Lecture 3

Summary of Lectures 1 & 2

1) Liouville integrability & superintegrability

2) spin Calogero-Moser systems

- $Ad: G \circlearrowright T^*G$, Hamiltonian
- $\mu: T^*G \to \mathfrak{g}^*$, $(x, g) \mapsto x - Ad^*_{g^{-1}}(x)$

$$S(\mathcal{O}) = \mu^{-1}(\mathcal{O})/G \subset T^*G/Ad_G$$

$$\cup$$

$$S(\mathcal{O})_{reg} \cong (\mathcal{O}/\!/H \times T^*H_{reg})/W$$

$\uparrow$

phase space for spin CM system

Hamiltonians $= C(\mathfrak{g}^*)^G \hookrightarrow C(T^*G)^G$ restr. to $S(\mathcal{O})_{reg}$

4) Superintegrability

We should find Poisson manifold $\mathcal{P}(\mathcal{O})$ with Poisson projections p_1, p_2 surjective

$$S(\mathcal{O}) \xrightarrow[p_1]{} \mathcal{P}(\mathcal{O}) \xrightarrow[p_2]{} B(\mathcal{O})$$

s.t. $\dim S(\mathcal{O}) = \dim \mathcal{P}(\mathcal{O}) + \dim B(\mathcal{O})$

i.e., we should prove that

$Z(C(B(\mathcal{O})), C(S(\mathcal{O})))$ has maximal rank

Theorem 1) Such $\mathcal{P}(\mathcal{O})$ is

$$\mathcal{P}(\mathcal{O}) = (\mu_L \times \mu_R)(\mu^{-1}(\mathcal{O}))/Ad_G =$$

$$= \{(x, y) \mid x - y \in \mathcal{O},\ Gx = Gy\}$$

2) $p_1(G(x,g)) = G(x, -Ad^*_{g^{-1}}(x))$

3) $p_2(G(x,y)) = Gx (= Gy)$

<u>Proof</u> (h/w)

$B(\mathcal{O}) = \pi(S(\mathcal{O}))$

$$\begin{array}{ccc} T^*G/G & \supset & S(\mathcal{O}) \\ \pi \downarrow & & \pi \downarrow \\ \mathfrak{g}^*/G & \supset & B(\mathcal{O}) \end{array}$$

$\pi = p_2 \circ p_1$

Explicitly:

$B(\mathcal{O}) = \{ \mathcal{O}' \subset \mathfrak{g}^*/G \mid \mathcal{O} - \mathcal{O}' \subset \mathcal{O} \}$

$= \{ \mathcal{O}' \in \mathfrak{g}^*/G \mid y - y' \in \mathcal{O}, \forall y, y' \in \mathcal{O}' \}$

<u>Fibers of p_2:</u>

$\mathcal{O}' \in B(\mathcal{O})$, $\qquad p_2^{-1}(\mathcal{O}) \subset \mathcal{P}(\mathcal{O})$

$$\mathcal{P}(\mathcal{O},\mathcal{O}') = p_2^{-1}(\mathcal{O}') = \{(x,y)\in \mathfrak{g}^*\times\mathfrak{g}^* \mid x-y\in\mathcal{O},\ \ x,y\in\mathcal{O}'\} = \{(x,y,z)\in \mathcal{O}'\times(-\mathcal{O}')\times\mathcal{O} \mid x+y+z=0\}/G = (\mathcal{O}'\times(-\mathcal{O}')\times\mathcal{O})//G$$

Hamiltonian reduction

5) Explicit solution to dynamics

$$\begin{tikzcd}
T^*G \simeq \mathfrak{g}^*\times G \arrow[d] & \ni & (x,g) \arrow[d, mapsto] \\
\mathfrak{g}^* & \ni & x
\end{tikzcd}$$

$$H\in C(\mathfrak{g}^*)^G \subset C(T^*G)$$

__Lemma__ The Hamiltonian flow generated by H on $T^*G \simeq \mathfrak{g}^*\times G$ passing through (x,g) at $t=0$ is

$$(x(t), g(t)) = (x, e^{t\nabla H(x)} g)$$

$$(\nabla H(x), y) = \frac{d}{dt} H(e^{ty} x)\Big|_{t=0}$$

Killing form

sl_n: $H = \frac{1}{k} tr(x^k)$, $\nabla H(x) = x^{k-1}$,

Now we can project this to

T^*G/G :

$$G(x, g) \xmapsto{\ t\ } G(x(t), g(t)) = G(x, e^{\nabla H(x) t} g)$$

Restricts to $S(\mathcal{O})$.

