Supercurves, supersymmetric curves and their moduli spaces

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Outline

1. Introduction and motivation
   - Geometry and Physics
   - Superschemes and morphisms
2. Geometry of SUSY curves
   - SUSY curves
   - SUSY curves over ordinary schemes
3. Moduli of SUSY curves
   - Local structure of the supermoduli
   - Global construction of the supermoduli
   - Punctures

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- Various approaches (Kostant-Leites, De Witt, Rogers).
  - After Kostant and Manin, the Kostant model prevailed. Moreover, the definition can be also adapted for holomorphic and algebraic varieties (or schemes).
- Differentiable supermanifods have locally graded coordinates $(z_1, \dots, z_m, \theta_1, \dots, \theta_n)$, $|z_i| = 0$ (even), $|\theta_j| = 1$ (odd). The algebra of superfunctions is the $\mathbb{Z}_2$-graded algebra

$$\bigwedge\nolimits_{\mathcal{C}}\langle\theta_1, \dots, \theta_n\rangle$$

where $\mathcal{C} = \mathcal{C}^\infty(z_1, \dots, z_m)$.

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- For the bosonic string, these are computed by integrating the Polyakov measure on a compactification of the moduli spaces of algebraic curves (or Riemann surfaces).
- The compactification introduces poles in the measure, fermions were introduce to compensate them.
- Since then, the moduli of SUSY curves has attracted a lot of attention. We will come again to this point.

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We give definitions in the algebraic case. For simplicity, all algebraic varieties are noetherian and locally of finite type over $\mathbb{C}$.

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A superscheme $\mathcal{X}$ is locally split if $\mathcal{E} = \mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}_X$-module and $\bigwedge \mathcal{E} \simeq G_{\mathcal{J}}\mathcal{O}_{\mathcal{X}}$.

Morphisms

Superschemes are graded-commutative locally ringed spaces.
A morphism $f\colon \mathcal{X}=(X,\mathcal{O}_\mathcal{X})\to\mathcal{Z}=(Z,\mathcal{O}_\mathcal{Z})$ is a morphism of graded-commutative locally ringed spaces: It is given by a continuous map $f\colon X\to Z$ and a (homogeneous of degree 0) local morphism of graded-commutative algebras $f_\sharp\colon \mathcal{O}_\mathcal{Z}\to\mathcal{O}_\mathcal{X}$.
The projection $\mathcal{O}_\mathcal{X}\to\mathcal{O}_X=\mathcal{O}_\mathcal{X}/\mathcal{J}\to 0$ induces a closed immersion of superschemes.

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- Differentiable supermanifolds are always split (non canonically)

First properties

Familiar notions of fibre products, diagonal of a morphism, cotangent (graded differentials) and tangent (graded derivations) sheaves, proper, flat, faithfully flat, separated, smooth, étale morphisms, and many others are readily extended for superschemes.

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If $\mathcal{X}$ is locally split, one has $0 \to \mathcal{E} = \mathcal{J}/\mathcal{J}^2 \to i^*\Omega_{\mathcal{X}} \to \Omega_X \to 0$. This gives

$$(i^*\Omega_{\mathcal{X}})_0 \xrightarrow{\sim} \Omega_X\,, \quad \mathcal{E} \xrightarrow{\sim} (i^*\Omega_{\mathcal{X}})_1\,.$$

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Definition

A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of superschemes is smooth of relative dimension (m, n) if it is flat and for every (closed) point $y \in \mathcal{Y}$ the fibre $\mathcal{X}_y$ of f over y is a smooth scheme of dimension (m, n).

Supercurves

Definition

A supercurve is a (proper and smooth) superscheme of dim $(1,1)$.

Now $\mathcal{L} = \mathcal{J}/\mathcal{J}^2 = \mathcal{J}$ is a line bundle, and one has $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$, that is, every supercurve is split.

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When $\mathcal{S} = S$ is an ordinary scheme, we still have

$$\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$$

for a line bundle $\mathcal{L}$ on $\mathcal{X}$ as above.

If $\mathcal{S}$ has fermionic part, this may fail to be true.

