A minimal introduction to conic programming

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Handout version

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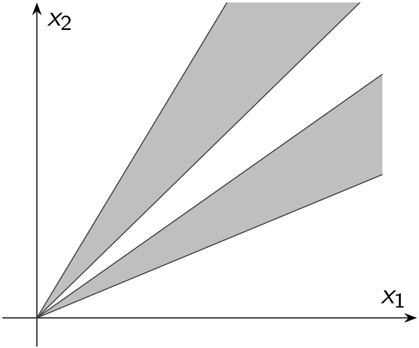
Proper cones

A set $K \subseteq \mathbb{R}^n$ that is

- closed under taking rays
  $x \in K, \lambda \in \mathbb{R}_{\geq 0} \implies \lambda x \in K$

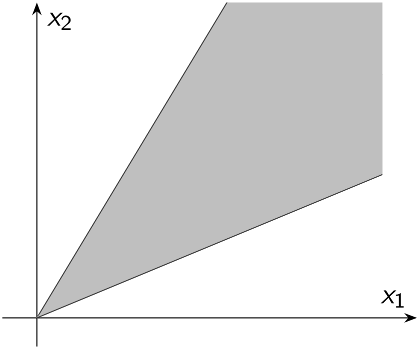


is called a **cone**.

Proper cones

A set $K \subseteq \mathbb{R}^n$ that is

- closed under taking rays
  $x \in K, \lambda \in \mathbb{R}_{\geq 0} \implies \lambda x \in K$
- convex
  $x, y \in K, \lambda \in [0, 1]$
  $\implies \lambda x + (1 - \lambda) y \in K$



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Proper cones

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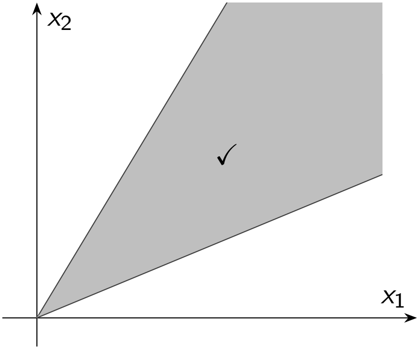
- closed under taking rays
  $x \in K, \lambda \in \mathbb{R}_{\geq 0} \implies \lambda x \in K$
- convex
  $x, y \in K, \lambda \in [0, 1]$
  $\implies \lambda x + (1 - \lambda)y \in K$
- closed $\partial K \subseteq K$
- solid $\mathrm{int}(K) \neq \emptyset$
- pointed $x, -x \in K \implies x = 0$

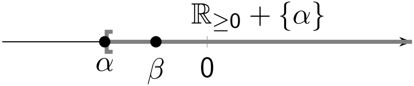
is called a **proper cone**.



Conic inequalities

Proper cones can
generalize linear inequalities!

$$\alpha \leq \beta \iff \beta - \alpha \in \mathbb{R}_{\geq 0}$$
$$\iff \beta \in \mathbb{R}_{\geq 0} + \{\alpha\}$$



Conic inequalities

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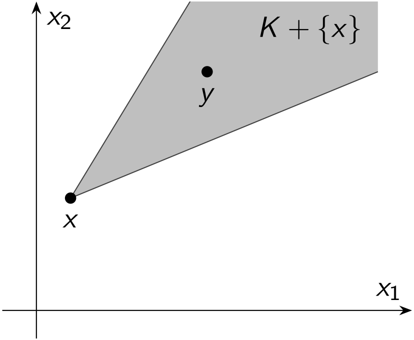
$$\alpha \leq \beta \iff \beta - \alpha \in \mathbb{R}_{\geq 0}$$
$$\iff \beta \in \mathbb{R}_{\geq 0} + \{\alpha\}$$

$$x \preceq_K y \iff y - x \in K$$
$$\iff y \in K + \{x\}$$



- $x \preceq_K y$ is a **convex constraint**
  (has a convex feasible region)

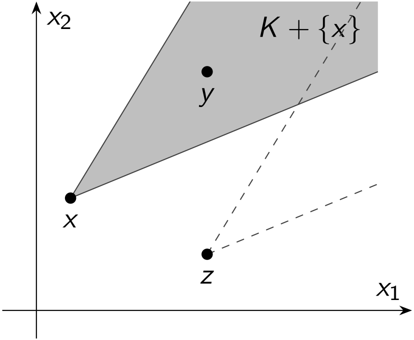
Conic inequalities

Proper cones can
generalize linear inequalities!