6) <u>Special case: $\mathcal{O} \subset sl_n^*$, rank = 1</u>

$$\mathcal{O}_{reg} = \left\{ \begin{array}{l} g\mu g^{-1},\ g \in SL_n,\ \mu \in n\times n \text{ traseless},\\ \qquad \text{rank}(\mu) = 1,\quad \mu = \text{digon-ble} \end{array} \right\}$$

- $$\mu_{ij} = \varphi_i \psi_j - \delta_{ij} \frac{(\varphi,\psi)}{n}, \qquad (\varphi,\psi) \neq 0$$

- $\mu_0^{-1}(\mathcal{O}_{reg})$: $\mu_{ii} = 0, \Rightarrow \varphi_i \psi_i = \frac{(\varphi,\psi)}{n}, \forall i$

- H acts as $s: \mu_{ij} \mapsto s_i \mu_{ij} s_j^{-1}$,

$$\varphi_i \mapsto s_i \varphi_i, \quad \psi_i \mapsto s_i^{-1} \psi_i$$

cross-section: $\varphi_i = 1, \quad \psi_i = \frac{(\varphi,\psi)}{n} = c$

$$\boxed{\mu_{ij} = c, \ i \neq j, \ \mu_{ii} = 0}$$

$$\Rightarrow G(x,g) = S_n(\tilde{x}, h)$$

$$\tilde{x} = \begin{bmatrix} p_1 & & \frac{c}{1-h_i h_j^{-1}} \\ & \ddots & \\ \frac{c}{1-h_i^{-1} h_j} & & p_n \end{bmatrix}, \quad h = \begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix}$$

Non spin Calogero-Moser-Sutherland model

$$I_k = \frac{1}{k} \operatorname{tr}(x^k) = \frac{1}{k} \operatorname{tr}(\tilde{x}^k)$$

$$I_2 = \frac{1}{2}\vec{p}^{\,2} + \sum_{i<j} \frac{c^2}{(1-h_i h_j^{-1})(1-h_i^{-1}h_j)}, \qquad h_i = e^{q_i},$$

7) Eplicit action-angle variables:

Assume $x \in (sl_n^*)_{reg}$,

$$G(x,g) = N(H_{reg})(d,s), \quad d = \begin{bmatrix} d_1 & & \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$d_i \neq d_j$

Moment map condition:

$$x - g^{-1} x g \in \mathcal{O}$$

For representatives d, s:

$$d - s^{-1} d s = \mu$$

or

$$s_{ij} d_j - d_i s_{ij} = \sum_{k=1}^{n} s_{ik} \mu_{kj}$$

or

$$s_{ij} = \frac{c\, s_{ii}}{x_i - x_j + c} \qquad (h/w)$$

<u>Thm</u> (d_i, s_{ii}) are action angle variables:

- $$\omega = \sum_{i=1}^{n} d_i \wedge \frac{ds_{ii}}{s_{ii}}$$

- $$d_i(t) = d_i \, , \quad s_{ii}(t) = e^{d_i t} s_{ii}$$

Solution to equations of motion

<u>Step 1</u> "direct spectral transform"

$$U \begin{bmatrix} p_1 & & \dfrac{c}{sh\left(\frac{q_j - q_i}{2}\right)} \\ & \ddots & \\ \dfrac{c}{sh\left(\frac{q_i - q_j}{2}\right)} & & p_n \end{bmatrix} U^{-1} = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$U \begin{bmatrix} e^{q_1} & & 0 \\ & \ddots & \\ 0 & & e^{q_n} \end{bmatrix} U^{-1} = \begin{bmatrix} s_{11} & & 0 \\ & \ddots & \\ 0 & & s_{nn} \end{bmatrix} \begin{bmatrix} 1 & & \dfrac{c}{d_i - d_j + c} \\ & \ddots & \\ \dfrac{c}{d_j - d_i + c} & & 1 \end{bmatrix}$$

Step 2 Inverse spectral transform:

