Mathematical Institute

Higgs bundles and mirror symmetry 3

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MIRROR SYMMETRY

RECALL....

- $p : \mathcal{M} \to \mathcal{B}$

  fibre abelian variety A

- mirror $p^{\vee} : \mathcal{M}^{\vee} \to \mathcal{B}$

  fibre $A^{\vee}$ = degree 0 line bundles on A

- complex Lagrangian $L \subset \mathcal{M}$

  $(L \cap A)^0 \subset A^{\vee}$ = line bundles trivial on $L \cap A$

RECALL....

- $p : \mathcal{M} \to \mathcal{B}$

  fibre abelian variety A

- mirror $p^\vee : \mathcal{M}^\vee \to \mathcal{B}$

  fibre $A^\vee =$ degree 0 line bundles on A

- complex Lagrangian $L \subset \mathcal{M}$

  $(L \cap A)^0 \subset A^\vee =$ line bundles trivial on $L \cap A$

- $\mathbf{C}^*$-invariant $\Rightarrow$

  $L \cap A =$ union of translates of an abelian subvariety

  (usually zero-dimensional)

- $E \in (L \cap A)^0 \subset \mathcal{M}^\vee$ line bundle on A trivial on $L \cap A$

  $H^0(L \cap A, E)$: basis vector for each component of $L \cap A$

- universal bundle U on family $A \times A^\vee \sim$

  relative Fourier-Mukai:

- suppose $L^\vee \subseteq \mathcal{M}^\vee$ is a hyperkähler submanifold, then ..

- .. projection $P : L \times_{p(L)} \mathcal{M}^\vee \to L^\vee$ defines $\mathbf{V} = P_* U$

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is this hyperholomorphic?

HYPERHOLOMORPHIC BUNDLES

- connection with curvature of type $(1,1)$ wrt I, J, K

- 4 dimensions = anti-self-dual

- $\Leftrightarrow$ holomorphic bundle on twistor space

DIRAC-HIGGS BUNDLE

THE TANGENT BUNDLE OF $\mathcal{M}$

- Levi-Civita connection is hyperholomorphic

- Higgs bundle tangent space $(\dot{A}, \dot{\Phi}) \in \Omega^{01}(\mathfrak{g}) \oplus \Omega^{10}(\mathfrak{g})$

- $\bar{\partial}_A \dot{\Phi} + [\dot{A}, \Phi] = 0$ modulo $(\dot{A}, \dot{\Phi}) = (\bar{\partial}_A \psi, [\psi, \Phi])$

- elliptic complex

  $0 \to \Omega^{00}(\mathfrak{g}) \to \Omega^{01}(\mathfrak{g}) \oplus \Omega^{10}(\mathfrak{g}) \to \Omega^{11}(\mathfrak{g}) \to 0$

- tangent space to $\mathcal{M}$ = first cohomology group

- Dolbeault version of hypercohomology

- sequence of sheaves $\mathcal{O}(\mathfrak{g}) \stackrel{\mathrm{ad}\,\Phi}{\to} \mathcal{O}(\mathfrak{g} \otimes K)$

- tangent space to $\mathcal{M}$ = first hypercohomology group $\mathbb{H}^1$

- varies holomorphically over $\mathcal{M}$ with complex structure I

- Higgs bundle equations $F_A + [\Phi, \Phi^*] = 0 \Rightarrow$ flat connection

- variation: $d_A(\dot{A} + \dot{\Phi} + \dot{\Phi}^*) + [\Phi + \Phi^*, \dot{A} + \dot{\Phi} + \dot{\Phi}^*] = 0$

- tangent space to $\mathcal{M}$ = first de Rham cohomology group H^1 of flat connection

- varies holomorphically over $\mathcal{M}$ with complex structure J

- Hodge theory for elliptic complex

  $0 \to E_0 \xrightarrow{d} E_1 \xrightarrow{d} E_2 \to 0$

- $d + d^* : E_0 \oplus E_2 \to E_1$

- same operator for each complex $-$"Dirac" operator $\mathbf{D}$

- coker $\mathbf{D}$ defines a hyperholomorphic bundle over $\mathcal{M}$

- replace $\mathfrak{g}$ by any representation of G

- hypercohomology of sequence of sheaves: $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

- coker $\mathbf{D}$ defines a hyperholomorphic bundle over $\mathcal{M}$

- "Dirac-Higgs bundle" (if a universal bundle over $\mathcal{M} \times \Sigma$ exists)