Definition of SUSY curve

Definition

A SUSY curve over a superscheme $\mathcal{S}$ of genus g is a relative supercurve $\pi\colon \mathcal{X} \to \mathcal{S}$ of genus g endowed with a superconformal structure, that is, a locally free subsheaf of rank $(0,1)$ of the relative tangent sheaf, $\mathcal{D} \hookrightarrow \Theta_{\mathcal{X}/\mathcal{S}}$, such that the composition

$$\mathcal{D} \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D} \xrightarrow{[\ ,\ ]} \Theta_{\mathcal{X}/\mathcal{S}} \to \Theta_{\mathcal{X}/\mathcal{S}}/\mathcal{D}$$

is an isomorphism of $\mathcal{O}_{\mathcal{X}}$-modules.

That is, $\mathcal{D}$ is totally non-integrable.

SUSY curves and spin curves

Locally, there exist superconformal relative graded coordinates (z, θ) such that

$$\mathcal{D} = \langle D \rangle \,, \qquad D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z} \,; \quad D \otimes D \mapsto \overline{\frac{\partial}{\partial z}}$$

When $\mathcal{S} = S$ is a scheme, so that $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$, one has

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$\implies$ For an ordinary scheme S:

$$\left\{\begin{matrix}\text{SUSY curves}\\ (\mathcal{X}\to S, \mathcal{D})\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Relative Spin curves}\\ (X\to S, \mathcal{L})\end{matrix}\right\}$$

Morphisms of SUSY curves

$(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$, $(\pi'\colon \mathcal{X}' \to \mathcal{S}, \mathcal{D}')$ SUSY curves over $\mathcal{S}$.
A morphism of SUSY curves over $\mathcal{S}$ is a morphism $\phi\colon \mathcal{X} \to \mathcal{X}'$ of $\mathcal{S}$ superschemes that preserves the superconformal structure, i.e. such that $\phi_*\mathcal{D} \subseteq \mathcal{D}'$.

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$(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$, $(\pi'\colon \mathcal{X}' \to \mathcal{S}, \mathcal{D}')$ SUSY curves over $\mathcal{S}$.
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Then, a SUSY curve has always a non trivial automorphism.

Infinitesimal deformations

SUSY curve $\pi\colon \mathcal{X} \to \mathcal{S}$.
We define a sheaf $\mathcal{G}$ on X by

$$\mathcal{G}(U) = \{D' \in \mathcal{D}er(\mathcal{O}_\mathcal{X}) \,|\, [D', D] \in \mathcal{D}(U), \text{for every } D \in \mathcal{D}(U)\}$$

$$\mathcal{G}_\pi = \mathcal{G} \cap \Theta_{\mathcal{X}/\mathcal{S}}$$

Proposition (LeBrun-Rothstein)

$\mathcal{G}_\pi \simeq \mathcal{D} \otimes \mathcal{D}$ as sheaves of $\mathcal{O}_\mathcal{S}$-modules.

Then,

$$\mathcal{G}_\pi \simeq \kappa_\pi^{-1} \oplus \kappa_\pi^{-1/2}, \text{(even and odd parts)}$$

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It follows that if $g \geq 2$, one has:

- $\pi_* \mathcal{G}_\pi = 0$.
- $R^1\pi_*\mathcal{G}_\pi$ is a locally free sheaf of rank $(3g-3, 2g-2)$.

Infinitesimal deformations, II

Consider $\mathbb{C}[\epsilon_0, \epsilon_1]$ ($|\epsilon_i| = i$, $\epsilon_0^2 = \epsilon_1^2 = \epsilon_0\epsilon_1 = 0$) and $\mathcal{S}[\epsilon_0, \epsilon_1] = \mathcal{S} \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[\epsilon_0, \epsilon_1]$

Definition

An infinitesimal deformation of a SUSY curve $\pi \colon \mathcal{X} \to \mathcal{S}$ is a relative SUSY curve $\tilde{\pi} \colon \tilde{\mathcal{X}} \to \mathcal{S}[\epsilon_0, \epsilon_1]$ such that

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "\pi"] & \tilde{\mathcal{X}} \arrow[d, "\tilde{\pi}"] \\ \mathcal{S} \arrow[r, hook] & \mathcal{S}[\epsilon_0, \epsilon_1] \end{tikzcd}$$

is cartesian.

