Solutions to exercises

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1 Exercises

Exercise 1. *Let $\varphi : (M, \omega_1) \to (N, \omega_2)$ then φ preserves the volume.*

Solution 1. Denote the dimenion of N by $2n$. As φ is a symplectomorphism, it follows that $\varphi^*(\omega_2^n) = (\varphi^*\omega_2)^n = \omega_1^n$, hence φ is volume preserving.

Exercise 2. *Prove that there is a volume preserving embedding $\varphi : B^{2n}(a) \to Z^{2n}(A)$ for $A < a$. Prove that this is not true for linear symplectomorphisms (for the standard symplectic structure of course).*

Solution 2. For simpler notations, we work in dimension 4, although the proof is identical in higher dimension. Consider the diffeomorphism given by $\varphi(x_1, y_1, x_2, y_2) = (\epsilon x_1, \epsilon y_1, \frac{1}{\epsilon}x_2, \frac{1}{\epsilon}y_2)$. This is volume-preserving. For ϵ small this provides a suitable embedding. To check that this does not work for symplectomorphisms, notice that a general linear symplectomorphism writes down $\varphi(x_1, y_1, x_2, y_2) = (\epsilon x_1, \frac{1}{\epsilon}y_1, \delta x_2, \frac{1}{\delta}y_2)$. When $a > A$ then ϵ needs to be smaller than 1 to embed the x_1-coordinate in the cylinder. But then the y_1-coordinate is not contained anymore in the cylinder.

Exercise 3. *Check that $\omega_0(u, v) = g_0(J_0u, v)$ and that $g_0(J_0u, J_0v) = g(u, v)$ where ω_0 denotes the standard symplectic structure, J_0 the standard complex and g_0 the standard inner product on $\mathbb{R}^{2n}$. Check that if $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves ω_0 and J_0 then φ also preserves g_0.*

Solution 3. This exercise if a matter of writing down the definitions.

Exercise 4. *Prove that if S is a proper complex submanifold of $B^{2n}(r)$ (for the standard complex structure, and here the ball is the ball of radius r, in contrast to the ball that we saw in the lecture that had radius $\sqrt{r/\pi}$) passing through the origin, then $\pi r^2 \le area_{g_0}(S)$. To prove this follow the following hints:*

1. *Let $\Sigma(r) \subset B^{2n}(r) \subset \mathbb{C}^n$ an orientable surface whose boundary lies in $S^{2n-1}(r)$. Denote the symplectic area of Σ by $A_\omega(r)$ and the Riemannian length of $\partial\Sigma$ by $L(r)$. Prove that the following inequalities hold:*
$$\frac{d}{dr}A_\omega(r) \ge L(r) \ge \frac{2A_\omega(r)}{r}.$$
2. *Prove that if a loop in $S^{2n-1}(r)$ has length strictly less than $2\pi r$ then it is contained in a hemisphere of $S^{2n-1}(r)$.*
3. *Conclude the exercise.*

Solution 4. 1. We prove the two inequalities seperately. For the lower bound on $L(r)$, we first use Stokes theorem: it follows that $A_\omega(r) = \int_{\Sigma(r)} \omega = \int_{\gamma(r)} \lambda$ where $\gamma(r)$ is the boundary of (r) and λ is the Liouville form $\lambda = \frac{1}{2}\sum_{i=1}^n (x_i dy_i - y_i dx_i)$. Hence dividing by r we obtain

$$\frac{A(r)}{r} = \int_{\gamma(r)} \frac{1}{r}\lambda = \int_{\gamma(r)} \omega(\frac{1}{r}V_\lambda, \cdot) = \int_{[0,2\pi]} \omega(\frac{1}{r}V_\lambda, \dot{\gamma}(r))dt = \int_{[0,2\pi]} \langle -J_0\frac{1}{r}V_\lambda, \dot{\gamma}(r)\rangle dt \le \frac{1}{2}L(r).$$

Here we denoted V_λ the Liouville vector field (i.e. $\mathcal{L}_{V_\lambda}\lambda = \lambda$)). In the last passage we used that the Liouville vector field is radially expanding from the origin of the disk. For the other inequality, we would like to estimate $A(r+\epsilon) - A(r)$ from below. Take a tubular neighbourhood around $\gamma(r)$, being a local parametrization for $\Sigma(r)$ around $\gamma(r)$. We can assume it covers $\gamma(r+\epsilon)$ as well. Note that this tubular neighbourhood is trivial because the surface is orientable. Let us denote it by $\Sigma(\theta, t)$ where $\theta \in [0, 2\pi]$ is a parametrization for $\gamma(r)$ and t the normal direction. Let us compute the area of the shell $A(r+\epsilon) - A(r)$ using Fubini's theorem:

$$A(r+\epsilon) - A(r) = \int_{[0,2\pi]\times[0,1]} \Sigma(\theta, t) d\theta dt = \int_{[0,2\pi]} \int_{[0,1]} \Sigma(\theta, t) dt d\theta \geq L(r)\epsilon$$

where we used that for fixed θ, $\Sigma(\theta, [0,1]) \geq \epsilon$. Now taking the limit $\epsilon \to 0$, we obtain that $\frac{d}{dr}A(r) \geq L(r)$.

