

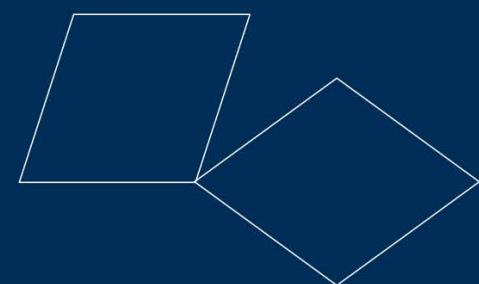


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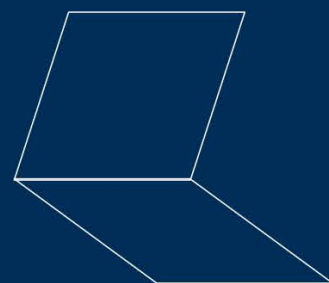
Higgs bundles and mirror symmetry 1

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Oxford
Mathematics



HODGE THEORY

- Σ compact Riemann surface $g > 1$
- G Lie group
- $H^1(\Sigma, G) =$ equivalence classes of flat G -bundles
- ... apply Hodge theory

- trivial bundle
- connection $\nabla_A = d + A$, $A \in \Omega^1(\Sigma, \mathfrak{g})$
- curvature $= dA + \frac{1}{2}[A, A] = 0$
- gauge equivalence: $g : \Sigma \rightarrow G$,

$$\nabla_A \mapsto g^{-1} \nabla_A g$$

THE ABELIAN CASE $G = \mathbb{C}^*$

- $H^1(\Sigma, \mathbb{C}^*) = H^1(\Sigma, \mathbb{C}) / H^1(\Sigma, \mathbb{Z}) \cong (\mathbb{C}^*)^{2g}$ (holonomy)
- Hodge theory: $H^1(\Sigma, \mathbb{C}) = H^{10} \oplus H^{01}$
- flat connection $d + \alpha^{10} + \alpha^{01}$ $\alpha^{10} = adz$ holomorphic

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- $\beta, \gamma \in H^{10}$, flat $U(1)$ -connection $\nabla_A = d + \beta - \bar{\beta}$
- $\nabla_A + \gamma + \bar{\gamma} = d + \alpha^{10} + \alpha^{01}$
if $\beta + \gamma = \alpha^{10}, -\beta + \gamma = \overline{\alpha^{01}}$

- flat $U(1)$ connection $\nabla_A = d + \beta - \bar{\beta}$
- $\bar{\partial}_A = \bar{\partial} - \bar{\beta}$ holomorphic line bundle
- flat $\mathbb{C}^*$ -connection $\cong$
holomorphic line bundle $+$ holomorphic 1-form
- $H^1(\Sigma, \mathbb{C}^*) \cong \text{Jac}(\Sigma) \times H^0(\Sigma, K)$

- $(\gamma, \nabla_A) \in H^0(\Sigma, K) \times \text{Jac}(\Sigma) \cong H^1(\Sigma, \mathbb{C}^*)$ *not holomorphic*
- $\int_{\Sigma} \beta \wedge \bar{\beta} + \gamma \wedge \bar{\gamma}$ flat metric
- $\int_{\Sigma} \bar{\beta} \wedge \gamma$ holomorphic symplectic structure

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- hyperkähler metric, complex structures I, J, K

$$I^2 = J^2 = K^2 = IJK = -1$$

- $I: H^0(\Sigma, K) \times \text{Jac}(\Sigma) \cong T^*\text{Jac}(\Sigma)$

$$J: H^1(\Sigma, \mathbb{C}^*) \cong (\mathbb{C}^*)^{2g}$$

THE ABELIAN CASE $G = \mathbf{R}^*$

- $0 \rightarrow H^1(\Sigma, \mathbf{R}) \rightarrow H^1(\Sigma, \mathbf{R}^*) \rightarrow \mathbf{Z}_2^{2g} \rightarrow 0$
- $\alpha^{10} + \overline{\alpha^{10}} \in H^1(\Sigma, \mathbf{R})$
- $p : H^0(\Sigma, K) \times \text{Jac}(\Sigma) \rightarrow H^0(\Sigma, K)$
- $H^1(\Sigma, \mathbf{R}^*) = 2^{2g}$ *holomorphic* sections of p
 $H^1(\Sigma, \mathbf{R}^*) =$ *real* points of $(\mathbf{C}^*)^{2g}$

