# *Mass, Scalar Curvature, &*

# *Kähler Geometry, II*

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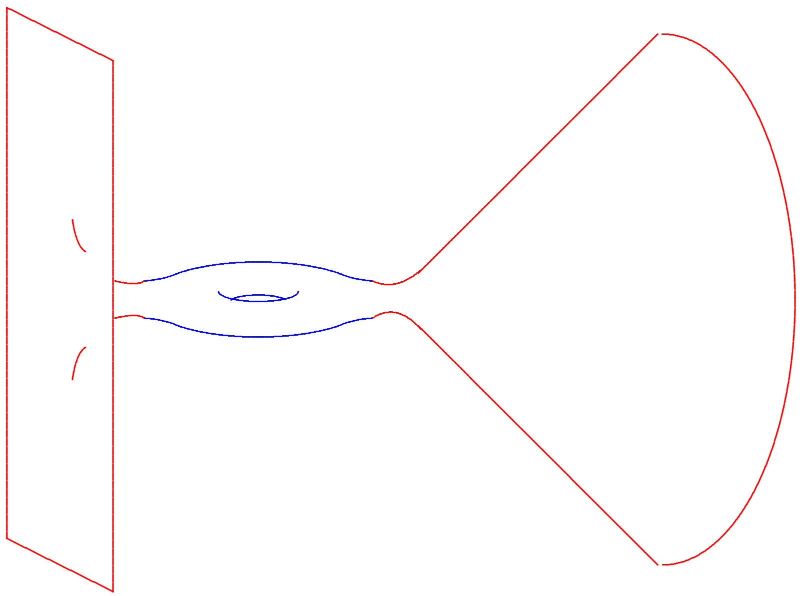
Seminario de Geometría
ICMAT, November 5, 2018

**Definition.** *Complete, non-compact n-manifold* $(M^n, g)$ *is* asymptotically locally Euclidean



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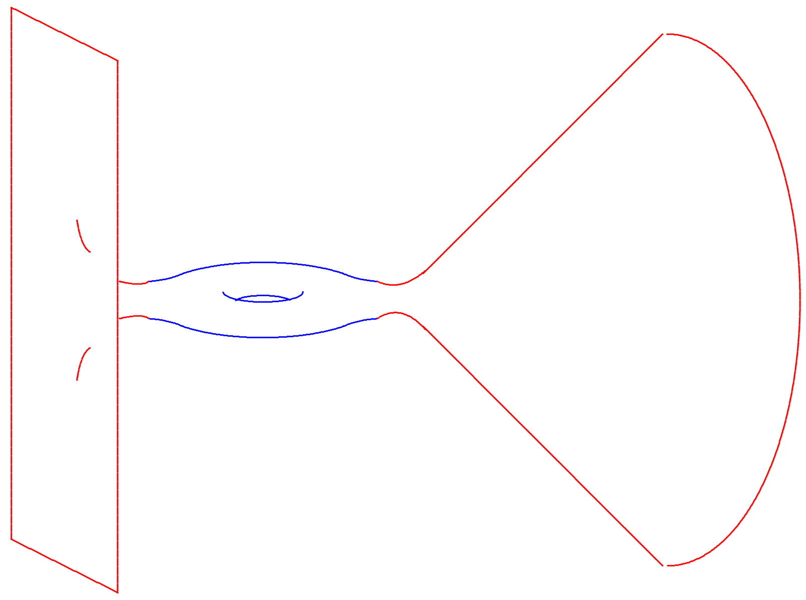

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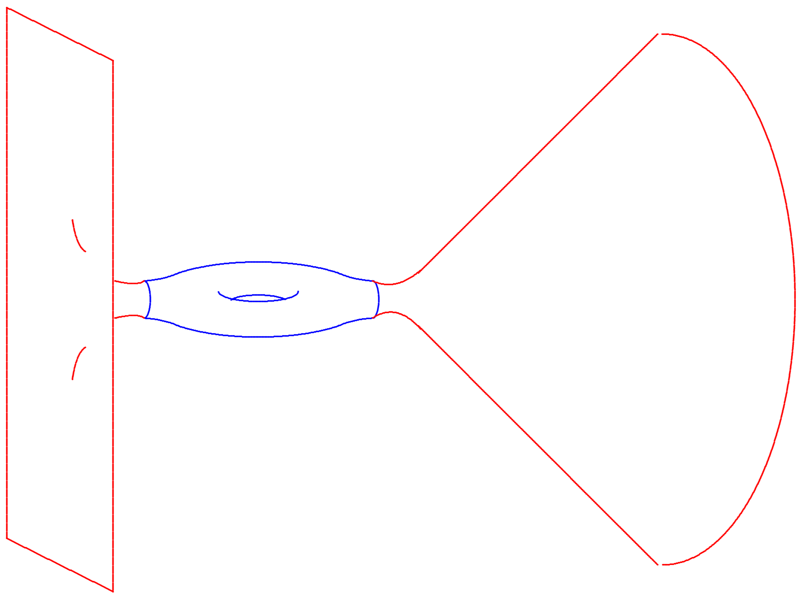