2. After scaling, we may assume without loss of generality that we are on the unit sphere and that the closed curve has length less than 2π. Pick any point P on the curve, travel half way around the curve to the point Q, and let N, the north pole, be the point half-way between P and Q. N is uniquely defined because the distance between P and Q is less than π. N determines an equator and we are left to show that the curve lies entirely in the northern hermisphere. Assume by contradiction it does not, so let E be a point on the curve where it crosses the equator. We claim that $d(E, P) + d(E, Q) = \pi$. Indeed, if we had chosen P through the equatorial plane to P' on the other side, P' is antipodal to Q; hence, $d(E, P') + d(E, Q) = \pi$. However, for any point X on the curve, $d(P, X) + d(X, Q)$ must be less than π, which is a contradiction.

3. As the surface is holomorphic, we saw in the lecture that the symplectic and the Riemannian area coincide. By construction, it passes through the origin, hence $L(r) \geq 2\pi r$ by the second point in the exercise. By the first point, we get $\frac{d}{dr}A(r) \geq 2\pi r$ and now integrate over $[0, r]$ to obtain that $A(r) \geq \pi r^2$.

Exercise 5. *Prove that if $J : V \to V$ is a complex structure on a vector space V, then* $\dim V = 2n$.

Solution 5. Compute the determinant of J^2: On the one hand, $\det(J^2) = \det(-1) = (-1)^{\dim(V)})$. On the other hand $\det(J^2) = (\det(J))^2$, hence $\dim(V)$ must be even.

Exercise 6. *Let (M, ω) be a symplectic manifold and denote by $\mathcal{J}(M, \omega) = \{J$ almost complex structure compatible with ω Prove that $\mathcal{J}$ is contractible following the following hints: Consider a symplectic vector space (V, Ω) of dimension $2n$ and a Lagrangian subspace L_0, i.e. $\dim L_0 = n$ and $\Omega|_{L_0} = 0$. Denote by $\mathcal{L}(V, \Omega, L_0)$ the set of Lagrangian subspaces that are transverse to L_0. Denote by G_0 the space of all positive inner products on L_0. Consider the map*

$$\begin{aligned} \Phi : \mathcal{J}(V, \Omega) &\to \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0) \\ J &\mapsto (JL_0, G_J|_{L_0}) \end{aligned}$$

Show that

1. *Φ is well-defined.*

2. *Φ is a bijection.*

3. *$\mathcal{L}(V, \Omega, L_0)$ and $\mathcal{G}(L_0)$ are contractible.*

Solution 6. 1. We need to check that $JL_0 \in \mathcal{L}(V, \Omega, L_0)$ and $G_J|_{L_0} \in \mathcal{G}(L_0)$. The latter one is trivial (restriction of inner product to any subspace is and inner product). To check the first one, observe that if $v \in L_0$ and $w \in L_0^\omega = L_0$, then $\omega(Jv, Jw) = \omega(v, w) = 0$, so $(JL_0)^\omega \subset JL_0$. As J is an automorphism of V, we have that $\dim JL_0 = \dim L_0 = n$. To see that the intersection is transverse, it is sufficient to check that $L_0 \cap JL_0 = \{0\}$ as we have that $\dim JL_0 = \dim L_0 = n$. Let $u \in L_0 \cap JL_0$. Then $u = Jv$ for some $v \in L_0$. Compute $g_J(u, u) = \omega(u, Ju) = \omega(u, -v) = \omega(v, u) = 0$, hence $u = 0$.

2. First, let us check that Φ is surjective. Take $(L, G) \in \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0)$ and let us define a J. For $v \in L_0$ consider the orthogonal $v^\perp$ which is a $(n-1)$-dimensional subspace of L_0. Its symplectic orthogonal

