Higgs bundles and mirror symmetry 2

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ICMAT Minicourse November 12th-16th 2018

DUALITY

- Higgs bundle fibration $p : \mathcal{M} \to \mathcal{B}$

- generic fibre abelian variety A

- = complex torus + positive line bundle H

- SYZ mirror symmetry $\sim$ replace A by its dual $A^\vee$

= moduli space of degree zero holomorphic line bundles on A

- $x \in A$, $L_x(y) = x + y$

- translation action $L_x : A \to A$

- $L_x^* H \otimes H^{-1}$ degree zero line bundle

- $A \mapsto A^\vee$ surjective, finite kernel

- $A = \text{Jac}(S)$ isomorphism (A principally polarized)

- spectral curve $\pi : S \to \Sigma$

- $\deg \pi_* L = \deg V = 0$ if $L = U \otimes \pi^* K^{(n-1)/2}$, $\deg U = 0$

- $\Lambda^n V$ trivial if $\mathrm{Nm}(U) = 0$

- $\mathrm{Jac}(S) \sim$ linear equivalence of divisors

  $\mathrm{Nm}(x_1 + \ldots + x_k) = \pi(x_1) + \ldots + \pi(x_k)$

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- $\ker \mathrm{Nm} \overset{\mathrm{defn}}{=} \mathrm{P}(S, \Sigma) =$ Prym variety

THE GROUP $SU(n)$

- $U \in \mathsf{P}(S, \Sigma) \Rightarrow \Lambda^n V$ trivial

- structure group $SU(n)$

- $\Phi \in H^0(\Sigma, \mathfrak{sl}(n) \otimes K) \Rightarrow a_1 = 0 \in H^0(\Sigma, K)$

- generic fibre for $SU(n)$ Higgs bundles $\cong$ Prym variety

- $\mathsf{Nm} : \mathsf{Jac}(S) \to \mathsf{Jac}(\Sigma)$ is dual to $\pi^* : \mathsf{Jac}(\Sigma) \to \mathsf{Jac}(S)$

- so $\mathsf{P}(S, \Sigma)^\vee \cong \mathsf{Jac}(S)/\pi^* \mathsf{Jac}(\Sigma)$

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- $\mathsf{Jac}(S)/\mathsf{Jac}(\Sigma) \cong \mathsf{P}(S, \Sigma)/(\mathsf{P}(S, \Sigma) \cap \pi^* \mathsf{Jac}(\Sigma))$

- $\mathsf{Nm}\, \pi^* x = nx \Rightarrow$

$$\mathsf{P}(S, \Sigma)^\vee \cong \mathsf{P}(S, \Sigma)/\pi^* H^1(\Sigma, \mathbf{Z}_n)$$

THE GROUP $Sp(m)$

- E rank $2m$ symplectic vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, S^2E \otimes K)$

- eigenvalues $\pm\lambda_i$

- spectral curve $S \subset |K|$:

  $0 = \det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

- involution $\sigma(x) = -x$

- $\pi : S \to \Sigma$
- $E = \pi_* L$
- $x : L \to L \otimes \pi^* K$ and $\Phi = \pi_* x$
- .... where $L = U\pi^* K^{m-1/2}$ and $\sigma^* U \cong U^*$

- $p : S \to \bar{S} = S/\sigma$

- $\sigma^* U \cong U^* \Leftrightarrow U \in \mathsf{P}(S, \bar{S})$

- abelian variety = Prym $\mathsf{P}(S, \bar{S})$

- dual $\mathsf{P}(S, \bar{S})^\vee \cong \mathsf{P}(S, \bar{S})/p^* H^1(\bar{S}, \mathbf{Z}_2)$

LANGLANDS DUALITY

Invent. math. 153, 197–229 (2003)
DOI: 10.1007/s00222-003-0286-7

Inventiones mathematicae

Mirror symmetry, Langlands duality, and the Hitchin system

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Oblatum 12-VII-2002 & 2-XII-2002
Published online: 10 April 2003 –

Abstract. Among the major mathematical approaches to mirror symmetry are those of Batyrev-Borisov and Strominger-Yau-Zaslow (SYZ). The first is explicit and amenable to computation but is not clearly related to the physical motivation; the second is the opposite. Furthermore, it is far from obvious that mirror partners in one sense will also be mirror partners in the other. This paper concerns a class of examples that can be shown to satisfy

COMMUNICATIONS IN
NUMBER THEORY AND PHYSICS
Volume 1, Number 1, 1–236, 2007

Electric-Magnetic Duality And The Geometric Langlands Program

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- $\mathcal{M}(G)$ Higgs bundle moduli space

- hyperkähler

- holomorphic Lagrangian fibration

- ....its mirror is $\mathcal{M}({}^L G)$ where ${}^L G$ is the Langlands dual group

R.Donagi & T.Pantev, *Langlands duality for Hitchin systems,* Invent. math. **189** (2012), 653–735.