VECTOR REPRESENTATION

- $\mathcal{O}(V) \stackrel{\Phi}{\rightarrow} \mathcal{O}(V \otimes K)$

- $0 \rightarrow H^1(\ker \Phi) \rightarrow \mathbb{H}^1 \rightarrow H^0(\mathrm{coker}\, \Phi) \rightarrow 0$

VECTOR REPRESENTATION

- $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

- $0 \to H^1(\mathrm{ker}\,\Phi) \to \mathbb{H}^1 \to H^0(\mathrm{coker}\,\Phi) \to 0$

- open covering $U_\alpha, \ldots$

  $\theta_{\alpha\beta}$ holomorphic section of V on $U_\alpha \cap U_\beta$

  ψ_α on U_α

- $\Phi\theta_{\alpha\beta} = \psi_\beta - \psi_\alpha \Rightarrow$ class in $\mathbb{H}^1$

- project to cokernel $\Rightarrow \bar{\psi}_\beta = \bar{\psi}_\alpha$

VECTOR REPRESENTATION

- $\mathcal{O}(V) \stackrel{\Phi}{\to} \mathcal{O}(V \otimes K)$

- $0 \to H^1(\ker \Phi) \to \mathbb{H}^1 \to H^0(\operatorname{coker} \Phi) \to 0$

- $\det \Phi = 0$ on $x = 0$ and $\operatorname{coker} \Phi \cong L \qquad (\pi_* L = V)$

  so $\quad \mathbb{H}^1 \cong \bigoplus_{x_i \in S \cap \{x=0\}} L_{x_i}$

  ↑

- Dirac-Higgs bundle $\mathbf{V}$ hyperholomorphic

UNIVERSAL BUNDLE AT $x \in \Sigma$

HYPERKÄHLER QUOTIENT

- hyperkähler manifold M, triholomorphic action of G

- $a \in \mathfrak{g} \Rightarrow$ vector field $X_a \Rightarrow$ Hamiltonian function f_a

  for $\omega_1, \omega_2, \omega_3$

- moment map $\mu : M \to \mathfrak{g}^* \otimes \mathbf{R}^3$

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- $\mu^{-1}(0)$ principal G-bundle P over $\mu^{-1}(0)/G =$ hyperkähler quotient

- $\Rightarrow$ hyperholomorphic connection on P

- $\mathcal{A}$ = affine space of connections on a G-bundle over Σ

- $T^*\mathcal{A} = \mathcal{A} \times \Omega^1(\Sigma, \mathfrak{g})$

  = ∞-dimensional flat hyperkähler manifold

- group $\mathcal{G}$ of gauge transformations acts

- $\mu^{-1}(0)$ principal $\mathcal{G}$-bundle $\mathcal{P}$ over $\mu^{-1}(0)/\mathcal{G}$

  = moduli space of Higgs bundles

.... hyperholomorphic $\mathcal{G}$-connection

- universal G-bundle on $\mathcal{M} \times \Sigma$
- point $x \in \Sigma \Rightarrow$ principal G-bundle P_x over $\mathcal{M}$
- evaluation homomorphism $\mathrm{ev}_x : \mathcal{G} \to G$
- ... associated connection hyperholomorphic

REAL FORMS AS BAA-BRANES

- complex structure I: moduli space of (stable) pairs (A, Φ)

  $G = U(n)$ vector bundle V, $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$

- complex structure J: flat G^c-connection

  $\nabla_A + \Phi + \Phi^*$ (representations $\pi_1(\Sigma) \to G^c$)

- complex structure K: flat G^c-connection

  $\nabla_A + i\Phi - i\Phi^*$

REAL FORM G^r

- $K \subset G^r$ maximal compact
- principal K^c-bundle
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$
- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

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$\mathbf{C}^*$-invariant

- Higgs field $\Phi \in H^0(\Sigma, \mathfrak{m} \otimes K)$
- holonomy of $\nabla + \Phi + \Phi^* \in G^r$

- moduli space of flat G^r-connections: Hom$(\pi_1, G^r)/G^r$
- fixed point set of involution on $\mathcal{M}$
- I-holomorphic, J, K-antiholomorphic
- BAA-brane

- $L =$ moduli space of flat G^r-connections

- $G^r =$ split real form e.g. $SL(n, \mathbf{R}), Sp(2m, \mathbf{R}), \ldots$

  $\Rightarrow L \cap A =$ 2-torsion points

$\Rightarrow$ support of mirror BBB-brane is the whole moduli space

$U(m,m) \subset GL(2m, \mathbf{C})$

- maximal compact $U(m) \times U(m)$

- bundle $V = V_+ \oplus V_-$ Higgs field $\Phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$

- characteristic class $c_1(V_+) \in H^2(\Sigma, \mathbf{Z})$

- $\Rightarrow$ different topological components

L.Schaposnik, *Spectral data for* $U(m,m)$ *Higgs bundles*, IMRN, **11** (2015) 3486 – 3498.

