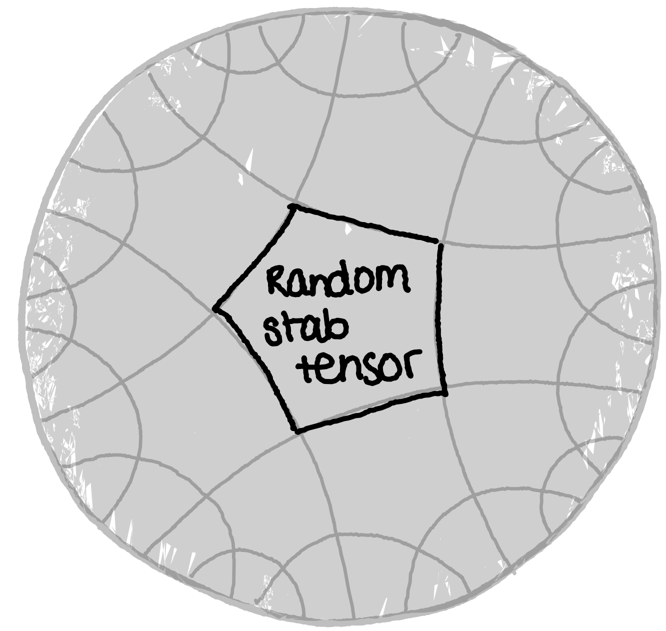
  2) Using Hamiltonian simulation techniques we can localise the resulting boundary Hamiltonian with "reasonable" resources



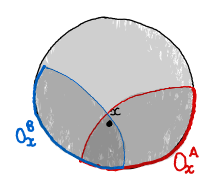
OUR TOY MODEL [AKC22]

→ Random stabilizer tensors on qudits

→ 2d/3d bulk tessellated by particular coxeter polytopes



① ERROR CORRECTION



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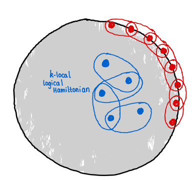

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$S(\rho_A) \propto |\gamma_A|$

First model to simultaneously achieve these features

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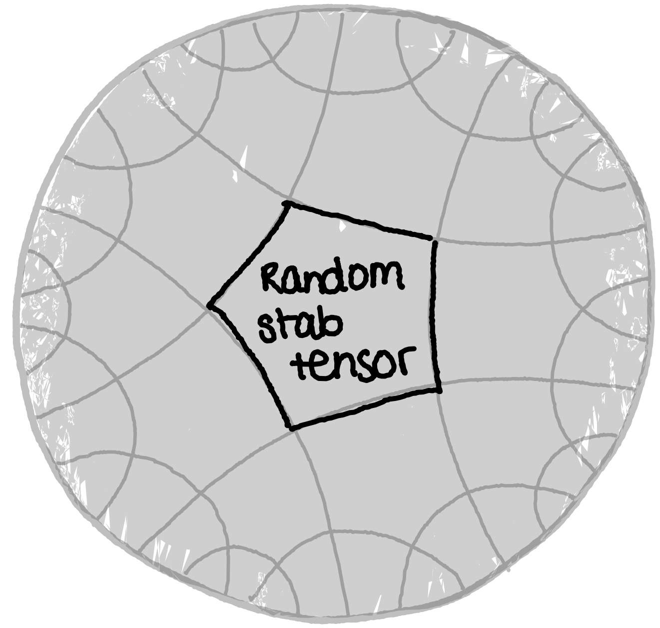

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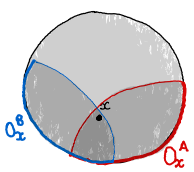

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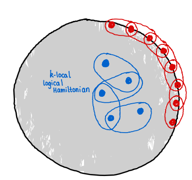

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NOTABLE DIFFERENCES TO ADS/CFT

- → Finite dimensional
- → No (discrete limit of) conformal symmetry
- → No spacetime
- → Bulk Hamiltonian doesn't have to be gravitational!

First model to simultaneously achieve these features

References

[Pas+15] Fernando Pastawski et al. “Holographic quantum error-correcting codes: toy models for the bulk/boundary correspondence". In: *Journal of High Energy Physics* 2015.6 (2015).

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[CMP18] Toby Cubitt, Ashley Montanaro and Stephen Piddock. “Universal Quantum Hamiltonians”. In: *Proceedings of the National Academy of Sciences* 115.38 (2018)

Thank you