- G and ${}^LG$ are Langlands dual groups if..

- ... their root systems are dual

- roots $\leftrightarrow$ coroots, characters $\leftrightarrow$ 1-parameter subgroups

- ${}^LU(n) = U(n)$

  ${}^LSU(n) = PSU(n) = SU(n)/\mathbf{Z}_n$

  ${}^LSp(m) = SO(2m+1)$

- ${}^{L}U(n) = U(n)$: $\mathsf{Jac}(S)^\vee \cong \mathsf{Jac}(S)$

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- ${}^{L}SU(n) = PSU(n) = SU(n)/\mathbf{Z}_n$

- $\mathbf{Z}_n$ action $(V, \Phi) \mapsto (V \otimes L, \Phi)$ where L^n is trivial

$H^1(\Sigma, \mathbf{Z}_n)$ quotient of $SU(n)$-moduli space, $PSU(n)$-Higgs bundles

- $\mathsf{P}(S, \Sigma)^{\vee} \cong \mathsf{P}(S, \Sigma)/\pi^* H^1(\Sigma, \mathbf{Z}_n)$

$Sp(2m, \mathrm{C})$ AND $SO(2m+1, \mathrm{C})$

$Sp(2m, \mathbf{C})$

- E rank $2m$ symplectic vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, S^2 E \otimes K)$

- eigenvalues $\pm\lambda_i$

- spectral curve $S \subset |K|$:

  $0 = \det(x - \Phi) = x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m}$

- involution $\sigma(x) = -x$

$SO(2m+1,\mathbf{C})$

- V rank $(2m+1)$ orthogonal vector bundle

- $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = H^0(\Sigma, \Lambda^2 V \otimes K)$

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- eigenvalues $0 \pm \lambda_i$

- kernel $\sim \Phi^m \in \Lambda^{2m} V \otimes K^m \cong V \otimes K^m$

- reducible spectral curve

  S

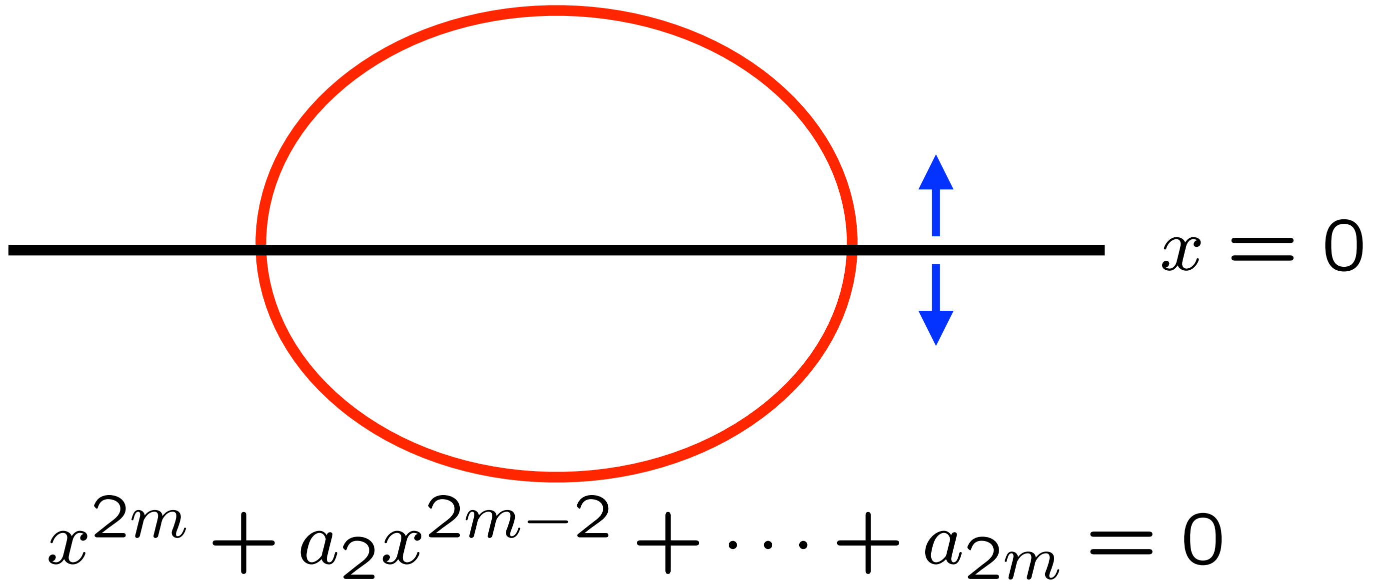
  $0 = \det(x - \Phi) = x(x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m})$

- $V = \pi_* L$ where ...