- spectral curve $\det(x-\Phi) = x^{2m} + a_2 x^{2m-2} + \ldots + a_{2m}$

- involution $\sigma(x) = -x$ on S

- $V = \pi_*(U\pi^* K^{(2m-1)/2})$, $U \in \mathrm{Jac}(S)$

- line bundle $U \in \mathrm{Jac}(S)$, $\sigma^* U \cong U$

- $L \cap A =$ fixed point set of σ

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- fixed points $a_{2m} = 0$ $4m(g-1)$ points

- $\sigma^* U \cong U$ action at fixed points ± 1

- action $+1$ everywhere $\Rightarrow U$ pulled back from $\bar{S} = S/\sigma$

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- action $+1$ everywhere $\Rightarrow U$ pulled back from $\bar{S} = S/\sigma$

- $L \cap A = 2^{4m(g-1)-1}$ copies of $\mathrm{Jac}(\bar{S})$

- $(L \cap A)^0 \cong \mathsf{P}(S, \bar{S})$

- mirror supported on the family of Prym varieties over $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^4) \oplus \cdots \oplus H^0(\Sigma, K^{2m})$

- $= Sp(m)$ moduli space in $U(2m)$ moduli space

- ... which is hyperkähler.

$U(m,m) \subset GL(2m, \mathbf{C})$

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- Lagrangians $L_0, L_1, \ldots$

- same support of the BBB-brane $\Rightarrow$

  they must differ through the hyperholomorphic vector bundle

- $\sigma^* U \cong U$

- action at fixed point set ± 1

- $c_1(V_+) \sim$ number of $+1$s (even)

- basis vectors for $H^0(L \cap A, E) \sim$

  even subsets of $4m(g-1)$ zeros of a_{2m}

- fixed points $x_1, \ldots, x_N$ $(N = 4m(g-1))$

  divisor $2x_i$ pulled back from $\bar{S}$

- $B = p^* \operatorname{Jac}(\bar{S})$

  translates $x_1 + \ldots + x_{2k} - 2kx_1 + B$

- $2x_i - 2x_1 \in B \Rightarrow$ independent of choice of x_1

- $x_1 + \ldots + x_{2k} \in \mathsf{P}(S, \bar{S})^\vee = \operatorname{Jac}(S)/p^* \operatorname{Jac}(\bar{S})$

- $\mathcal{M}^\vee = Sp(m)$ moduli space

- $E \in A^\vee = \mathsf{P}(S, \bar{S})$

- $\{x_1, \dots, x_\ell\} \subset S \cap \{x = 0\}$ defines

  $E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- vector space $\displaystyle\bigoplus_{\{x_1,\dots,x_\ell\} \subset S \cap \{x=0\}} E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- Dirac-Higgs bundle $\mathbf{V} = \bigoplus_{x_\ell \in S \cap \{x=0\}} E_{x_\ell}$

- $\Lambda^\ell \mathbf{V} = \bigoplus_{\{x_1,\ldots,x_\ell\} \subset S \cap \{x=0\}} E_{x_1} \otimes E_{x_2} \otimes \cdots \otimes E_{x_\ell}$

- sum over ℓ-element subsets

  induced hyperholomorphic connection

- no universal bundle for $Sp(m)$

- local ones differ by a line bundle $L_{\alpha\beta}$ on

  $U_\alpha \cap U_\beta \subset \mathcal{M}^\vee$ of order 2

- ℓ even $\Rightarrow \Lambda^\ell \mathbf{V}_\alpha = \Lambda^\ell \mathbf{V}_\beta$ well-defined

HECKE TRANSFORM

- rank 2 vector bundle V: local sections $\mathcal{O}(V)$

- $x \in \Sigma, \alpha \in V_x^*$

- $\ker \alpha : \mathcal{O}(V) \to \mathcal{O}_x$ is $\mathcal{O}(V')$

- $\Lambda^2 V' \cong \Lambda^2 V(-x)$

- $\alpha(v_1, v_2) = v_1$ at $z = 0$

- local basis of sections of V': $e_1 = (z, 0), e_2 = (0, 1)$

- $\Phi \in H^0(\Sigma, \mathrm{End}\, V \otimes K)$ $\quad \Phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