HIGGS BUNDLES

- $H^1(\Sigma, GL(n, \mathbb{C})) \sim \text{Hom}(\pi_1(\Sigma), GL(n, \mathbb{C}))/GL(n, \mathbb{C})$
- flat rank n vector bundle V
- irreducible reps $\Rightarrow$ smooth complex manifold

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- Hodge theory $\Rightarrow$ represent as $\nabla_A + \Phi + \Phi^*$

∇_A unitary connection, $\Phi \in H^0(\Sigma, \text{End } V \otimes K)$

(Corlette, Donaldson, Simpson, NJH,...)

- Higgs field: $\Phi \in H^0(\Sigma, \text{End } V \otimes K)$
- $\nabla_A + \Phi + \Phi^*$ flat $\Rightarrow F_A + [\Phi, \Phi^*] = 0$

Higgs bundle equations

- stability $\Rightarrow$ existence

- moduli space $\mathcal{M}$ hyperkähler
- Kähler forms $\omega_1, \omega_2, \omega_3$ for complex structures I, J, K
- holomorphic symplectic form $\omega^c = \omega_2 + i\omega_3$ relative to I etc.
- $\Rightarrow$ covariant constant holomorphic volume form

Calabi-Yau

MIRROR SYMMETRY

- symplectic geometry: A-model
- complex geometry: B-model
- + Calabi-Yau

Mirror symmetry is T -duality

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SYZ MIRROR SYMMETRY

- Calabi-Yau manifold M^n : ω symplectic form,
 Ω = real part of a holomorphic n -form
- special Lagrangian fibration: $p : M \rightarrow B$
(ω, Ω vanish on fibres)
- fibre M_x is a torus
- mirror = fibration by dual torus =
moduli space of flat $U(1)$ -bundles over M_x

- (M, ω) real symplectic
- $p : M \rightarrow B$ Lagrangian (proper $\Rightarrow$ fibres are tori)
- $V \in T_x B \Rightarrow$ section U of normal bundle of M_x
 $\Rightarrow i_U \omega$ closed 1-form on M_x
 $\Rightarrow T_x B \cong H^1(M_x, \mathbf{R})$
- $x, y \in$ contractible neighbourhood $H^1(M_x, \mathbf{R}) \cong H^1(M_y, \mathbf{R})$
 $\Rightarrow$ flat connection on TB

- if $\omega = d\alpha$ then $d\alpha = 0$ on M_x
- integrate α on $H_1(M_x, \mathbf{Z})$: flat coordinates x_i
- $i_U\Omega$ well-defined and closed
- $[i_U\Omega] \in H^{n-1}(M_x, \mathbf{R}) \cong H^1(M_x, \mathbf{R})^*$
- $T_F^* \cong p^*TB \cong T_F \Rightarrow$ metric on B

Hessian form
$$g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}$$

- flat connection on $H_1(M_x, \mathbf{R}) \Rightarrow$
horizontal subspace for $p : M \rightarrow B$

- $TM \cong T_F \oplus p^*TB$

- $TM \cong T_F \oplus T_F^*$ symplectic

$$TM \cong T_F \oplus T_F = T_F \otimes \mathbf{C} \text{ complex}$$

- metric $\Rightarrow$ Kähler metric on $\hat{M}$

$$\omega = \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i \wedge dy_j$$

- semi-flat metric –

Lagrangian fibres are orbits of Killing vector fields

ISSUES

- Compact Calabi-Yau threefold with a SL fibration?
- (dimension 2 – elliptic K3 – hyperkähler)
- semi-flat metric $\sim$ large complex structure limit?
- semi-flat metric not Calabi-Yau in general

