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Title: "Kippenhahn's Theorem for Joint Numerical Ranges and Quantum States"

This is joint work with Daniel Plaumann (TU Dortmund) and Rainer Sinn (FU Berlin).

Abstract. Kippenhahn's Theorem [1] asserts that the numerical range of a matrix is the convex hull of an algebraic curve. A higher-dimensional generalization of the numerical range is the joint numerical range

$$W = \{(\mathrm{tr}(\rho A_1), \ldots, \mathrm{tr}(\rho A_n)) : \rho \in \mathcal{B}\}$$

of finitely many hermitian $d \times d$ matrices $A_1, \ldots, A_n$, where $\mathcal{B}$ is the convex set of $d \times d$ density matrices. Chien and Nakazato [2] have shown that the analogous assertion, W is the convex hull of an affine variety, fails for $n = 3$.

Here we show that W is the convex hull of a semi-algebraic set. First, we discuss a known, analogous statement regarding the dual convex cone to a hyperbolicity cone (Example 3.15 of [3]). Secondly, we prove that the class of convex bases of these dual cones is closed under linear operations (up to affine isomorphism).

The result offers a new geometric method to analyze quantum states. The joint numerical range represents the set of mean values of simple measurements associated with (possibly non-commuting) hermitian matrices, the set of measurement probabilities of a positive operator valued measure, a set of reduced density matrices (quantum marginals), etc. Abstractly, W represents the state space of the operator system spanned by $A_1, \ldots, A_n$. Recently, Schwonnek and Werner [4] showed that the singularities of the Wigner distributions for $A_1, \ldots, A_n$ lie in the mentioned semi-algebraic set.

[1] R. Kippenhahn, *Über den Wertevorrat einer Matrix*, Mathematische Nachrichten **6** (1951), 193–228.

[2] M.-T. Chien and H. Nakazato, *Joint numerical range and its generating hypersurface*, Linear Algebra and its Applications **432** (2010), 173–179.

[3] R. Sinn, *Algebraic boundaries of convex semi-algebraic sets*, Mathematical Sciences (2015) 2: 3. `https://doi.org/10.1186/s40687-015-0022-0` (open access)

[4] R. Schwonnek and R. F. Werner, *Properties of the Wigner distribution for n arbitrary operators.* `arXiv:1802.08343 [quant-ph]`