This is a list of problems suggested by participants of the workshop "Profinite Rigidity" held in Madrid from June 26 to June 30, 2023.

1. Profinite Rigidity and profinite properties

Problem 1 (Remeslennikov)**.** *Is a finitely generated free group profinitely rigid?*

Comment (Organizers)**.** Given the result of [JZ22] that a finitely generated residually finite group with the same profinite completion as the free group is parafree, it seems worth trying to understand the structure of parafree groups. In particular given the result of [JZ22] and the standard correspondence between profinite completion of subgroups, a natural question that arises is the following:

Problem 2 (Organizers)**.** *Let Γ be a finitely generated parafree group. Suppose that every finite index subgroup of Γ is also parafree, is Γ free? A positive answer to this question would imply a positive answer to Problem 1.*

In [Bau05], Baumslag raised several questions about parafree groups, one of which is:

Problem 3 (Baumslag)**.** *Is a finitely generated parafree group hyperbolic?*

Given the recent developments surrounding the role of surface subgroups and virtual special groups in geometric group theory, the following questions seem natural to explore.

Problem 4 (Organizers)**.** *Does a finitely generated parafree group contain a surface subgroup? Assuming hyperbolicity, can one make this surface group quasi-convex.*

Problem 5 (Organizers)**.** *Is a finitely generated parafree group virtually special?*

Comment (Organizers)**.** Assuming Problem 4 (indeed that the surface subgroup is quasi-convex) and Problem 5 then this again implies a positive answer to Problem 1. (Since the full profinite topology would be induced on the surface subgroup thereby giving a profinite group of cohomological dimension 2 in the free profinite group, a contradiction).

Problem 6 (A well-known problem)**.** *Is a surface group profinitely rigid?*

Problem 7 (A well-known problem)**.** *Are all Fuchsian and Kleinian groups profinitely rigid?*

Problem 8 (A well-known problem)**.** *Is the group* $\mathrm{SL}(n, \mathbb{Z})$ *profinitely rigid?*

Problem 9 (Lubotzky, Reid)**.** *Is the group* $\mathrm{SL}(2, \mathbb{Z}[1/p])$ *profinitely rigid? What about* $\mathrm{SL}(3, \mathbb{Z}[1/p])$*?*

Comment (Organizers)**.** *It follows from* [CWLRS] *that* $\mathrm{SL}(4, \mathbb{Z}[1/p])$ *is not profinitely rigid.*

Problem 10 (Zalesski)**.** *Let G be a residually finite, finitely generated one-ended group. Can $\widehat{G}$ be isomorphic to the profinite completion of a residually finite finitely generated not one-ended group?*

Problem 11 (Piwek, Wykowski)**.** *Is bi-orderability a profinite property?*

Problem 12 (Kionke, Lubotzky, Reid)**.** *Is Property τ a profinite property?*

Comment (Organizers)**.** *Aka* [Aka12] *shows that Property T is not a profinite property (see also* [CWLRS]*) but the groups in question do have τ.*

Comment (Kionke)**.** *Property τ is not a profinite property amongst finitely generated residually finite groups. Kassabov discusses a counterexample in the set of slides* [Kas08] *and* [KS23, Theorem 1.5] *provides a Grothendieck Pair $G \hookrightarrow H$ with G amenable and H having Property τ.*

2. Three manifold groups.

Comment (Ma)**.** In Liu's talk, he discussed his results about profinite rigidity of finite-volume hyperbolic three manifold groups among finitely generated three manifold groups, up to finite ambiguity [Liu23]. One would expect that there are no finite ambiguity at all for hyperbolic three manifold groups, but one could also try to answer a weaker question.