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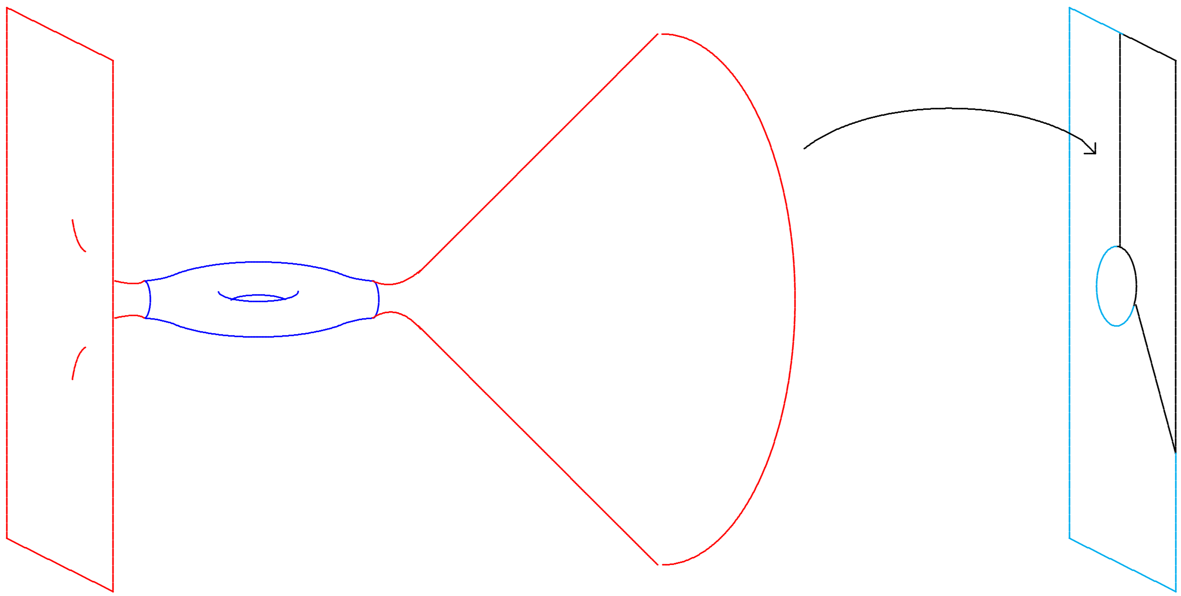

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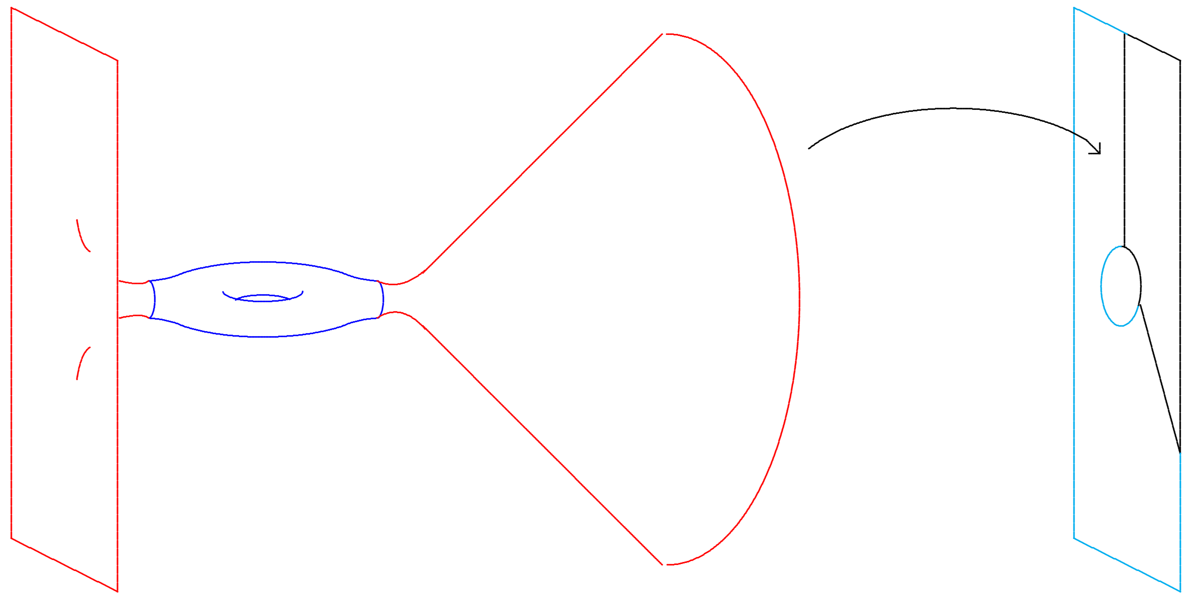

