Lecture 4

Here we will constuct superintegrable with phase spaces

$$\mathcal{M}^G_\Sigma(\mathcal{C}_1,\dots,\mathcal{C}_n)$$

$n = |\partial\Sigma|$, $\mathcal{C}_i \subset G/Ad_G$ conj. classes.

1) Choose cycles $C_1,\dots,C_k \subset \Sigma$

Define the subalgebra

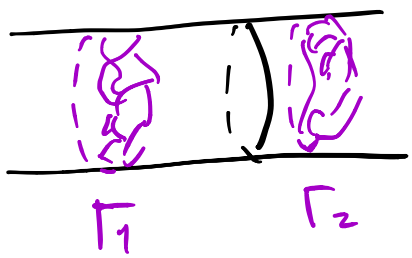
$$I(C) \subset A = C(\mathcal{M}^G_\Sigma(\mathcal{C}_1,\dots,\mathcal{C}_k))$$

as the algebra spanned by graph functions conctractible to $C_1,\dots,C_k$

Thm $I(C)$ is Poisson commutative

Proof. $f_{[\Gamma_1]}$, $f_{[\Gamma_2]}$ for Γ_1, Γ_2 as above.

$[\Gamma_1]$ & $[\Gamma_2]$ have representatives $\Gamma_1 \cap \Gamma_2 = \emptyset$



$\Rightarrow \{ f_{[\Gamma_1]}, f_{[\Gamma_2]} \} = 0$

Now, let us construct the centralizer of $I(C)$ in A

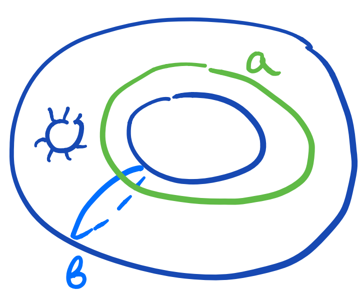
Thm. $Z(I(C), A) =$ span of $f_{[\Gamma]}$

where $[\Gamma]$ are graphs nonintersecting $C_1, \dots, C_k$

Proof. Clearly such functions $\in$ $\in Z(I(C), A)$. They also span $Z(I(C), A)$ (Artamonov, R)

Thm. This is a superintegrable system

Example



$$m^G_\Sigma = (\langle a, b\rangle \to G)/G = (G \times G)/G$$

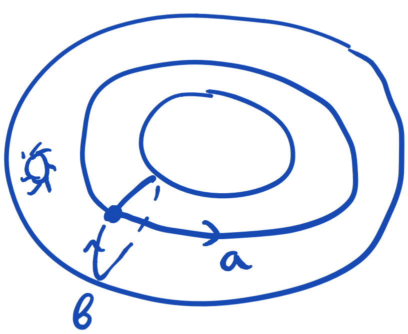
Poisson structure on $\mathcal{M}_\Sigma^G$

(analog of $(\mathfrak{g}^* \times G)/G$)

we know.

We do not know Poisson structure on $G \times G$ (analog of T^*G) which would reduce to A&B on $G\times G/G$

(i) Fock & Rosly:



. choose $B \subset G$, $H \subset B$
Borel Cartan

$$r \overset{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{r} x^i \otimes x_i + \sum_{\alpha \in \Delta_+} e_\alpha \otimes e_{-\alpha} \in \mathfrak{g}^{\otimes 2}$$

$a \mapsto x$

$b \mapsto y$

$$\{x_1, x_2\} = r_{12} x_1 x_2 - x_1 x_2 r_{21} + x_1 r_{21} x_2 - x_2 r_{12} x_1,$$

$\{x_1, y_2\} =$

$\{y_1, y_2\} =$

action of G by conjugation is admissible

$\Rightarrow \quad G\times G/\mathrm{Ad}_G$ is also Poisson

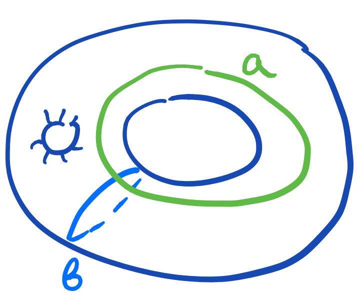
<u>Theorem</u>. This is the A&B Poisson bracket.

(ii) quasi Poisson structures on $C(G\times G)$

Jacobi identity is not satisfied, but holds for $C(G\times G)^G$.

(Alekseev, Meinrenken, ...)

<u>A superintegrable system</u>



choose $C = b$

$$\begin{array}{ccc} G\times G & \qquad & T^*G \\ \downarrow & & \downarrow \\ (G\times G)/\mathrm{Ad}_G & & T^*G/G \end{array}$$