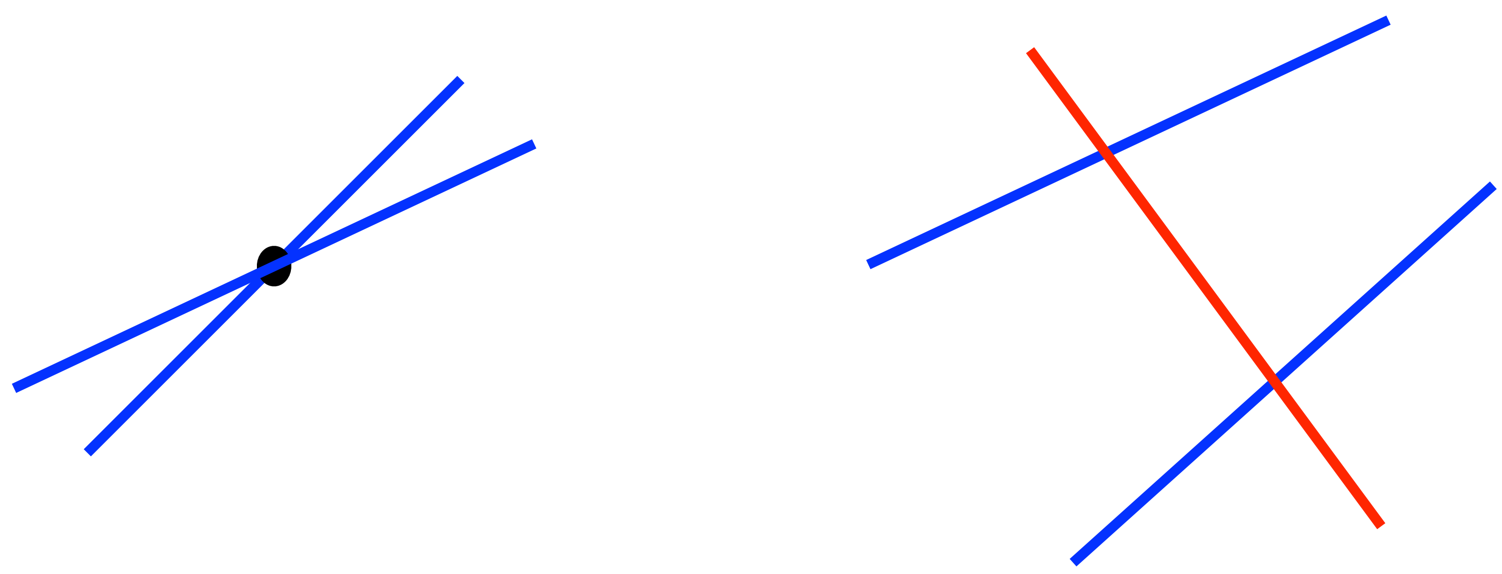
$$\Phi' = \begin{pmatrix} a & bz^{-1} \\ cz & d \end{pmatrix}$$

- if $\Phi(0)(e_2) = \lambda e_2$, get new Higgs bundle (V', Φ')