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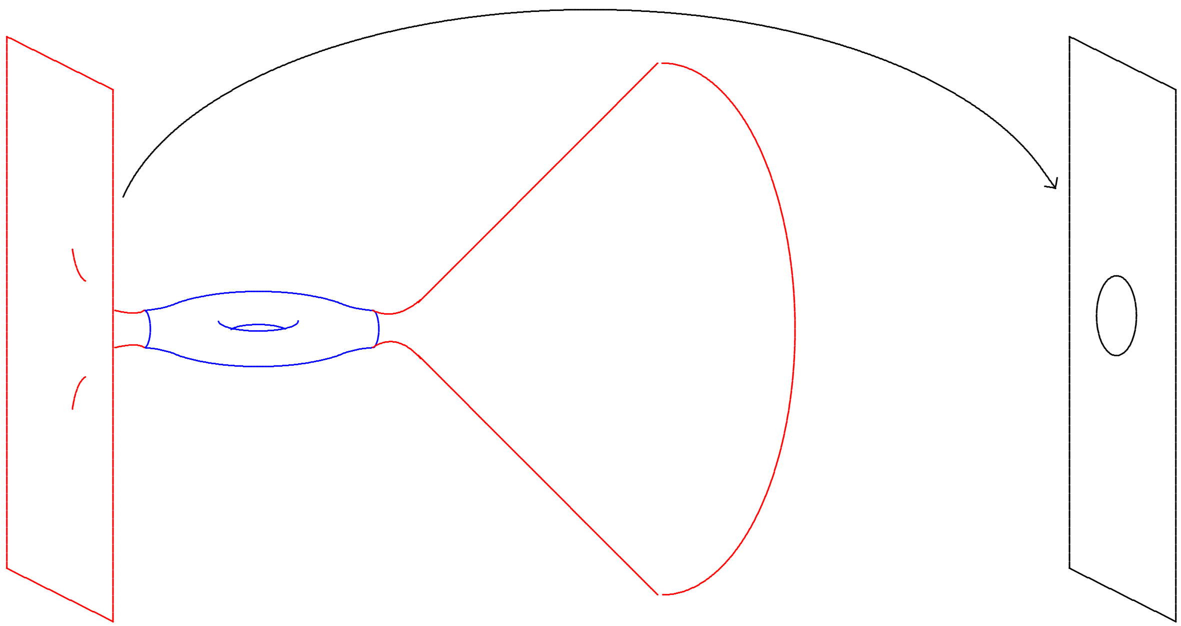

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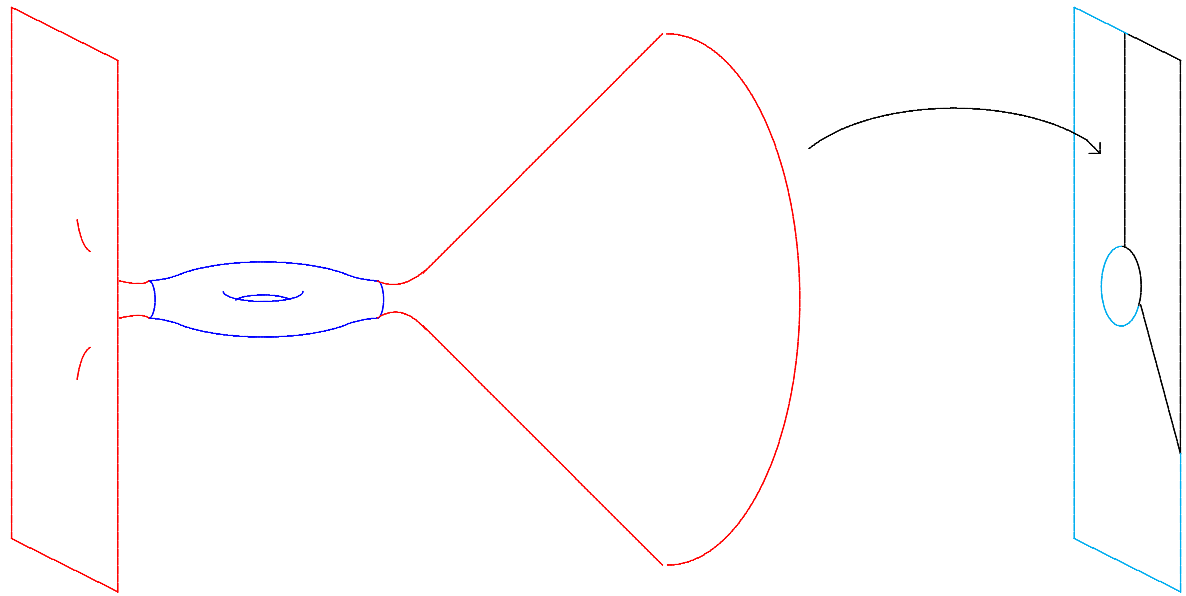