HIGGS BUNDLES

- moduli space hyperkähler $\Rightarrow$ Calabi-Yau
- there is a natural special Lagrangian fibration
- the semi-flat metric is hyperkähler
- noncompactness replaces large complex structure limit

THE INTEGRABLE SYSTEM

- algebraic curve Σ
- holomorphic G^c -principal bundle P
- section $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K) = \text{Higgs field}$
- $+$ stability condition

- $\Rightarrow$ reduction to maximal compact G
- $\Rightarrow G$ -connection A
- $F_A + [\Phi, \Phi^*] = 0$
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- holomorphic symplectic manifold

THE FIBRATION

- hyperkähler moduli space $\mathcal{M}(G^c)$ (G^c semisimple)

$$\dim_{\mathbb{C}} = 2(g - 1) \dim G^c$$

- principal G^c -bundle, $\Phi \in H^0(\Sigma, \mathfrak{g} \otimes K)$

- invariant polynomials $p_1, \dots, p_\ell$ on $\mathfrak{g}$

$$p_m(\Phi) \in H^0(\Sigma, K^{d_m})$$

- fibration $\mathcal{M}^{2k} \rightarrow \mathbb{C}^k$

- these k functions Poisson-commute
- integrable system
- generic fibre abelian variety A
- $G^c = GL(n, \mathbf{C})$ $\det(x - \Phi) = 0$ spectral curve S
- fibre = $\text{Jac}(S)$

- Hyperkähler manifold + holomorphic Lagrangian fibration wrt I
- $\Rightarrow (\omega_2 + i\omega_3)$ vanishes on fibres
- ω_2 vanishes and real or imaginary part of $(\omega_3 + i\omega_1)^k$ vanishes
- $\Rightarrow$ special Lagrangian fibration for complex structure J

$$\omega = \omega_2, \quad \Omega = (\omega_3 + i\omega_1)^k$$

$$G = GL(n, \mathbb{C})$$

- $\det(x - \Phi) = x^n + a_1 x^{n-1} + \dots + a_n, \ a_k \in H^0(\Sigma, K^k)$
- $(a_1, \dots, a_n) \in H^0(\Sigma, K) \oplus \dots \oplus H^0(\Sigma, K^n) = \mathcal{B}$ base

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- spectral curve S : $\det(x - \Phi) = 0$
 $\pi : S \rightarrow \Sigma$ degree n
- x eigenvalue of $\Phi \in H^0(\Sigma, \text{End } V \otimes K), \quad x \in H^0(S, \pi^* K)$
- x embeds S in the cotangent bundle K of Σ

- line bundle L on S
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- $\pi : S \rightarrow \Sigma$ degree $n \Rightarrow H^0(\pi^{-1}(U), L) = H^0(U, V)$
 V rank n vector bundle

- $x \in H^0(S, \pi^* K)$
- $x : H^0(\pi^{-1}(U), L) \rightarrow H^0(\pi^{-1}(U), L \otimes \pi^* K))$
- $= \Phi : H^0(U, V) \rightarrow H^0(U, V \otimes K)$ i.e. $\Phi \in H^0(\Sigma, \text{End } V \otimes K)$

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- fibre for a smooth curve $\sim$
line bundles L on S : Jacobian $\text{Jac}(S)$

- Higgs bundle moduli space $p : \mathcal{M} \rightarrow B$
- *generic* fibre smooth abelian variety
- $B^{\text{reg}} \sim$ smooth spectral curve

THE SEMI-FLAT METRIC

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- $p : M \rightarrow B$ holomorphic Lagrangian fibration
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- $p : M \rightarrow B$ holomorphic Lagrangian fibration
- ... base B has a “special Kähler structure”
- flat affine connection ∇ such that ...
 - (i) Kähler form ω is covariant constant
 - (ii) complex structure $I \in T \otimes T^*$ is locally ∇X

D.Freed, *Special Kähler manifolds*, Commun.Math.Phys. **203** (1999) 31–52.