Problem 13 (Ma)**.** *Do we have any uniform control about the finite ambiguity? That is, does there exist a number N such that for any finite-volume hyperbolic manifold group Γ and any finitely generated three manifold group G with $\widehat{\Gamma} \cong \widehat{G}$, G has at most N choices up to isomorphism?*

Problem 14 (A well-known question)**.** *Among hyperbolic three manifolds of finite volume, is the volume a profinite invariant? What about if one replaces hyperbolic three manifolds by higher rank locally symmetric spaces?*

Boileau [Boi18] conjectures that any prime knot K has the property that the profinite completion of $\pi_1(S^3 \setminus K)$ determines K amongst prime knots: if J is a prime knot and $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ have isomorphic profinite completions, then J and K are isotopic. Hence all knot invariants should be the same. A natural question is therefore:

Problem 15 (Simon)**.** *Which knot invariants of finite type (aka Vassiliev invariants) depend only on the profinite completion of the knot group?*

Comment (Simon)**.** For example we know by results of Jun Ueki that the coefficients of the Alexander polynomial are profinitely rigid.

Problem 16 (Organizers)**.** *Let K be a hyperbolic knot, and $A_0(K)$ the component of the A-polynomial associated to the component of the* $\mathrm{PSL}(2,\mathbb{C})$*-character variety containing the character of the faithful discrete representation. Prove that $A_0(K)$ is a profinite invariant amongst hyperbolic knots.*

Comment (Organizers)**.** *The Alexander polynomial can be read off of from the A-polynomial.*

Problem 17 (Organizers)**.** *Is being Haken a profinite invariant?*

Comment (Organizers)**.** Since having positive first Betti number is a profinite invariant, the problem really reduces to the follwing:
Suppose that M is a closed irreducible 3-manifold that is a Haken rational homology 3-sphere, and N a closed irreducible 3-manifold with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. Is N Haken?
Since having a nontrivial JSJ decomposition is a profinite invariant (see [Wil18]), together with Wilkes's work [Wil17] on Seifert fibered spaces, the case of interest is when M is hyperbolic.
Note that by [CWLRS] splitting as an amalgamated product is not a profinite invariant.

3. Lattices in Lie groups and arithmetic groups

Comment (Stover)**.** It is a classical result of Malcev that any finitely generated matrix group is residually finite (RF). Deligne, however, produces lattices in nonlinear Lie groups that are not RF (hence, not linear), e.g., lifts of $\mathrm{Sp}(2g,\mathbb{Z})$ to sufficient finite coverings of $\mathrm{Sp}(2g,\mathbb{R})$ for $g > 1$. On the other hand, all lattices in all connected covers of $\mathrm{PSL}(2,\mathbb{R})$ are RF, despite $\mathrm{SL}(2,\mathbb{R})$ being the only one that is linear.

Problem 18 (Stover)**.**

(1) *Classify the nonlinear Lie groups that contain a lattice that is not RF.*
(2) *For which Lie groups are* all *lattices non-RF?*
(3) *A key case, in particular for its possible connection to the existence of Gromov hyperbolic groups that are not RF, is $G = \mathrm{PU}(n,1)$, $n > 1$. Are all lattices in all connected covers of G RF?*
(4) *Specific examples where group presentations are known (due to Cartwright and Steger) are fake projective planes. Is the preimage of a fake projective plane group in the universal cover of* $\mathrm{PU}(2,1)$ *residually finite?*
(5) *Is a lattice in a connected Lie group RF if and only if it is linear? This is true for finite extensions of lattices in linear Lie groups.*

Problem 19 (Spitler)**.** *It is known that many distinct pairs of arithmetic groups can have isomorphic congruence completions, for example* $\mathrm{SL}(3,\mathcal{O}_K)$ *and* $\mathrm{SL}(3,\mathcal{O}_L)$ *where $\mathcal{O}_K$ and $\mathcal{O}_L$ are the rings of integers of locally-equivalent number fields;* $\mathrm{Spin}(n,n;\mathbb{Z})$ *and* $\mathrm{Spin}(n+4,n-4;\mathbb{Z})$*; and even pairs of subgroups of* $\mathrm{SL}(3,Z)$*. On the other hand certain arithmetic groups, such as* $\mathrm{SL}(3,\mathbb{Z})$ *itself, certain hyperbolic triangle groups, and Bianchi groups, can be determined by their congruence completions. Can one describe which arithmetic groups Γ are such that any other arithmetic group Λ with the same congruence completion must have $\Lambda \leq \Gamma$?*