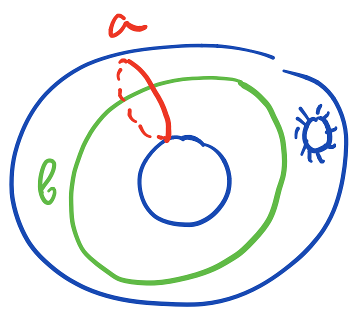
find $U(t)$, $q_i(t)$, s.t.

$$\begin{bmatrix} s_{11}e^{d_1 t} & & 0 \\ & \ddots & \\ 0 & & s_{nn}e^{d_n t} \end{bmatrix} \begin{bmatrix} 1 & \ddots & \frac{c}{d_i-d_j+c} \\ & \ddots & \\ \frac{c}{d_j-d_i+c} & \ddots & 1 \end{bmatrix} = U(t) \begin{bmatrix} e^{q_1(t)} & & 0 \\ & \ddots & \\ 0 & & e^{q_n(t)} \end{bmatrix} U(t)^{-1}$$

$$\begin{array}{ccc} (p, q) & \longmapsto & (d, s) \\ \downarrow & & \downarrow \\ (p(t), q(t)) & \longleftarrow\!\!\mid & (d, s e^{dt}) \end{array}$$

Similar to solving KdV by direct and inverse spectral problems.

Spin Calogero-Moser-Sutherland type systems and moduli spaces

1) 

$\pi_1(T \setminus \text{disc}) = \langle a, b \rangle$

More general: Σ, $\partial\Sigma \neq \emptyset$, $n = |\partial\Sigma| > 1$

$\pi_1(\Sigma)$ - free group on $2g+n-1$ generators

- Representation variety:

$$R(\Sigma, G) = \pi_1(\Sigma) \to G \quad (\text{base poin } p_0)$$

- Character variety (moduli space of flat connections):

$$\mathcal{M}_\Sigma^G = R(\Sigma, G)/G$$

Poisson structure on $\mathcal{M}_\Sigma^G$.

(i) Symplectic structure on M^G_Σ, $\partial\Sigma = \phi$

Σ - oriented, compact surface.

$\partial\Sigma = \phi$, Fix a principal G-bundle over Σ, $E \to \Sigma$

Can choose it to be trivial $E = G \times \Sigma$

(a) $\mathrm{Conn}^G_\Sigma = \{$ connections on $E \}$

in general Conn^G_Σ is an affine vector space over $\Omega^1(\Sigma, \mathfrak{g})$. Because E is trivial, we have trivial connection and

$$\mathrm{Conn}^G_\Sigma \simeq \Omega^1(\Sigma, \mathfrak{g})$$

(b) Symplectic form on Conn^G_Σ:

(Σ - oriented)

$$\omega_\Sigma^G = \int_\Sigma DA \wedge DA$$

$$DA \in \Omega^1(\Omega^1(\Sigma, \mathfrak{g}))$$

(c) Hamiltonian action of the gauge group $\mathcal{G}_\Sigma = \mathrm{Maps}(\Sigma, G)$ on Conn_Σ^G :

$$g : A \mapsto g^{-1} A g + g^{-1} dg$$

The moment map:

$$\mu : A \longrightarrow F_A = dA + \frac{1}{2}[A \wedge A]$$

(d) The Hamiltonian reduction:

- $\mu^{-1}(0) = \{A \in \mathrm{Conn}_\Sigma^G \mid F_A = 0\}$
  the space of flat connections
  (coisotropic in Conn_Σ^G)

- $\mu^{-1}(0)/G_\Sigma = \mathcal{M}_\Sigma^G$ - moduli space of flat connections $\simeq$

  $\simeq$ character variety

($\Rightarrow$)

<u>Thm</u> (Atiyah & Bott) There is a natural symplectic structure on $\mathcal{M}_\Sigma^G$ (when $\partial\Sigma = \emptyset$).