- on $x^{2m} + a_2 x^{2m-2} + \cdots + a_{2m} = 0$

  $L = U\pi^* K^m$ and $U \in P(S, \bar{S})$

- on $x = 0 \cong \Sigma$, $L = K^m$



- $x = 0$ fixed point set of σ

- $\sigma^* U \cong U^* \Rightarrow$ trivialization of U^2 on $x = 0$

- $K^m \cong UK^m \Rightarrow \pm 1$ choice

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- $x = 0 \Leftrightarrow a_{2m}(z) = 0$: $4m(g-1)$ points

- $2^{4m(g-1)}$ covering of $P(S, \bar{S})$

- overall ± 1

  $+$

  $SO(2m+1)$-bundle spin/non-spin

- $2^{4m(g-1)-2}$ covering of $P(S, \bar{S})$

- $= P(S, \bar{S})/p^* H^1(\bar{S}, \mathbf{Z}_2)$

  $=$ dual of $P(S, \bar{S})$

BRANES

- symplectic geometry

  A-brane = Lagrangian submanifold + flat vector bundle

- holomorphic geometry

  B-brane = complex submanifold + holomorphic bundle

- ... or generalizations, sheaves etc.

SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n: ω symplectic form,

  Ω = real part of a holomorphic n-form

- special Lagrangian fibration: $p : M \to B$

  (ω, Ω vanish on fibres)

- fibre M_x is a torus

- mirror = fibration by dual torus =

  moduli space of flat $U(1)$-bundles over M_x

- $W \subset T_x M$ Lagrangian subspace

- suppose $W = V \oplus H \subset T_F M \oplus p^* TB \cong p^*(T^* B \oplus TB)$

- Lagrangian $\Rightarrow V = H^0$ (annihilator)

- then $V^0 \oplus H \cong H \oplus iH$ complex

- hyperkähler: complex structures I, J, K

- symplectic forms $\omega_1, \omega_2, \omega_3$

- BAA-brane = holomorphic Lagrangian submanifold wrt I + flat connection

- BBB-brane = hyperkähler submanifold + hyperholomorphic bundle

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mirror symmetry is supposed to interchange these

BBB-BRANES

- hyperkähler submanifold

- strong condition

- Kähler $\Rightarrow$ second fundamental form S complex linear

  $S(IX, JY) = IS(X, JY) = IJS(X, Y) = JIS(X, Y) = 0$

- $\Rightarrow$ totally geodesic

EXAMPLES

- points, whole manifold
- Higgs bundles for subgroup $H \subset G$
- pull-back of Higgs bundles from a map $f : \Sigma \to C$
- fixed point set of a triholomorphic automorphism of $\mathcal{M}$

  e.g.

  i) induced action of finite group holomorphic action on Σ

  ii) $V \mapsto V \otimes L$ where L^n trivial.

BAA BRANES

- $\mathcal{N} \subset \mathcal{M}$ = moduli space of stable bundles $(V, 0)$
- torus fibres of the integrable system
- $T^*\mathcal{N} \subset \mathcal{M}$ open embedding ....
- .... closure of conormal bundle of submanifold of $\mathcal{N}$

BAA BRANES

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any

SYZ MIRROR SYMMETRY

- $p : \mathcal{M} \to \mathcal{B}$ integrable system

  $L \subset \mathcal{M}$ complex Lagrangian submanifold

- $p : L \to p(L)$ suppose generically $p(L) \subset \mathcal{B}^{\text{reg}}$

  $L \cap A \subset A$ compact subvariety of abelian variety

- **Define** $(L \cap A)^0 \subset A^\vee =$ line bundles on A trivial on $L \cap A$

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- .... $=$ support of a BBB-brane?

EXAMPLE

- $L =$ a single fibre A
- $p(L)$ a point $a \in \mathcal{B}$
- $(L \cap A) = A \Rightarrow (L \cap A)^0 = 0$
- $\Rightarrow$ BBB-brane is a single point

EXAMPLE

- $L =$ (closure of) a cotangent fibre of $T^*\mathcal{N}$
- $(L \cap A)$ finite set $\Rightarrow (L \cap A)^0 = A^\vee$
- $\Rightarrow$ BBB-brane is the whole manifold
- How general is this?