is then $(n+1)$-dimensional. It is easy to check that $(v^{\perp})^{\omega} \cap L$ is 1-dimensional. Let $Jv \in (v^{\perp})^{\omega} \cap L$ be the unique vector field such that $\omega(v, Jv) = 1$. Repeat this construction for a every $v_i, i = 1, \ldots, n$ where $\{v_i\}$ form a G-orthonormal basis of L_0. It is clear that the Jv_i are linearly independent. We now left to check that the J that we defined satisfies that $\Phi(J) = (L, G)$. This follows from the construction. We now check that Phi is injective: suppose that $\Phi(J) = \Phi(J')$. Then $JL_0 = J'L_0 := L$ and $G_J|_{L_0} = G'_J|_{L_0} =: G$. Take an G-orthonormal basis $v_1, \ldots, v_n$ of L_0. Using the same notation as before we have $v_i \perp = \langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \rangle$ and $(v_i^{\perp})^{\omega} = \langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n, u_i \rangle$ where $u_i \in L$. Since $Jv_i \in L$, $\omega(v_i, Jv_i) = 1$ and $G(v_j, Jv_i) = \omega(v_j, JJv_i) = -\omega(v_j, v_i) = 0$ for all $j \neq i$, it follows that Jv_i is the oly vector in $(v_i^{\perp})^{\omega} \cap L$ satisfying $\omega(v_i, Jv_i) = 1$, so $J = J'$.

3. $\mathcal{G}(L_0)$ is contractible since $tG_1 + (1-t)G_2$ is still an inner product on L_0. To see that $\mathcal{L}(V, \omega, L_0)$ is contractible we identify it with the vector space of all symmetric $n \times n$-matrices (which is convex, hence contractible). Fix a compatible almost complex structure J compatible with ω. Observe that if L is an n-dimensional subspace of V transverse to L_0, we have that L is the graph of a linear map $S : JL_0 \to L_0$ where $L_0 = \langle v_1, \ldots, v_n \rangle$ and $L = \langle Jv_1 + SJv_1, \ldots, Jv_n + SJ_n \rangle$. In this basis, the linear map S is expressed as a symmetric $n \times n$ matrix that we denote by A. Indeed, we have that $A_{ij} = \omega(SJv_i, Jv_j)$. Then using that L_0 and L are Lagrangian we have that $0 = \omega(Jv_i + SJv_i, Jv_j + SJv_j) = \omega(Jv_i, Jv_j) + \omega(Jv_i, SJv_j) + \omega(SJv_j, Jv_j) + \omega(SJv_i, SJv_j) = -\omega(SJv_j, Jv_i) + \omega(SJv_i, Jv_j)$, so A is symmetric. Similarily, when A is symmetric than L is Lagrangian.

Exercise 7. *Let (M, ω) be a symplectic manifold and J a compatible almost complex structure. Show that if $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic curve that then $\partial_s u$ and $\partial_t u$ are orthogonal and have same length for the compatible metric. Prove that $Area_{g_J}(u) = \int_{\Sigma} u^*\omega$.*

Solution 7. We did this during the lecture. Here is the computation. Take holomorphic coordinates on (Σ, j) given by $z = s + it$. Applying ∂_s to the equation $J \circ du = du \circ j$, we obtain $J(\partial_s u) = \partial_t u$. Then, using compatibility of J, we obtain

1. $g_J(\partial_s u, \partial_t u) = \omega(\partial_s u, J\partial_t u) = \omega(\partial_s u, -\partial_s u) = 0$,
2. $g_J(\partial_s u, \partial_s u) = \omega(\partial_s u, J\partial_s u) = \omega(J\partial_t u, -\partial_t u) = g_J(\partial_t u, \partial_t u)$.

Hence the area is compute integrating $\|\partial_s u\|\|\partial_t u\| = \|\partial_s u\|^2 = \omega(\partial_s u, J\partial_s u) = \omega(\partial_s u, \partial_t u)$. Here the norm is of course the norm induced by the metric g_J.

Exercise 8. *Let (Σ, j) be a Riemann surface and (M, ω) be symplectic equipped with a compatible almost complex structure. For a smooth map $u : \Sigma \to M$ the Dirichlet energy is defined to be*

$$E(u) = \frac{1}{2} \int_{\Sigma} |du|_J^2 dvol_{\Sigma}.$$

Prove that

$$E(u) = \int_{\Sigma} |\overline{\partial}_J(u)|_J^2 dvol_{\Sigma} + \int_{\Sigma} u^*\omega$$

where $\overline{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ j)$.

Solution 8. Take conformal coordinates $z = s + it$, we may assume without loss of generality that Σ is an open subset of $\mathbb{C}$. In this case

$$\frac{1}{2}|du|_J^2 \mathrm{dvol}_{\Sigma} = \frac{1}{2}(|\partial_s u|_J^2 + |\partial_t u|_J^2) ds \wedge dt = \frac{1}{2}|\partial_s u + J\partial_t u|_J^2 ds \wedge dt - \langle \partial_s u, J\partial_t u \rangle_J ds \wedge dt = |\overline{\partial}_J(u)|_J^2 \mathrm{dvol}_{\Sigma} + \omega(\partial_s u, J\partial_t u) ds \wedge dt.$$