Problem 20 (Spitler)**.** *If Γ is an arithmetic lattice in the rank-1 Lie group $G(\mathbb{R})$, when is the inclusion representation the only rigid representation of Γ in $G(\mathbb{C})$ up to Galois conjugation?*

Problem 21 (Kammeyer, Spitler)**.** *If Γ is an arithmetic lattice in the rank-1 Lie group $G(\mathbb{R})$ (other than* $\mathrm{PSL}(2,\mathbb{R})$ *or* $\mathrm{PSL}(2,\mathbb{C})$ *where the answer is already known), can proper subgroups of Γ be distinguished from Γ by their profinite completions? When the subgroup is of infinite index? When it is of finite index?*

Problem 22 (Spitler)**.** *If Γ is a higher-rank (S-)arithmetic group, can Γ contain a Grothendieck subgroup (i.e. be the larger group in a Grothendieck pair)? (especially if Γ has the congruence subgroup property, like $\mathrm{SL}(n, \mathbb{Z})$ for $n \geq 3$).*

Problem 23 (Kammeyer)**.** *Do lattices in the rank-1 Lie group $F_{4(-20)}$ have CSP?*

Problem 24 (Kammeyer)**.** *Do anisotropic A_1-forms have CSP?*

4. Abstract subgroups of profinite groups

Problem 25. *(Jaikin-Zapirain) Is a free profinite group locally indicable?*

Problem 26 (Nikolov)**.** *Does every perfect finitely generated profinite group contain a dense finitely generated perfect abstract subgroup?*

Comment (Nikolov)**.** I can prove the answer is YES for finitely generated vitually solvable perfect profinite groups. It will be great of course to generalise this to virtually prosolvable groups as this will cover many projective groups and give an example for Problem 26.

Problem 27. *(Jaikin-Zapirain) Let F be a finitely generated group and $w \in F$ an element which is not a proper power in F. Baumslag showed that if $n \in \mathbb{N}$ is coprime with a prime p, then the natural map of $G = F *_{x^n=w} \langle x \rangle$ to the free pro-p group $F_{\widehat{p}}$ is injective. Can G be embedded into $\mathrm{SL}(2, \mathbb{C})$?*

5. Mapping class groups, proposed by Biao Ma

Mapping class groups of surfaces are tied closely with hyperbolic three manifolds. Some notable progress on profinite properties of mapping class groups has been made recently partially because of the success we have in understanding profinite rigidity of hyperbolic three manifolds. However, profinite aspects of mapping class groups are still a big challenge for us. Interestingly, I noticed that there are no results/open problems have been mentioned during the workshop. To complete it, I will list very few of them.
One of the most important questions is

Problem 28 (A well-known problem)**.** *Are mapping class groups of surfaces are good in the sense of Serre?*

One could reduce this question to closed surfaces and the answer is yes for surfaces with genus one or two. The reader is referred to an answer given by Agol in Mathoverflow for the reduction.
In Lubotzky's talk, he talked about his results with Avni and Meiri on first-order rigidity of non-uniform higher rank arithmetic groups [ALM19]. One of the key ingredients of the proof is Margulis's Superrigidity. Since mapping class groups also exhibit certain superrigidity, one could ask

Problem 29. *Do mapping class groups of closed surfaces have first-order rigidity (among certain classes of groups)?*