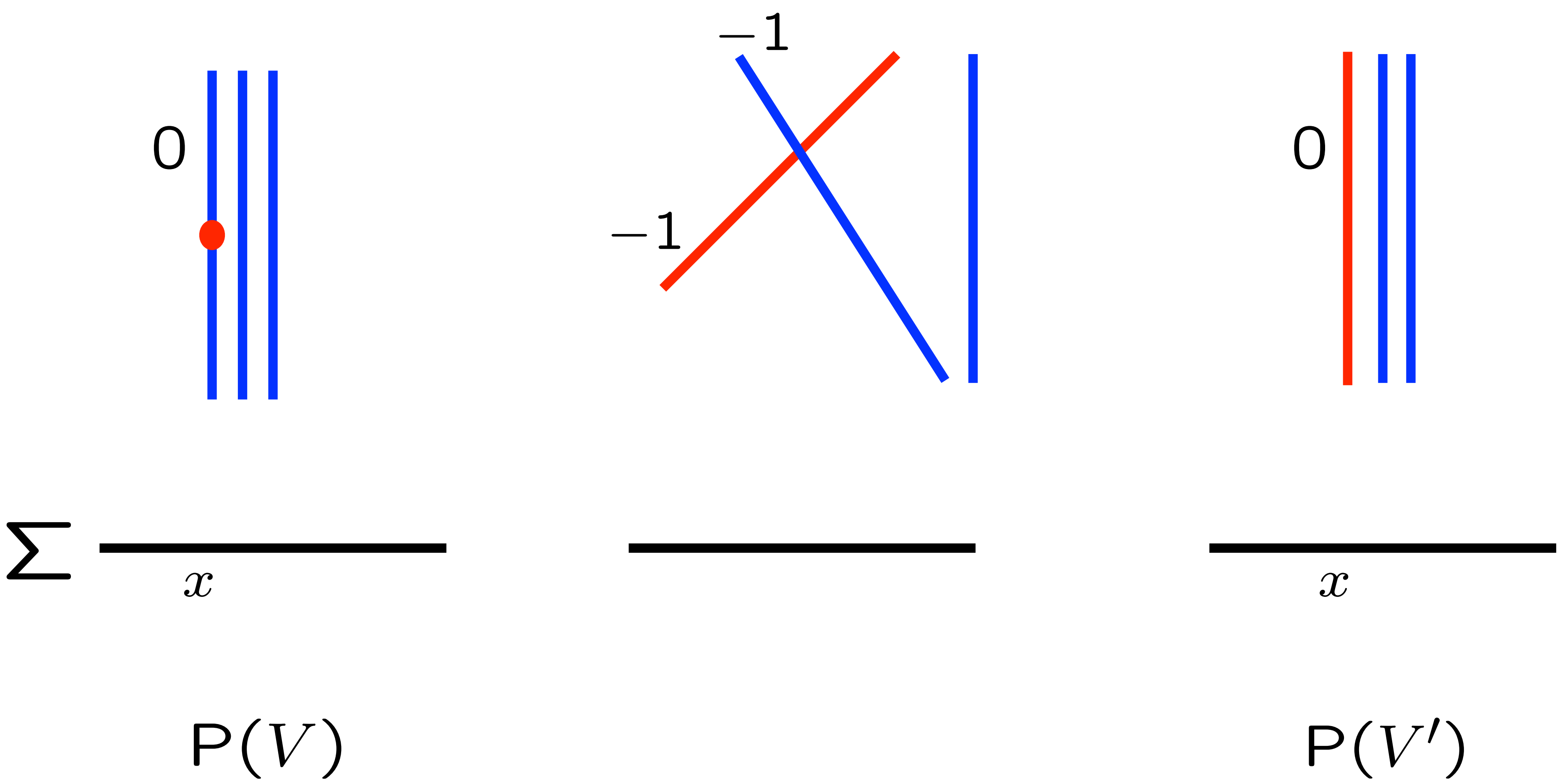
- projective bundle $\pi : \mathsf{P}(V) \to \Sigma$

- algebraic surface

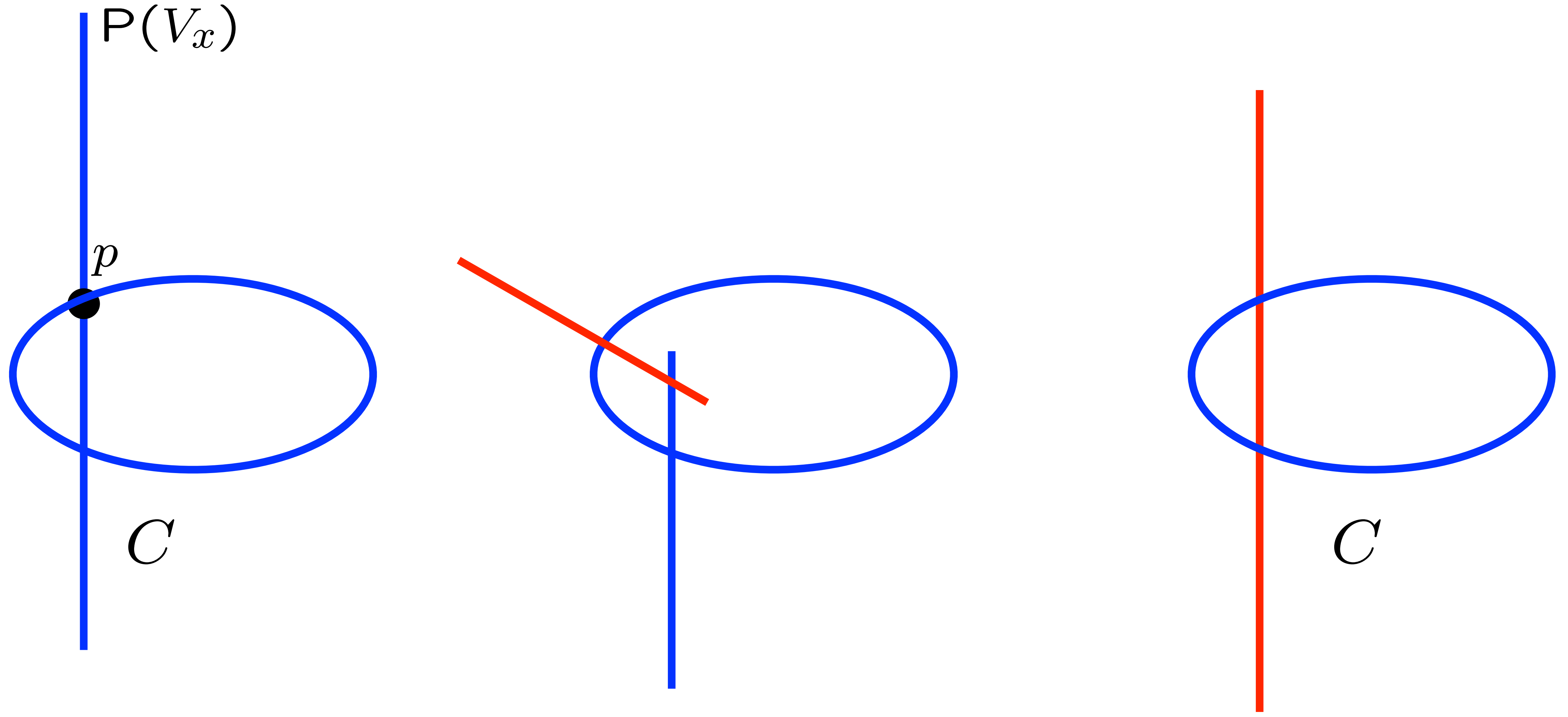
- blow up a point $\sim$ replace by P^1 of tangent directions