Sheaf of infinitesimal deformations on $\mathcal{S}$:

$$U \mapsto \mathcal{I}nf_{def}(\mathcal{X}/\mathcal{S})(U) = Inf_{def}(\mathcal{X}_{|U}/\mathcal{S}_{|U})$$

Kodaira-Spencer map

Then

$$\mathcal{I}nf_{def}(\mathcal{X}/\mathcal{S}) \simeq R^1\pi_*\mathcal{G}_\pi \, .$$

From

$$0 \to \mathcal{G}_\pi \to \mathcal{G} \xrightarrow{d\pi} \pi^*\Theta_{\mathcal{S}} \to 0 \, .$$

one gets

Definition

The (relative) Kodaira-Spencer map of $\pi\colon \mathcal{X} \to \mathcal{S}$ *is the composition*

$$ks_{rel}(\pi)\colon \Theta_{\mathcal{S}} \to R^1\pi_*(\mathcal{G}_\pi)$$

of the natural morphism $\Theta_{\mathcal{S}} \to \pi_*(\pi^*\Theta_{\mathcal{S}}) \simeq \Theta_{\mathcal{S}} \otimes \pi_*(\mathcal{O}_{\mathcal{X}})$ *with the connecting morphism* $\pi_*(\pi^*\Theta_{\mathcal{S}}) \to R^1\pi_*(\mathcal{G}_\pi)$.

Moduli functor of SUSY curves on superschemes

From now on all superschemes are locally split.
Presheaf of relative supercurves of genus g of superschemes:

$$\mathcal{Y} \rightsquigarrow SC_g(\mathcal{Y}) = \left\{ \begin{matrix} \text{Isomorphism classes of relative SUSY curves} \\ \pi\colon \mathcal{X} \to \mathcal{Y} \text{ of genus } g \end{matrix} \right\}$$

Definition

The sheaf of relative supercurves of genus g is the associated sheaf $\mathcal{SC}_g$ to SC_g for the étale topology of superschemes.

Moduli problem: To find superscheme $\mathcal{SM}_g = (SM_g, \mathcal{O}_{\mathcal{SM}_g})$ representing $\mathcal{SC}_g$: For every superscheme $\mathcal{Y}$,

$$\mathcal{SM}_g^\bullet(\mathcal{Y}) := \mathrm{Hom}_{ss}(\mathcal{Y}, \mathcal{SM}_g) \xrightarrow{\sim} \mathcal{SC}_g(\mathcal{Y})$$

where Hom_{ss} means superscheme morphisms.

Moduli functor of SUSY curves on schemes

If the supermoduli $\mathcal{SM}_g$ exists, its underlying ordinary scheme SM_g verifies

$$\mathrm{Hom}_{sch}(Y, SM_g) = \mathrm{Hom}_{ss}(Y, \mathcal{SM}_g) \simeq \mathcal{SC}_g(Y)$$

where Hom_{sch} means scheme morphisms.

$\implies$ SM_g is the *moduli space for the functor* $\mathcal{SC}_{g|Sch}$.

Let's see that $\mathcal{SC}_{g|Sch}$ is representable on schemes.

- From now, we assume that curves have genus $g \geq 2$ and a n-level structure ($n \geq 3$) so that they have no automorphisms but the identity.

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- From now, we assume that curves have genus $g \geq 2$ and a n-level structure ($n \geq 3$) so that they have no automorphisms but the identity.
- Then there exist a fine moduli space M_g and a universal relative genus g curve $\pi_g \colon X_g \to M_g$ representing the moduli functor $\mathcal{M}_g$ of curves of genus g on schemes.

Moduli functor of SUSY curves on schemes, II

For every degree d there exist the relative Jacobian (or Picard scheme) $\rho_d \colon J^d = J^d(X_g/M_g) \to M_g$ and a universal 'degree d l.b. class' $\Upsilon_d \in \operatorname{Pic}(J^d \times_{M_g} X_g)/J^d)$.

- $\mu_2 \colon J^{g-1} \to J^{2g-2}$ = 'raising to two'.

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One has

$$\begin{array}{ccc} X_g := X_g \times_{M_g} SM_g & \to & X_g \\ \downarrow{\scriptstyle \pi_g} & & \downarrow{\scriptstyle \pi_g} \\ SM_g & \xrightarrow{\rho} & M_g \end{array}$$

Moduli scheme of Spin curves on schemes

Moreover, if $\Upsilon_s = \Upsilon_{g-1|SM_g} \in \mathrm{Pic}(X_g/SM_g)$,

$$\Upsilon_s^2 = [\kappa_s] \quad \left(\kappa_s = \kappa_{X_g/SM_g}\right).$$

Proposition

SM_g, endowed with the section Υ_s of $SC_g(SM_g)$, represents the functor SC_g of spin curves (or SUSY curves) on schemes.