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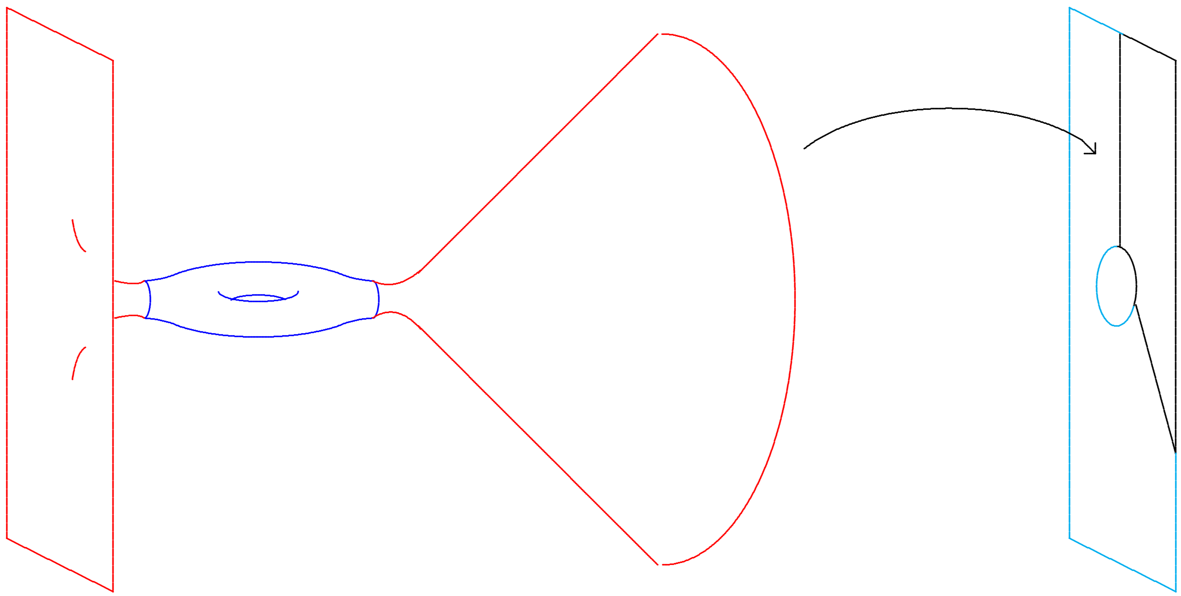

$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

$$g_{jk,\ell} = O(|x|^{-\frac{n}{2}-\varepsilon}), \quad s \in L^1$$

Why consider ALE spaces?

## Key examples:

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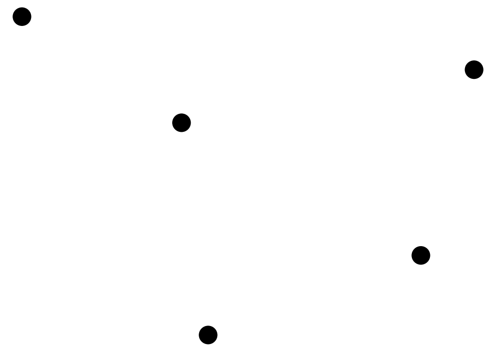
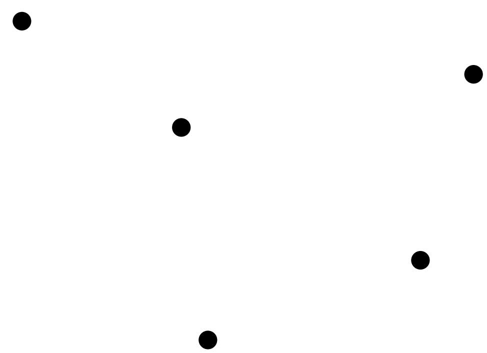
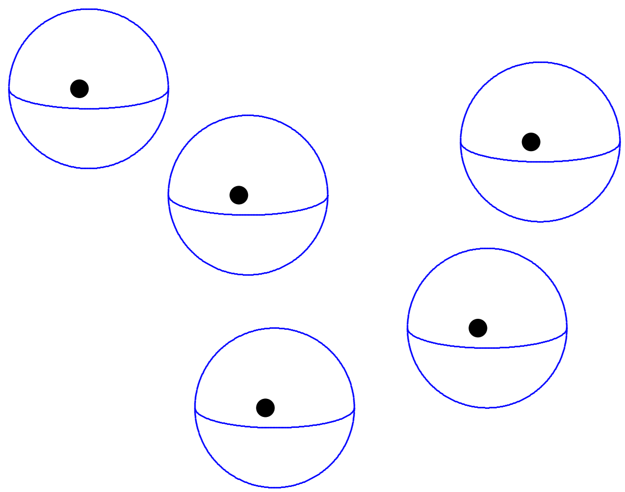


Data: $\ell$ points in $\mathbb{R}^3$.



Data: $\ell$ points in $\mathbb{R}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$V = \sum_{j=1}^{\ell} \frac{1}{2\varrho_j}$$



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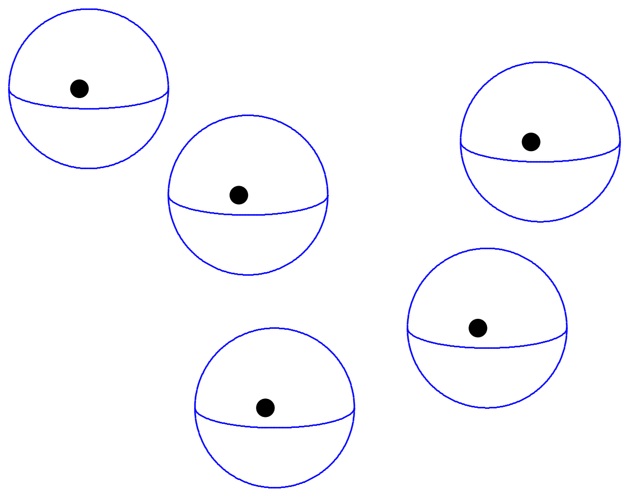