Notice that Avni-Lubotzky-Meiri's proof uses more ingredients than just superrigidity, the above question will be potentially harder than Problem 28.
Finally we discuss some questions on profinite properties of mapping class groups of surfaces arise from the Dehn-Nielsen-Baer theorem. Let S be a closed, connected, orientable surface of genus g and $\mathbb{M}(S)$ be the mapping class group of S. By the Dehn-Nielsen-Baer theorem, $\mathbb{M}(S)$ is a index two subgroup of $\mathrm{Out}(\pi_1(\mathrm{S}))$. Denote the profinite completion of $\pi_1(S)$ by $\hat{\pi}$. Then $\mathbb{M}(S)$ can be mapped injectively into $\mathrm{Out}(\hat{\pi})$. Denote the injection by

$$\phi : \mathbb{M}(S) \to \mathrm{Out}(\hat{\pi}).$$

Problem 30. *Can we describe the closure of $\mathrm{Im}(\phi)$ precisely? What is the relation between the closure of $\mathrm{Im}(\phi)$ and the profinite completion of $\mathbb{M}(S)$?*

There are some works in this direction, for instance [Bog20] for certain subgroups, but much is unknown. The action of $\mathbb{M}(S)$ on $\hat{\pi}$ seems to be quite interesting, so we conclude our problem list by

Problem 31. *Develop a framework to understand the action of $\mathbb{M}(S)$ on $\hat{\pi}$ and use it to understand $\mathbb{M}(S)$.*

6. Other problems

Problem 32 (Lubotzky, Wykowski)**.** *Is residual finiteness preserved under elementary equivalence?*

Problem 33 (Morales, Wykowski)**.** *Are one-relator groups good in the sense of Serre?*

Comment (Souza)**.** There is a plethora of torsion-free finitely generated abstract groups which are not free-by-cyclic but are virtually free-by-cyclic. Some of those give rise to profinite examples of not free-by-cyclic but virtually free-by-cyclic profinite groups through the profinite completion. For $p = 2$, the pro-2 completion of the Baumslag double along the word $[a, b]^2$ is an example of a pro-2 group that is not free-by-$\mathbb{Z}_2$ but has an index 2 free-by-$\mathbb{Z}_2$ subgroup.

Problem 34 (Souza)**.** *Let p be an odd prime. Is there a torsion-free topologically finitely generated pro-p group G that is not free-by-$\mathbb{Z}_p$ but virtually free-by-$\mathbb{Z}_p$? Can G be the pro-p completion of an abstract torsion-free virtually free-by-cyclic abstract group?*

Comment (Souza)**.** It is known that the Poincaré conjecture is equivalent to statement that for every closed orientable surface Σ_g of genus g the action of the group $\mathrm{Aut}(\pi_1\Sigma_g)$ on the set of all surjections $\pi_1(\Sigma_g) \to F_g \times F_g$ is transitive, where F_g denotes the free group on g generators (see [Hem04, Thm. 14.6]). It's pro-p version is still open.

Problem 35 (Pro-p Poincaré Conjecture)**.** *Let G be the pro-p completion of $\pi_1(\Sigma_g)$. Is the action of* $\mathrm{Aut}(G)$ *transitive on the set of all continuous surjections $G \to \widehat{F_g} \times \widehat{F_g}$, where $\widehat{F_g}$ denotes the pro-p completion of F_g?*

Problem 36 (Meiri)**.** *Let G be a free product of cyclic groups. Let g be an element which is a 1-commutator (i.e., of the form $xyx^{-1}y^{-1}$) in any finite quotient of G. Is g a 1-commutator in G?*

Comment (Meiri)**.** This is known to be true if G is a free group [Khe04]. Khelifs result was generalized to free products of cyclic groups of orders 2, 3 or infinity in [GMS22]. The method, is basically the same as the one of Khelif. However, when the factors are finite cyclic groups of higher order, the proof breaks. More generally, given a word w on d-letters, I am not familiar with an example of a free group F of rank at least d and an element g in F such that g is a w-value in every finite quotient of F but is not a w-value in F.

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