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- Then $(\pi_g \colon X_g \to SM_g, \Upsilon_s)$ is not a true 'supercurve' but a 'supercurve class'.

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However, there exists an *affine trivializing étale covering* $p\colon U \to SM_g$, such that, if $\pi_U\colon X_U = p^* X_g \to U$ and $\Upsilon = p^*\Upsilon_s$, there exist a line bundle $\mathcal{L}$ on X_U of relative degree $g-1$ with

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Since $U \to SM_g \to M_g$ is étale, the even Kodaira-Spencer map of $\pi_U\colon \mathcal{X}_U \to U$ is an isomorphism.

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- They are infinitesimal deformations of the 'universal relative supercurve class' $(\mathcal{X}_g \to SM_g, \Upsilon_s)$.

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Then, the candidate to 'local supermoduli' is

$$\mathcal{U} = (U, \bigwedge \pi_{U*}(\kappa_U^{3/2}))\,.$$

One has $\dim \mathcal{U} = (3g-3, 2g-2)$.

Two results of LeBrun-Rothstein

Proposition

$\pi\colon \mathcal{X} = (X, \mathcal{O} \oplus \mathcal{L}) \to V$, SUSY curve over an affine scheme whose even ks map is an isomorphism. Then there is a SUSY curve $\bar{\pi}\colon \mathfrak{X} \to \mathcal{V}$ over the superscheme $\mathcal{V} = (V, \bigwedge \pi_*(\kappa_\pi^{3/2}))$ extending $\mathcal{X}$ and whose ks map is an isomorphism.

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$\mathfrak{X}\to\mathcal{V}$, *SUSY curve whose ks map is an isomorphism.*
$\mathcal{X}=(X,\mathcal{O}\oplus\mathcal{L})\to V$ *its restriction to ordinary scheme* V. *For every morphism of schemes* $\varphi\colon Y\to V$*:*

$$\left\{\begin{matrix}\text{morphisms of superschemes}\\ \mathcal{Y}\to\mathcal{V}\text{ extending}\\ \varphi\colon Y\to V\end{matrix}\right\}\to\left\{\begin{matrix}\text{classes of SUSY curves}\\ \bar{\mathfrak{X}}/\mathcal{Y},\text{ extending}\\ \varphi^*\mathcal{X}/Y\end{matrix}\right\}$$

$$\phi\mapsto\phi^*\mathfrak{X}/\mathcal{Y}\,.$$

Construction of the supermoduli

- By the first result, the 'locally universal' SUSY curve $\pi_U\colon \mathcal{X}_U = (X_U, \mathcal{O}_{X_U} \oplus \mathcal{L}) \to U$ can be extended to a SUSY curve
$$\pi_U\colon \mathfrak{X}_U \to \mathcal{U} = (U, \bigwedge \pi_{U*}(\kappa_U^{3/2}))$$
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whose ks map is an isomorphism.
- By the second result, there is an isomorphism $\mathcal{U} \xrightarrow{\sim} SC_g \times_{SC_{g|\{\mathrm{Sch}\}}} U$ of étale presheaves on superschemes. Since $\mathcal{U}$ is a sheaf, the second member is a sheaf, so it coincides with its associated sheaf $\mathcal{SC}_g \times_{SM_g} U$, that is
$$\mathcal{U} \xrightarrow{\sim} \mathcal{SC}_g \times_{SM_g} U\,.$$

Construction of the supermoduli, II

The trivial étale equivalence relation $(p_1, p_2)\colon U \times_{SM_g} U \rightrightarrows U$ induces an étale equivalence relation of sheaves on superschemes

$$\begin{array}{ccccc} \mathcal{SC}_g \times_{SM_g} U \times_{SM_g} U & \overset{(p_1,p_2)}{\rightrightarrows} & \mathcal{SC}_g \times_{SM_g} U & \xrightarrow{p} & \mathcal{SC}_g \\ \| & & \| & & \\ \mathcal{U} \times_{SM_g} U & & \mathcal{U} & & \end{array}$$

with categorial quotient $\mathcal{SC}_g$.