$$\alpha \leq \beta \iff \beta - \alpha \in \mathbb{R}_{\geq 0}$$
$$\iff \beta \in \mathbb{R}_{\geq 0} + \{\alpha\}$$

$$x \preceq_K y \iff y - x \in K$$
$$\iff y \in K + \{x\}$$

- $\preceq_K$ is a partial order
- $x \preceq_K y$ is a **convex constraint**
  (has a convex feasible region)



Conic programs

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & c^T x \\ \text{subject to} \quad & Ax = b, \\ & Gx \preceq_K h \end{aligned}$$

$$Ax = b \quad \Longleftrightarrow \quad \begin{bmatrix} +A \\ -A \end{bmatrix} x \preceq_{\mathbb{R}^{2\dim b}_{\geq 0}} \begin{pmatrix} +b \\ -b \end{pmatrix}$$

(possibly $K = K_1 \times \cdots \times K_\ell$ is a *product cone*)

Convex conic = convex?

"We can write any convex optimization problem [...] as a conic program[.]
The power of conic programming, however, lies in the fact that
we only need a few classes of convex cones
to express a wide variety of optimization problems."

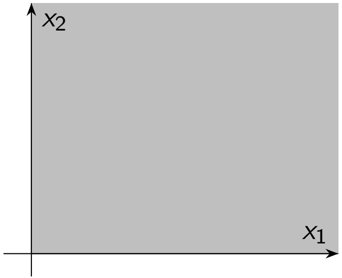
D. de Laat: "A ten page introduction to conic optimization" (2015)

A few classes of convex cones

The nonnegative orthant

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2_{\geq 0}$$

$$\mathbb{R}^n_{\geq 0} = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; x_i \geq 0 \; \forall i \leq n \right\}$$

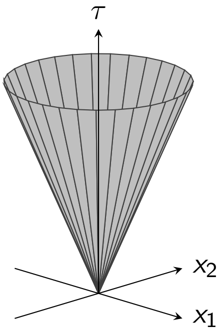


A few classes of convex cones

The second order cone

$$\mathcal{Q}^n = \left\{ \begin{pmatrix} \tau \\ x \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \|x\| \leq \tau \right\}$$

$$\begin{pmatrix} \tau & x_1 & x_2 \end{pmatrix}^T \in \mathcal{Q}^3$$



$$
\begin{aligned}
& \text{min. } \|Ax - b\| \\
\equiv \quad & \text{min. } \tau \quad \text{s.t. } \|Ax - b\| \leq \tau \\
\equiv \quad & \text{min. } \tau \quad \text{s.t. } \begin{pmatrix} \tau \\ Ax - b \end{pmatrix} \in \mathcal{Q}^{n+1} \\
\equiv \quad & \text{min. } \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix} \quad \text{s.t. } \begin{bmatrix} -1 & 0 \\ 0 & -A \end{bmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix} \preceq_{\mathcal{Q}^{n+1}} \begin{pmatrix} 0 \\ b \end{pmatrix}
\end{aligned}
$$

Recall: Second order cone (SOC)

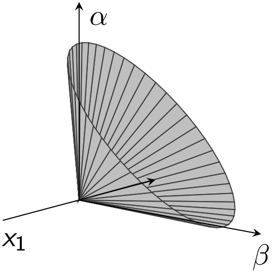
$$
\mathcal{Q}^n = \left\{ \begin{pmatrix} \tau \\ x \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \|x\| \leq \tau \right\}
$$

A few classes of convex cones

The *rotated* second order cone

$$\mathcal{Q}_r^n = \left\{ \begin{pmatrix} \alpha \\ \beta \\ x \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \|x\|^2 \leq 2\alpha\beta, \\ \alpha, \beta \geq 0 \end{array} \right\}$$

$$\begin{pmatrix} \alpha & \beta & x_1 \end{pmatrix}^T \in \mathcal{Q}_r^3$$



A few classes of convex cones

Convex quadratic programming

$$
\begin{aligned}
& \min.\ x^T Q x + q^T x + r \quad \text{where } Q \in \mathbb{S}^n_+ \\
\equiv \quad & \min.\ \tau + q^T x + r \quad \text{s.t. } x^T R^T R x \leq \tau \quad \text{where } Q = R^T R \\
\equiv \quad & \min.\ \tau + q^T x + r \quad \text{s.t. } \|Rx\|^2 \leq \tau \\
\equiv \quad & \min.\ \tau + q^T x + r \quad \text{s.t. } \begin{pmatrix} \frac{1}{2} \\ \tau \\ Rx \end{pmatrix} \in \mathcal{Q}_r^{n+2}
\end{aligned}
$$