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$$g = Vh + V^{-1}\theta^2$$

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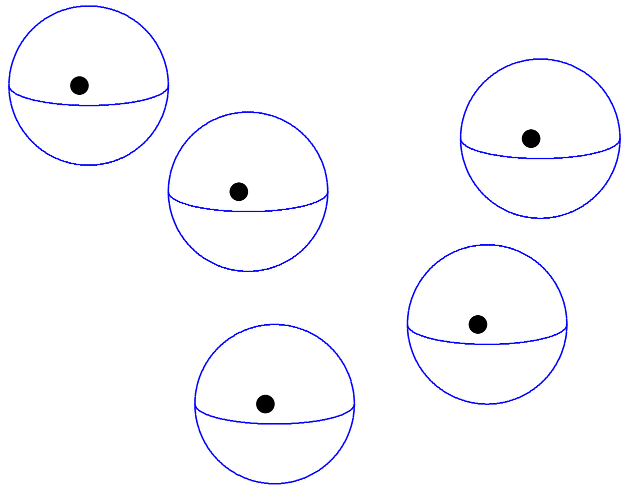

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$$g = V(dx^2 + dy^2 + dz^2) + V^{-1}\theta^2$$

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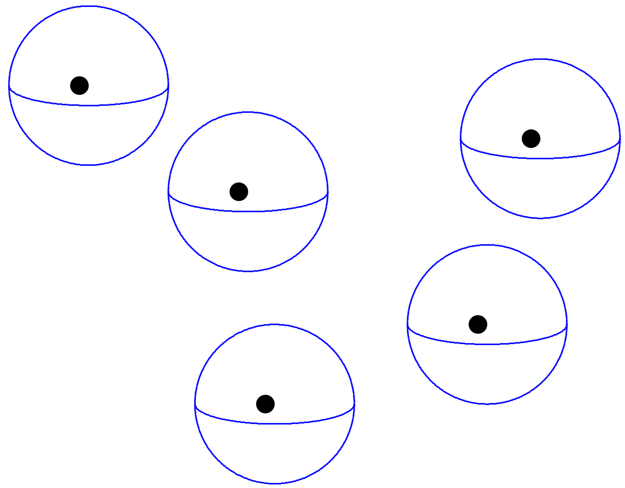

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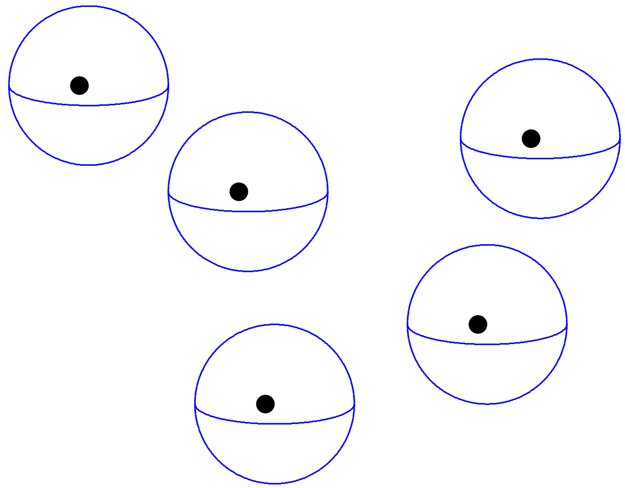

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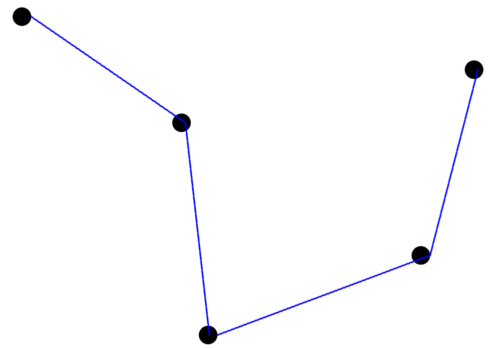

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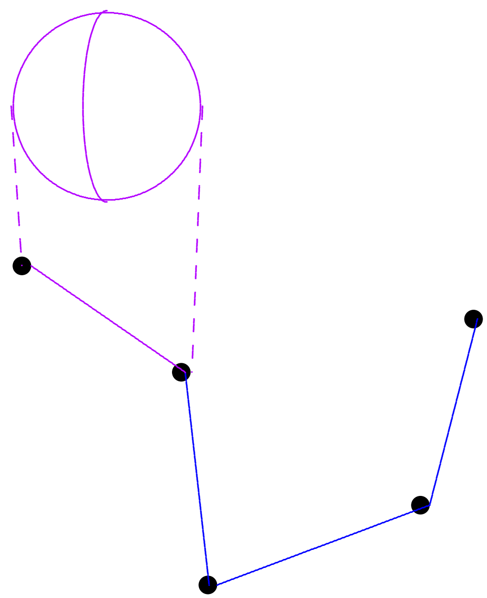

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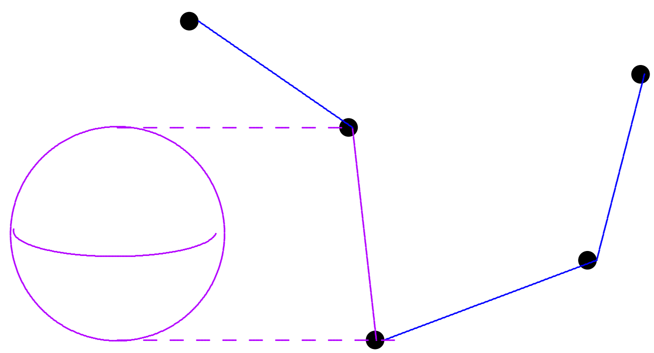