Proposition (D-HR-S)

The functor $\mathcal{SC}_g^m$ of relative supercurves of genus g is representable by an Artin algebraic superspace $\mathcal{SM}_g$, which is the categorical quotient of this étale equivalence relation of superschemes. Moreover

$$\dim \mathcal{SM}_g = \dim \mathcal{U} = (3g-3, 2g-2)\,.$$

A few references

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  - $\mathcal{SM}_g$ is non-projected (in particular non-split) for $g \geq 5$
  - $\mathcal{SM}_g^{(1)}$ is non-split for $g \geq 2$. (supermoduli of punctured SUSY curves).

Two kinds of punctures

There are two kinds of punctures on a SUSY curve, according to the different bosonic of fermionic fields that are inserted in the theory.

- Neveu-Schwartz (NS) punctures. These are merely unordered points, understood as the insertion points of bosonic operators.

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- Neveu-Schwartz (NS) punctures. These are merely unordered points, understood as the insertion points of bosonic operators.
- Ramond-Ramond (RR) punctures. These correspond to divisors where the superconformal structure degenerates and are related to the insertion of fermionic operators.

NS punctures

$(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$, SUSY curve.

Definition

A NS N-puncture is a unordered family $(x_1, \dots, x_N)$ *of (*$\mathcal{S}$*-valued) points of* $\pi\colon \mathcal{X} \to \mathcal{S}$ *(i.e. sections* $x_i\colon \mathcal{S} \hookrightarrow \mathcal{X}$ *of* π*).*

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Proposition (D-HR-S)

The functor of relative supercurves of genus g with N NS-punctures is representable by an Artin algebraic superspace $\mathcal{SM}_g^{(N)}$, whose underlying ordinary Artin algebraic space is the scheme $SM_g^{(N)} = \mathrm{Sym}_{SM_g}(X_g/SM_g)$.
Moreover

$$\dim \mathcal{SM}_g^{(N)} = (3g - 3 + N, 2g - 2 + N)\,.$$

RR punctures

$\pi\colon \mathcal{X} \to \mathcal{S}$, supercurve, $\mathcal{Z} \hookrightarrow \mathcal{X}$ positive superdivisor (codim $= (1,0)$).

Definition

$\pi\colon \mathcal{X} \to \mathcal{S}$ has a RR-puncture along $\mathcal{Z}$ if there is a locally free subsheaf of rank $(0,1)$ of the relative tangent sheaf, $\mathcal{D} \hookrightarrow \Theta_{\mathcal{X}/\mathcal{S}}$, such that the composition

$$\mathcal{D} \otimes \mathcal{D} \xrightarrow{[\,,\,]} \Theta_{\mathcal{X}/\mathcal{S}} \to \Theta_{\mathcal{X}/\mathcal{S}}/\mathcal{D}$$

induces an isom. of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{D} \otimes \mathcal{D} \xrightarrow{\sim} (\Theta_{\mathcal{X}/\mathcal{S}}/\mathcal{D})(-\mathcal{Z})$.

We also say that $(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$ is a RR-SUSY curve.

RR-SUSY curves

$(\pi\colon \mathcal{X} \to \mathcal{S}, \mathcal{D})$, a RR-SUSY curve.
If $\mathcal{S} = S$ is an ordinary scheme, $\mathcal{O}_{\mathcal{X}} = \mathcal{O}_X \oplus \mathcal{L}$, now on has

$$\mathcal{L} \otimes \mathcal{L} \xrightarrow{\sim} \kappa_{X/S} \otimes \mathcal{O}_X(Z) = \kappa_{X/S}(Z)\,, \quad \mathcal{L} \xrightarrow{\sim} \kappa_{X/S}(Z)^{1/2}\,.$$

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This forces $m = \deg Z$ to be even. In this case, one has:

$$\mathcal{G}_\pi \simeq (\Theta_{\mathcal{X}}/\mathcal{D})_0 \oplus [(\Theta_{\mathcal{X}}/\mathcal{D})_1 \otimes \mathcal{O}_X(-Z)]$$

$$Inf_{def}(\mathcal{X}/S) \simeq R^1\pi_*\mathcal{G}_\pi \simeq [R^1\pi_*(\kappa_{X/S}^{-1})] \oplus [R^1\pi_*(\kappa_{X/S}(Z)^{-1/2})]\,.$$

Proposition (HR)

The functor of relative RR-SUSY curves of genus g and $\deg \mathcal{Z} = m$ is representable by an Artin algebraic superspace of dimension

$$(3g-3+m, 2g-2+m/2)\,.$$