$\mathbb{C}^*$-INVARIANT LAGRANGIANS

- $\mathbf{C}^*$-action $(V, \Phi) \mapsto (V, \lambda\Phi)$

  generated by holomorphic vector field X

- on $T^*\mathcal{N}$ scales cotangent fibres

- in general preserves conormal bundles

- moves generic fibres ($0 \in \mathcal{B}$ only fixed point)

- Suppose L is a $\mathbf{C}^*$-invariant complex Lagrangian

- $i_X(\omega_2 + i\omega_3) = i_X\omega^c$ vanishes on L

- $i_X\omega^c$ nonzero on A

  (variation of Lagrangian A in the direction X)

- $i_X\omega^c$ holomorphic 1-form on A which vanishes on $L \cap A$

THE ALBANESE VARIETY

- M algebraic variety, $H^0(M, T^*)$ = holomorphic 1-forms

- integration over 1-cycle: $H_1(M, \mathbf{Z}) \to H^0(M, T^*)^*$

- quotient = $\mathrm{Alb}(M)$ the **Albanese variety**, an abelian variety

- for a curve $\mathrm{Alb}(C) \cong \mathrm{Jac}(C)$

- Universal property: a map $M \to A$ factors through

  the Albanese variety $M \to \mathrm{Alb}(M) \to A$

$$M \to \mathrm{Alb}(M) \qquad x \mapsto \int_e^x \alpha$$

- $i_X\omega|_A$ holomorphic 1-form on A which vanishes on $L \cap A$

- $\Rightarrow$ vanishes on B = image of $\mathrm{Alb}(L \cap A)$.

- $H_1(B, \mathbf{Z}) \subset H_1(A, \mathbf{Z})$

  over rationals extend a basis

- $i_X\omega^c$ vanishes on $B \Rightarrow$ periods vanish

- $\Rightarrow p(L)$ satisfies $2\dim_{\mathbf{C}} B = 2k$ real constraints

  $x_1 = x_2 = \cdots = x_{2k} = 0$

- k constraints $\Rightarrow \dim p(L) \leq \dim \mathcal{M}/2 - k$

- $L \cap A \subset B \Rightarrow \dim(L \cap A) \leq k$

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- L Lagrangian $\Rightarrow$

  $\dim \mathcal{M}/2 = \dim L = \dim(L \cap A) + \dim p(L) \leq k + (\dim \mathcal{M}/2 - k)$

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- $\Rightarrow \dim(L \cap A) = \dim B$

  $\Rightarrow$ each component of $L \cap A$ is an abelian subvariety

- linear constraints have rational coefficients

  $\Rightarrow$ locally constant

- $p(L)$ is locally defined by linear functions

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- SYZ mirror is fibred over $p(L)$ by abelian subvarieties B^0

- holomorphic symplectic, integrable system ...

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- $p(L)$ is locally defined by linear functions

- SYZ mirror is fibred over $p(L)$ by abelian subvarieties B^0

- holomorphic symplectic, integrable system ...

- but is it a hyperkähler submanifold?

ABELIAN SUBVARIETIES

EXAMPLE

- $B = \pi^* \text{Jac}(\Sigma) \subset \text{Jac}(S)$

- $\theta \in H^0(S, K)$ annihilates B if $a_1 = 0$

  $\det(x - \Phi) = x^n + a_1 x^{n-1} + \ldots + a_n$

- Lagrangian $L = \text{Jac}(\Sigma) \times H^0(\Sigma, K^2) \oplus \cdots \oplus H^0(\Sigma, K^n)$

- $B^0 = \text{P}(S, \Sigma) \Rightarrow$ mirror is $SU(n)$ moduli space

  $=$ hyperkähler submanifold

- $B \subset \mathsf{Jac}(S)$ abelian subvariety

- $S \subset \mathsf{Jac}(S), f : S \to \mathsf{Jac}(S)/B$

- image curve, f factors through normalization C and $\mathsf{Alb}(C)$

- $f : S \to C$

- $i_X\omega^c$vanishes on B
- $\Rightarrow i_X\omega^c$pulled back from A/B
- $\Rightarrow \theta = f^*\varphi$ for a 1-form φ on C

EXAMPLE $SU(2)$

- spectral curve $x^2 + a_2 = 0$, involution $x \mapsto -x$

- π^* Jac(Σ) $\subset B \subset$ Jac(S), involution on Jac(S)$/B$

  $\Rightarrow$ involution on C

  $\Rightarrow$ C is a spectral curve for a curve $\bar{\Sigma}$, $h : \Sigma \to \bar{\Sigma}$

- pull-back gives a hyperkähler submanifold.