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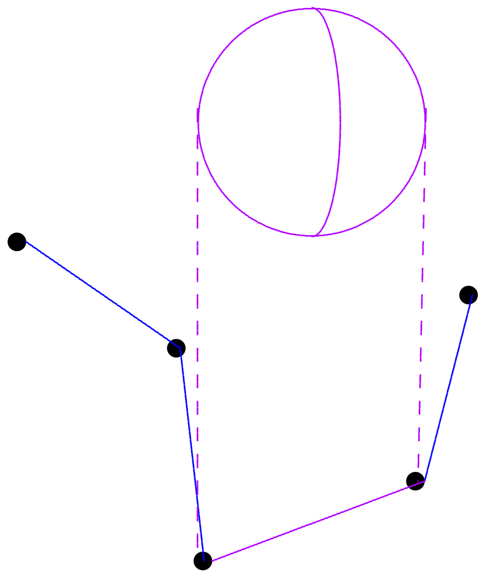

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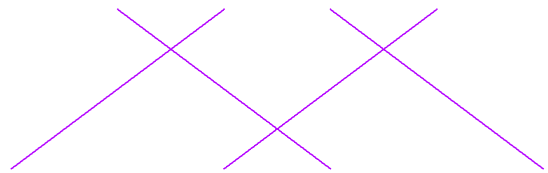

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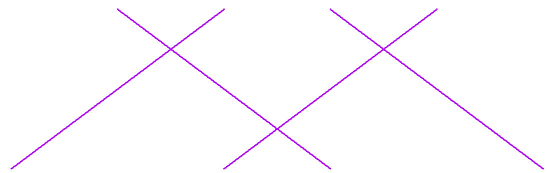

Configuration dual to Dynkin diagram $A_k$:

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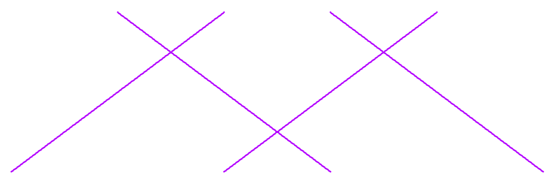

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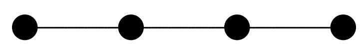

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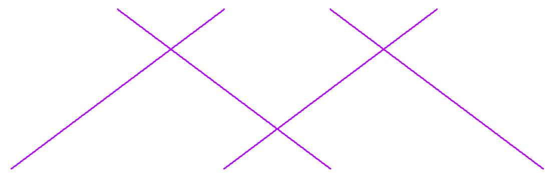

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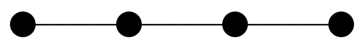

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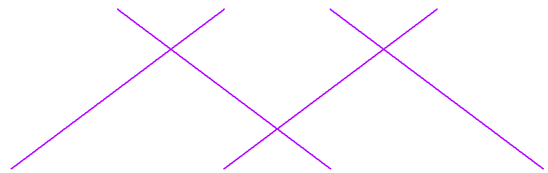

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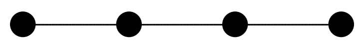

Diffeotype:

Plumb together $k$ copies of $T^*S^2$
according to diagram.

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The G-H metrics are hyper-Kähler, and were soon independently rediscovered by Hitchin.

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holonomy

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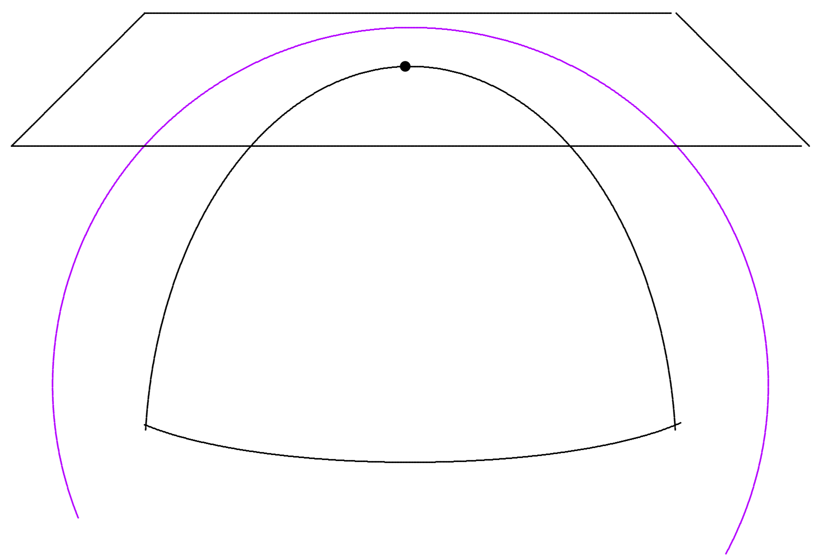