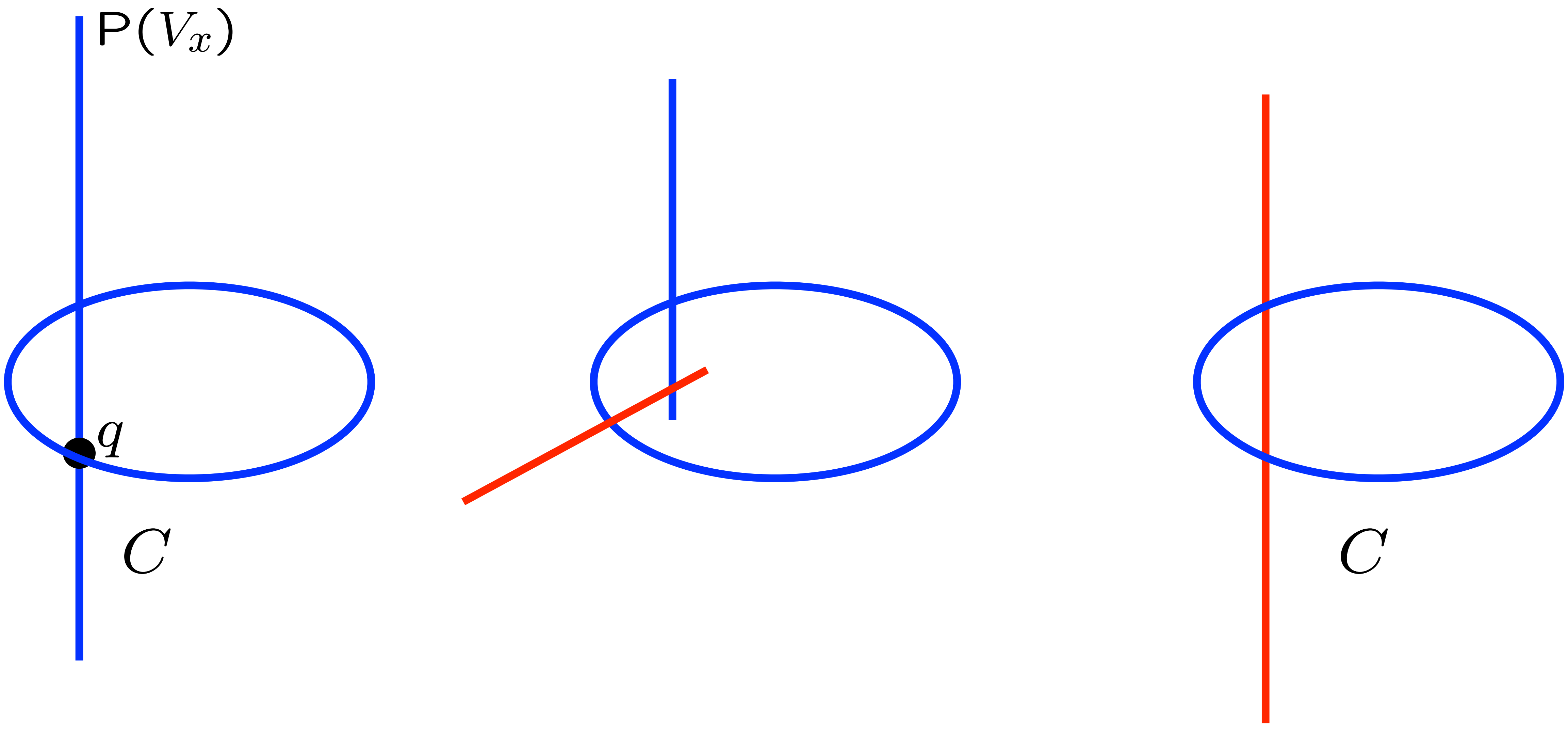


- Hecke transform = blowing up and down



- projective bundle $\pi : \mathsf{P}(V) \to \Sigma$
- Higgs field $\Phi \in H^0(\Sigma, S^2V^* \otimes K)$
- $\sim$ section s of line bundle $H^2\pi^*K$ on $\mathsf{P}(V)$
- divisor curve C of eigenspaces
- if smooth $C \cong S =$ curve of eigenvalues


$\mathsf{P}(V_x)$
p
C
C



$\mathsf{P}(V_x)$
q
C
C

- same curve $S \cong C$, $\pi : S \to \Sigma$

- (V, Φ) defined by $\pi_* L$

- Two Hecke transforms

  $$(V', \Phi') = \pi_* L(-p), \qquad (V'', \Phi'') = \pi_* L(-q)$$

- $\sim$ correspondence in $\mathcal{M} \times \mathcal{M}'$

