Introduction to J-holomorphic curves

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This is a 2-hours introduction to J-holomorphic curves and serves as a preparation for Urs Frauenfelder's course on the restricted three body problem, see [5] for the lecture notes.

J-holomorphic curves were introduced in the seminal paper by Gromov [6] in '85 an revolutionized the field of symplectic topology. The paper contains many results that are considered to be foundations to the field of symplectic topology. In this lecture, we will focus on the Gromov non-squeezing theorem, which gives as a good motivation why to study J-holomorphic curves and their moduli space. We will outline the original proof by Gromov and leave some exercises on the way. The correction of the exercises will be available later this week on my webpage. Some of the exercises are easy, and some are relatively hard.

I also include a bibliography for further information on J-holomorphic curves and related material. The standard references are Wendl's lectures notes [1] and the book by McDuff–Salamon [2]. For a short outreach text by Donaldson see [7]. Schlenk's lecture notes [3] explains the proof of Gromov's original paper. For a basic introduction on symplectic geometry see [4].

1 Exercises

Exercise 1 *Let $\varphi : (M, \omega_1) \to (N, \omega_2)$ then φ preserves the volume.*

Exercise 2 *Prove that there is a volume preserving embedding $\varphi : B^{2n}(a) \to Z^{2n}(A)$ for $A < a$. Prove that this is not true for linear symplectomorphisms (for the standard symplectic structure of course).*

Exercise 3 *Check that $\omega_0(u, v) = g_0(J_0 u, v)$ and that $g_0(J_0 u, J_0 v) = g(u, v)$ where ω_0 denotes the standard symplectic structure, J_0 the standard complex and g_0 the standard inner product on $\mathbb{R}^{2n}$. Check that if $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ preserves ω_0 and J_0 then φ also preserves g_0.*

Exercise 4 *Prove that if S is a proper complex submanifold of $B^{2n}(a)$ (for the standard complex structure) passing through the origin, then $a \leq area_{g_0}(S)$. To prove this follow the following hints:*

1. *Let $\Sigma(r) \subset B^{2n}(r) \subset \mathbb{C}^n$ an orientable surface whose boundary lies in $S^{2n-1}(r)$. Denote the symplectic area of Σ by $A_\omega(r)$ and the Riemannian length of $\partial\Sigma$ by $L(r)$. Prove that the following inequalities hold:*
$$\frac{d}{dr} A_\omega(r) \geq L(r) \geq \frac{2A(r)}{r}.$$
2. *Prove that if a loop in $S^{2n-1}(r)$ has length strictly less than $2\pi r$ then it is contained in a hemisphere of $S^{2n-1}(r)$.*
3. *Conclude the exercise.*

Exercise 5 *Prove that if $J : V \to V$ is a complex structure on a vector space V, then* $\dim V = 2n$.

Exercise 6 *Let (M, ω) be a symplectic manifold and denote by $\mathcal{J}(M, \omega) = \{J$ almost complex structure compatible with ω Prove that $\mathcal{J}$ is contractible following the following hints: Consider a symplectic vector space (V, Ω) of dimension $2n$ and a Lagrangian subspace L_0, i.e.* $\dim L_0 = n$ *and $\Omega|_{L_0} = 0$. Denote by $\mathcal{L}(V, \Omega, L_0)$ the set of*

Lagrangian subspaces that are transverse to L_0. Denote by G_0 the space of all positive inner products on L_0. Consider the map

$$\begin{aligned} \Phi : \mathcal{J}(V, \Omega) &\to \mathcal{L}(V, \Omega, L_0) \times \mathcal{G}(L_0) \\ J &\mapsto (JL_0, G_J|_{L_0}) \end{aligned}$$

Show that

1. *Φ is well-defined.*
2. *Φ is a bijection.*
3. *$\mathcal{L}(V, \Omega, L_0)$ and $\mathcal{G}(L_0)$ are contractible.*

Exercise 7 *Let (M, ω) be a symplectic manifold and J a compatible almost complex structure. Show that if $u : (\Sigma, j) \to (M, J)$ is a J-holomorphic curve that then $\partial_s u$ and $\partial_t u$ are orthogonal and have same length for the compatible metric. Prove that $Area_{g_J}(u) = \int_\Sigma u^*\omega$.*

References

[1] Wendl, Chris. *Lectures on holomorphic curves in symplectic and contact geometry.* arXiv preprint arXiv:1011.1690 (2010).

[2] McDuff, Dusa, and Dietmar Salamon. *J-holomorphic curves and symplectic topology.* Vol. 52. American Mathematical Soc., 2012.

[3] Schlenk, Felix. *Embedding problems in symplectic geometry.* Vol. 40. Walter de Gruyter, 2008.

[4] Da Silva, Ana Cannas. *Lectures on symplectic geometry.* Vol. 3575. Berlin: Springer, 2001.

[5] Frauenfelder, Urs, and Otto Van Koert. *The Restricted Three-Body Problem and Holomorphic Curves.* Birkhäuser, 2018.

[6] Gromov, Mikhael. *Pseudo holomorphic curves in symplectic manifolds.* Inventiones mathematicae 82.2 (1985): 307-347.

[7] Donaldson, Simon K. *What is a pseudoholomorphic curve?* Notices of the AMS 52.9 (2005).