Recall: Rotated second order cone

$$
\mathcal{Q}_r^n = \left\{ \begin{pmatrix} \alpha \\ \beta \\ x \end{pmatrix} \in \mathbb{R}^n \;\middle|\; \|x\|^2 \leq 2\alpha\beta \wedge \alpha, \beta \geq 0 \right\}
$$

A few classes of convex cones

Symmetric positive semidefinite matrices

$$\mathbb{S}^n_+ = \left\{ A \in \mathbb{S}^n \,\middle|\, x^T A x \geq 0 \;\forall x \in \mathbb{R}^n \right\}$$

(non-negative quadratic form)

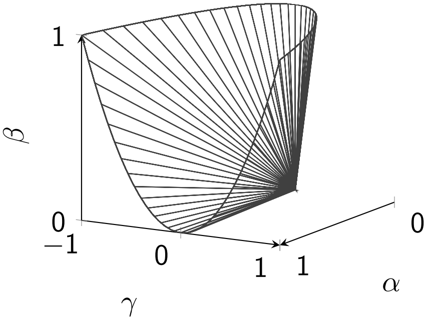
$$= \left\{ A \in \mathbb{S}^n \,\middle|\, \lambda_{\min}(A) \geq 0 \right\}$$

(non-negative eigenvalues)

$$= \left\{ A \in \mathbb{S}^n \,\middle|\, \det(A_{J,J}) \geq 0 \;\forall J \right\}$$

(non-negative principal minors)

$$\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix} \in \mathbb{S}^2_+$$



A few classes of convex cones

Hermitian positive semidefinite matrices

$$\mathbb{H}^n_+ = \left\{ A \in \mathbb{H}^n \,\middle|\, x^\dagger A x \geq 0 \;\forall x \in \mathbb{C}^n \right\}$$

(non-negative quadratic form)

$$= \left\{ A \in \mathbb{H}^n \,\middle|\, \lambda_{\min}(A) \geq 0 \right\}$$

(non-negative eigenvalues)

$$= \left\{ A \in \mathbb{H}^n \,\middle|\, \det(A_{J,J}) \geq 0 \;\forall J \right\}$$

(non-negative principal minors)

$$\begin{bmatrix} \alpha & \gamma + \delta i \\ \gamma - \delta i & \beta \end{bmatrix} \in \mathbb{H}^2_+$$

(insert 4D sketch above)

A few classes of convex cones
Eigenvalue bounds

$$
\begin{aligned}
& \lambda_{\max}(X) \le \tau \quad \text{for } X \in \mathbb{S}^n \\
\iff \quad & \sup_{\|v\|=1} v^T X v \le \tau \quad (X = Q^T D Q) \\
\iff \quad & \sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{v^T X v}{v^T v} \le \tau \\
\iff \quad & v^T(\tau I_n - X)v \ge 0 \quad \forall v \in \mathbb{R}^n \\
\iff \quad & \tau I_n - X \in \mathbb{S}^n_+ \\
\iff \quad & X \preceq \tau I_n \quad (\text{"linear matrix inequality"})
\end{aligned}
$$

Similarly, $\lambda_{\min}(X) \ge \tau \iff X \succeq \tau I_n$.

Recall: Positive semidefinite cone

$$\mathbb{S}^n_+ = \left\{ A \in \mathbb{S}^n \;\middle|\; x^T A x \ge 0 \; \forall x \in \mathbb{R}^n \right\}$$

Modeling languages

Conic programs

- extend linear programs
- express many convex objectives and constraints using few cones
- efficient numeric solution (using interior point methods)

Issues

- conic reformulation requires mathematical insights
- conic form can be cumbersome to write down
- resulting programs are hard to understand and update

Solution: optimization modeling languages, e.g.

- CVX, YALMIP in Matlab
- Convex.jl, JuMP in Julia
- CVXPY, PICOS, Pyomo in Python