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Lagrangian correspondence

$\Rightarrow$ Hecke transform maps a Lagrangian $L \subset \mathcal{M}$ to $L' \subset \mathcal{M}'$

EXAMPLE

- $\pi : S \to \Sigma$

  $$\pi_* \mathcal{O} = V = \mathcal{O} \oplus K^{-1} \quad \Phi = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix} \qquad a \in H^0(\Sigma, K^2)$$

- Lagrangian section of $p : \mathcal{M} \to \mathcal{B}$

  $\sim$ real form $SL(2, \mathbf{R})$

EXAMPLE

- $\pi : S \to \Sigma$

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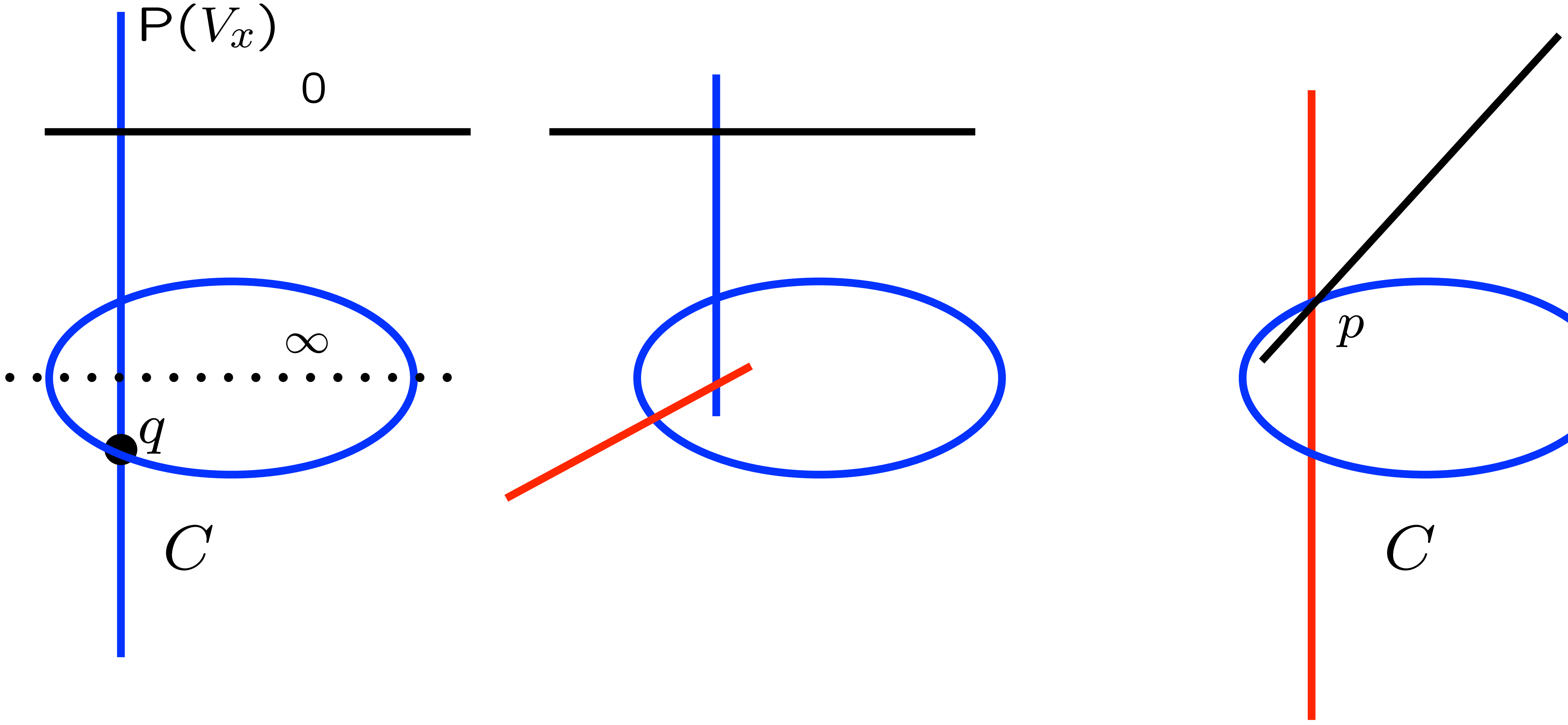
- Lagrangian section of $p : \mathcal{M} \to \mathcal{B}$