(ii) <u>Poisson structure:</u>

(a) <u>Graph functions on</u> $\mathcal{M}_\Sigma^G$

- $\Gamma \subset \Sigma$, graph with edges

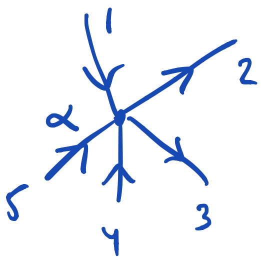
  $\bullet\!\longrightarrow\!\bullet$ solid, oriented

  $\bullet$- - - - - -$\bullet$ chords

- Coloring of $\Gamma \subset \Sigma$

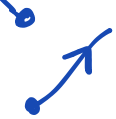
$\xrightarrow{\;V\;}$, solid egde $\rightsquigarrow$ f.d. representation V of G

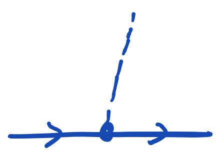
- - - - - - - , chord $\rightsquigarrow$ adjoin represent. in $\mathfrak{g}$



$\rightsquigarrow \quad \alpha : \left(V_1^{\varepsilon_1} \otimes \cdots \otimes V_5^{\varepsilon_5}\right)^G$

$\varepsilon_i = +$ if 

$\varepsilon_i = -$ if 



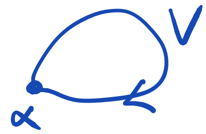
$$\sim \quad \pi_V : \underset{\text{ad}_{\mathfrak{g}}\text{-space}}{\overset{}{\mathfrak{g}}} \longrightarrow V \otimes V^* = \mathrm{End}(V)$$

We identified - - > - - $\sim$ - - < - -

$\mathrm{ad} \simeq \mathrm{ad}^*$ using the Killing form

- Graph functions on connections

$$f_\Gamma(A) = \left\langle \bigotimes_e \mathrm{hol}_e(A) , \bigotimes_v \alpha_v \right\rangle$$

Ex:  $\alpha \in (V \otimes V^*)^G \simeq \mathrm{End}(V)$

$$f_\Gamma(A) = \mathrm{tr}(\alpha\, \mathrm{hol}_e(A))$$

Theorem $f_\Gamma(A^g) = f_\Gamma(A)$, $A \in \mathrm{Conn}^G_\Sigma$

Theorem. (1) If $F_A = 0$,

$$f_\Gamma(A) = f_{[\Gamma]}(A)$$

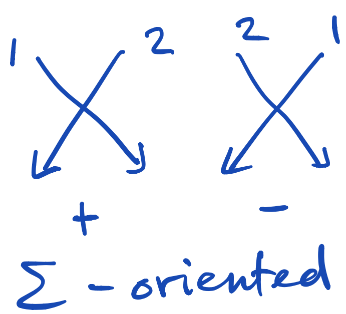
(2) f_Γ span $C(\mathcal{M}^G_\Sigma)$ in the algebraic case

(6) The Poisson bracket ($\partial\Sigma = \emptyset$)

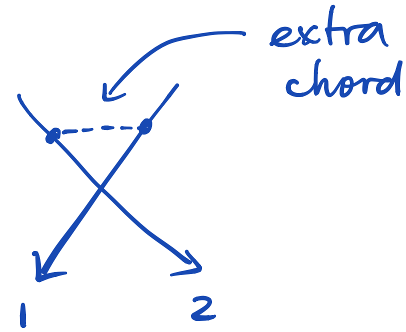
(Goldman; Turaev; Andersen, Mattes, R)

Theorem. Let $\{\cdot,\cdot\}$ be the Poisson bracket for the Atiyah-Bott sympl. structure on $\mathcal{M}^G_\Sigma$, then

$$\{f_{[\Gamma_1]}, f_{[\Gamma_2]}\} = \sum_{p\in\Gamma_1\cap\Gamma_2} \varepsilon_p \, f_{[\Gamma_1 \#_p \Gamma_2]} \qquad (\star)$$



; $\Gamma_1 \#_p \Gamma_2$



(c) Define the Poisson structure on M^G_Σ by the same formula $(\star)$ even when $\partial\Sigma \neq \emptyset$.

<u>Thm</u> (i) $(M^G_\Sigma, \star)$ is a Poisson variety.

(ii) Fix conjugacy class $\mathcal{C}_i \subset G/\mathrm{Ad}G$ for each boundary component $(\partial\Sigma)_i$

then $\mathcal{M}^G_\Sigma(\mathcal{C}_1, \dots, \mathcal{C}_n) \subset \mathcal{M}^G_\Sigma$ is a symplectic leaf

Symplectic manifolds $\mathcal{M}^G_\Sigma(\mathcal{C}_1, \dots, \mathcal{C}_n)$ will be phase spaces for superintegrable systems.