- if $\omega = d\alpha$ then $d\alpha = 0$ on M_x
- integrate α on $H_1(M_x, \mathbf{Z})$: flat coordinates
- complex algebraic Lagrangian fibration $p : M \rightarrow B$
Kähler class $[H] \in H^2(M, \mathbf{Z}) \Rightarrow$ flat symplectic form

$$\omega([\beta_1], [\beta_2]) = \int_{M_x} \beta_1 \wedge \beta_2 \wedge H^{m-1}$$

$$[\beta_1], [\beta_2] \in H^1(M_x, \mathbf{R})$$

- semi-flat metric is hyperkähler

$$\omega_2 = \sum \frac{\partial^2 \phi}{\partial x_j \partial x_k} dx_j \wedge dy_k$$

$$\omega_1 + i\omega_3 = \frac{1}{2} \sum \omega_{jk} d(x_j + iy_j) \wedge d(x_k + iy_k)$$

- defined over $B^{\text{reg}} =$ open set of base giving smooth S

- $\mathcal{M}$ non-compact
- instead of large complex structure limit
- ... consider $\|\Phi\|^2 \rightarrow \infty$
- hyperkähler metric is approximated by the semi-flat metric

(R.Mazzeo, J.Swoboda, H.Weiss, F.Witt, *Asymptotic geometry of the Hitchin metric*, arXiv:1709.03433)

THE CIRCLE ACTION

- Higgs bundle (A, Φ)

$$\bar{\partial}_A \Phi = 0, F_A + [\Phi, \Phi^*] = 0$$

- equations preserved under $(A, \Phi) \mapsto (A, e^{i\theta} \Phi)$

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$$\bar{\partial}_A \Phi = 0, F_A + [\Phi, \Phi^*] = 0$$

- equations preserved under $(A, \Phi) \mapsto (A, e^{i\theta} \Phi)$

- isometry of hyperkähler metric

$$\omega_1 \mapsto \omega_1, \omega_2 + i\omega_3 \mapsto e^{i\theta}(\omega_2 + i\omega_3)$$

- I -holomorphic $\mathbf{C}^*$ -action $(V, \Phi) \mapsto (V, \lambda \Phi)$

- Killing vector field X infinitesimal action
- $\mathcal{L}_X \omega_1 = 0, \mathcal{L}_X \omega_2 = \omega_3, \mathcal{L}_X \omega_3 = -\omega_2$
- $d(i_X \omega_2) = \omega_3, d(i_X \omega_3) = -\omega_2$
- periods = flat coordinates on B
- $\omega^c = \omega_2 + i\omega_3$
 $i_X \omega^c$ holomorphic 1-form

- fibre $\cong \text{Jac}(S)$

Abel-Jacobi map $S \rightarrow \text{Jac}(S)$

- $H^{10}(S) \cong H^{10}(\text{Jac}(S)), H_1(S, \mathbf{Z}) \cong H_1(\text{Jac}(S), \mathbf{Z})$

- $S \subset$ cotangent bundle of Σ

- periods of $i_X \omega^c \sim$ periods of θ

$\theta =$ canonical 1-form on cotangent bundle

- potential for special Kähler metric on space of curves:

$$K = -\frac{i}{4} \int_S \theta \wedge \bar{\theta}$$

- this defines the semi-flat metric on $\mathcal{M}$

(D.Baraglia & Z.Huang, *Special Kähler geometry of the Hitchin system and topological recursion*, arXiv:1707.04975v2)

- **Prop:** The moduli space of compact complex Lagrangian submanifolds of a holomorphic symplectic Kähler manifold has a natural special Kähler structure

(NJH, *The moduli space of complex Lagrangian submanifolds*, Asian Journal of Mathematics, **3** (1999) 77– 92.)

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- $S \subset T^*\Sigma$ Lagrangian
- in general $\alpha, \beta \in H^{1,0}(L)$, Kähler form ω

$$\int_L \alpha \wedge \bar{\beta} \wedge \omega^{n-1}$$

defines the flat Kähler form on moduli space

$\dim L = 1$ ω not needed