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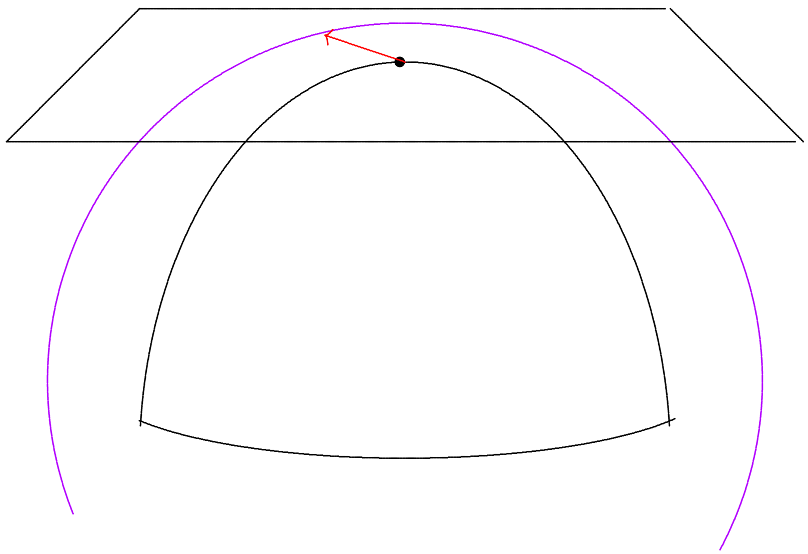

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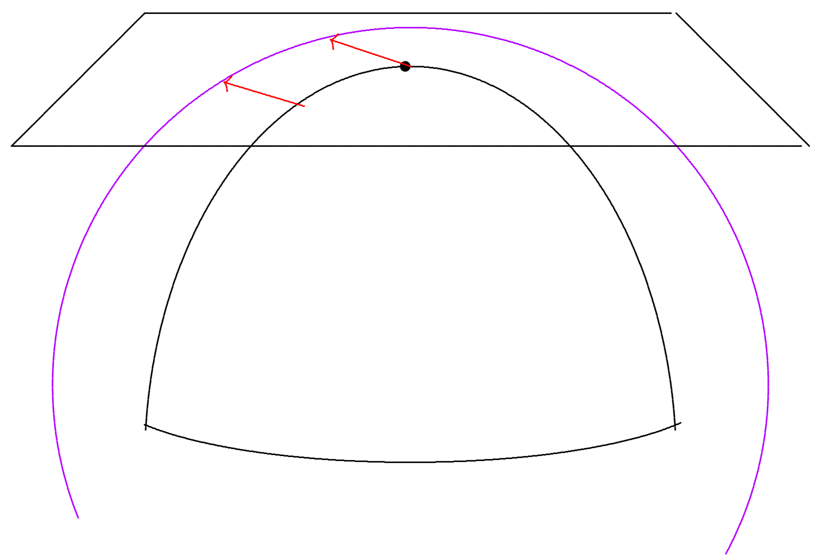

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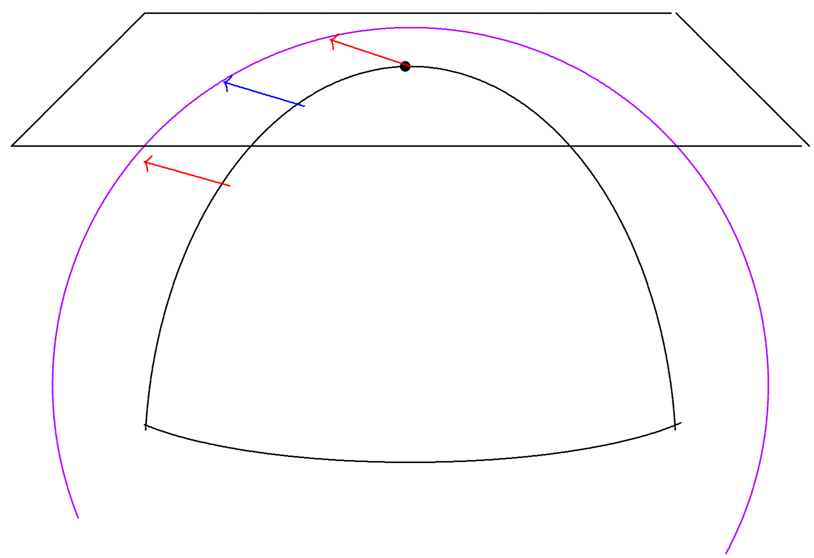

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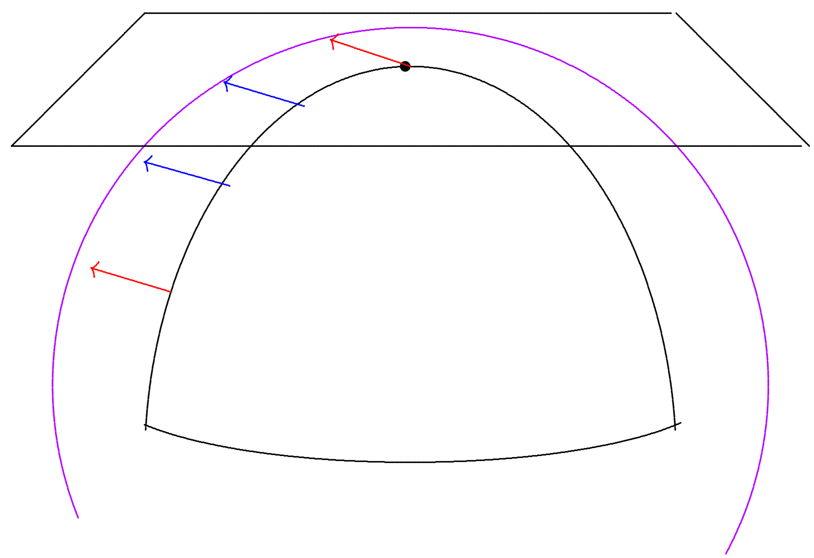