  $\sim$ real form $SL(2, \mathbf{R})$

- hyperholomorphic bundle = trivial line bundle

- $\mathsf{P}(\mathcal{O} \oplus K^{-1})$: ∞ section and zero section



$$\pi(p) = \pi(q) = x \in \Sigma$$

- section = distinguished subbundle

  $0 \to \mathcal{O}(-x) \to V \to K^{-1} \to 0$

  = eigenspace of Φ at $p \in S$

- $\mathcal{O}(-x) \stackrel{\Phi}{\to} V \otimes K \to K^{-1} \otimes K \cong \mathcal{O}$

  vanishes at x, so $\mathcal{O}(-x)$ is also an eigenspace at q

- $\mathbf{C}^*$-action, take fixed point

  ... upward flow is Lagrangian

B.Collier & R.Wentworth, *Conformal limits and the Bialynicki-Birula stratification of the space of lambda-connections*, arXiv:1808.01622

- fixed point $V = \mathcal{O}(-x) \oplus K^{-1}$ $\quad \boldsymbol{\Phi} = \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix}$

- Hecke-transformed Lagrangian =

- Higgs bundles $\mathcal{O}(-x) \to V \to K^{-1}$ such that

  $\mathcal{O}(-x) \stackrel{\Phi}{\to} V \otimes K \to K^{-1} \otimes K \cong \mathcal{O}$ vanishes at x

- What hyperholomorphic bundle replaces the trivial bundle?

- Lagrangian meets fibre A in 2 points $\sim \pi^{-1}(x)$

- $V_x = \pi_*(L)|_x \cong L_p \oplus L_q$

- hyperholomorphic bundle $=$ universal bundle $\mathbf{V}_x$ at x

- Lagrangian meets fibre A in 2 points $\sim \pi^{-1}(x)$

- $V_x = \pi_*(L)|_x \cong L_p \oplus L_q$

- hyperholomorphic bundle = universal bundle $\mathbf{V}_x$ at x

- in general k Hecke transforms $\Rightarrow \mathbf{V}_{x_1} \otimes \mathbf{V}_{x_2} \otimes \cdots \otimes \mathbf{V}_{x_k}$

$V = \mathcal{O} \oplus K^{-1}$

- conformal limit = opers (2nd order ODEs)

O.Dumitrescu, L.Fredrickson, G.Kydonakis, R.Mazzeo, M.Mulase, A.Neitzke, *Opers versus noabelian Hodge*, arXiv:1607.02172

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$$0 \to \mathcal{O}(-x) \to V \to K^{-1} \to 0$$

- conformal limit =

  2nd order ODEs with apparent singularity at x

B.Collier & R.Wentworth, *Conformal limits and the Bialynicki-Birula stratification of the space of lambda-connections*, arXiv:1808.01622