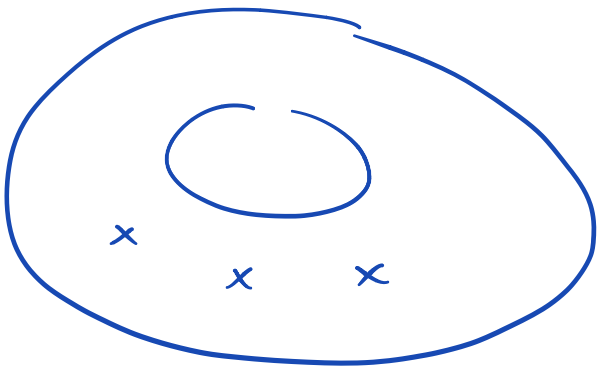
$$xyx^{-1}y^{-1} \in \mathcal{C}$$

$$\begin{array}{ccc}
(G\times G)/Ad_G & \supset & \mathcal{M}(\mathcal{C}) \\
\downarrow p_1 & & \downarrow p_1 \\
(G \tilde{\times}_{G/Ad_G} G)/Ad_G & \supset & \mathcal{P}(\mathcal{C}) \\
\downarrow p_1 & & \downarrow p_2 \\
G/Ad_G & \supset & \mathcal{B}(\mathcal{C})
\end{array}
\qquad
\begin{array}{ccc}
S(\mathcal{O}) & \subset & (\mathfrak{g}^* \times G)/G \\
\downarrow p_1 & & \downarrow p_1 \\
\mathcal{P}(\mathcal{O}) & \subset & (\mathfrak{g}^* \tilde{\times}_{\mathfrak{g}^*/G} \mathfrak{g}^*)/G \\
\downarrow p_2 & & \downarrow p_2 \\
\mathcal{B}(\mathcal{O}) & \subset & \mathfrak{g}^*/G
\end{array}$$

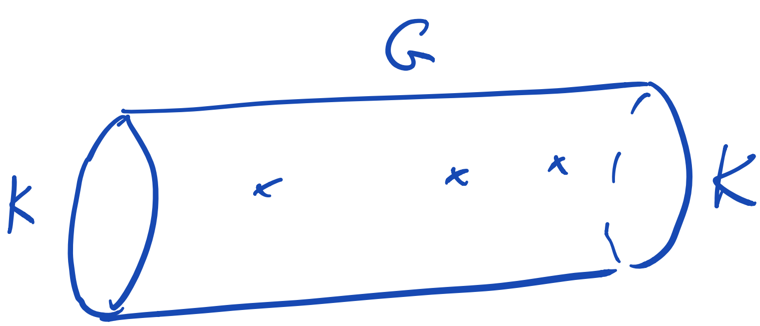
$$\begin{array}{c}
G(x,y) \\
\downarrow p_1 \\
G(x, y^{-1}x^{-1}y) \\
\downarrow p_2 \\
Gx
\end{array}
\qquad
\begin{array}{c}
G(x,g) \\
\downarrow p_1 \\
G(x, -y^{-1}xy) \\
\downarrow p_2 \\
Gx
\end{array}$$

"Nonlinear" deformation of spin CM system

Relativistic CM, Ruijsenaars-Schneider



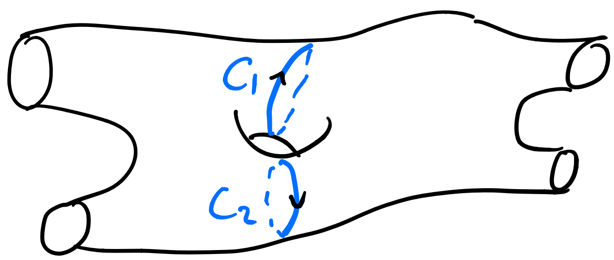
also very interesting example.
Related to dynamical KZ equation.... (Etingof, Schiffmann)
to integrable systems studied by (Chalykh, Fairon; ...)
(special multiparticle systems)



$K \subset G$

spherical functions.... (Stokman, R)

Universal integrable system on a surface.



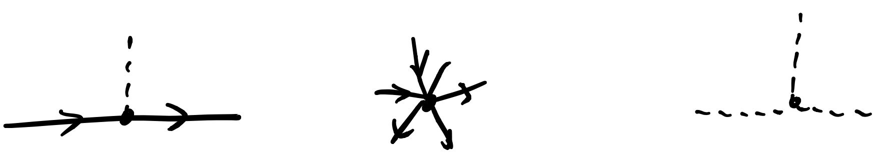
Σ oriented

$C = C_1 \cup C_2$

simple, oriented

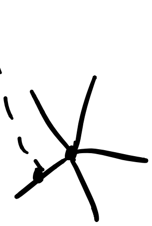
1) Algebra of chord diagrams

a) Chord diagrams: $\Gamma \subset \Sigma$



Relations:

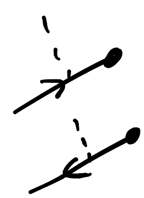
b) $\sum_i \pm$

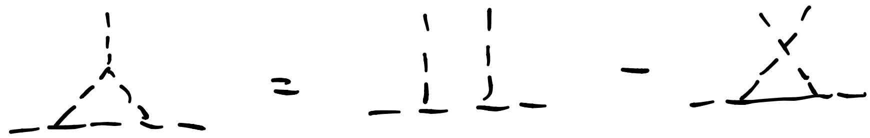


$= 0$

$+$

$-$



c) 

$[\Gamma]$ = the class of continuous deformations

2) Multiplication:

$$[\Gamma_1]\cdot[\Gamma_2] = [\Gamma_1 \cup \Gamma_2]$$

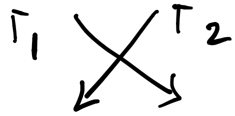
Γ_1 & Γ_2 are not joint at intersection points.

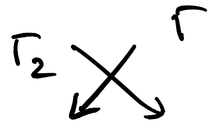
2) Poisson structure:

$$\{[\Gamma_1], [\Gamma_2]\} = \sum_{p \in \Gamma_1 \cap \Gamma_2} \varepsilon_p \; \Gamma_1 \#_p \Gamma_2$$

where

$\varepsilon_p = \begin{cases} +1, \\ -1, \end{cases}$

$+1$, 

-1, 

and

$\Gamma_1 \#_p \Gamma_2 =$