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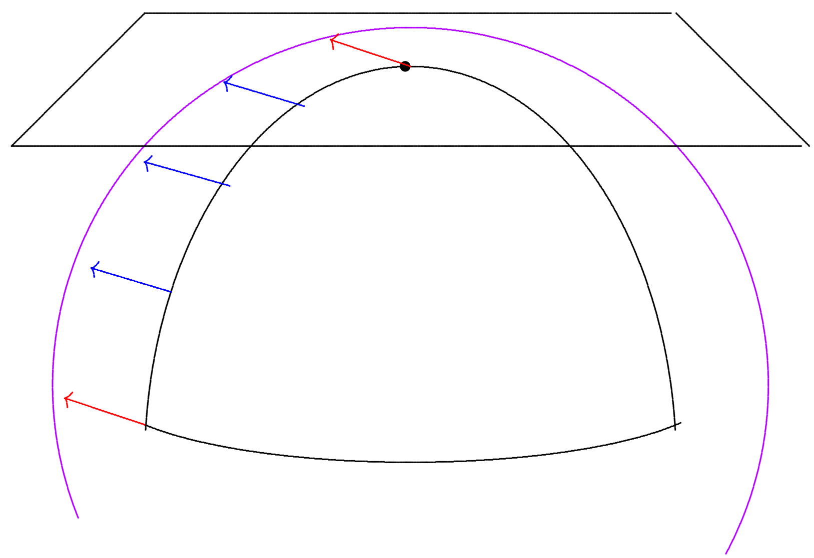

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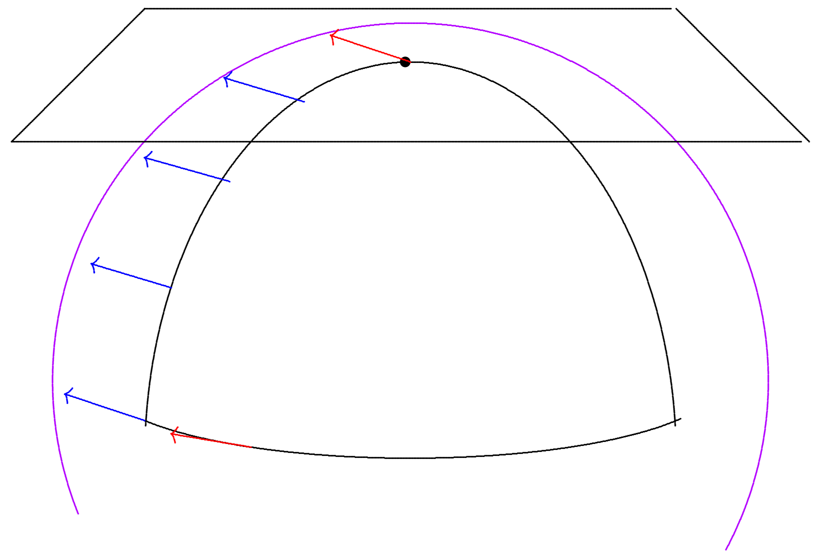

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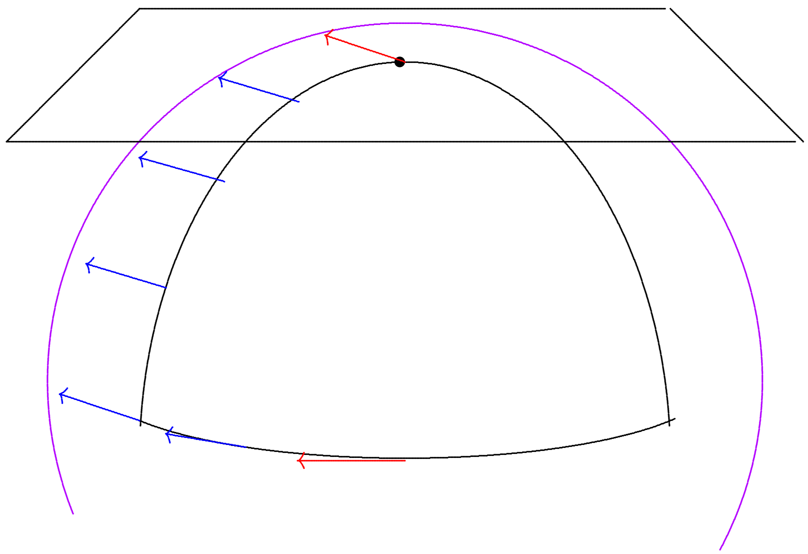

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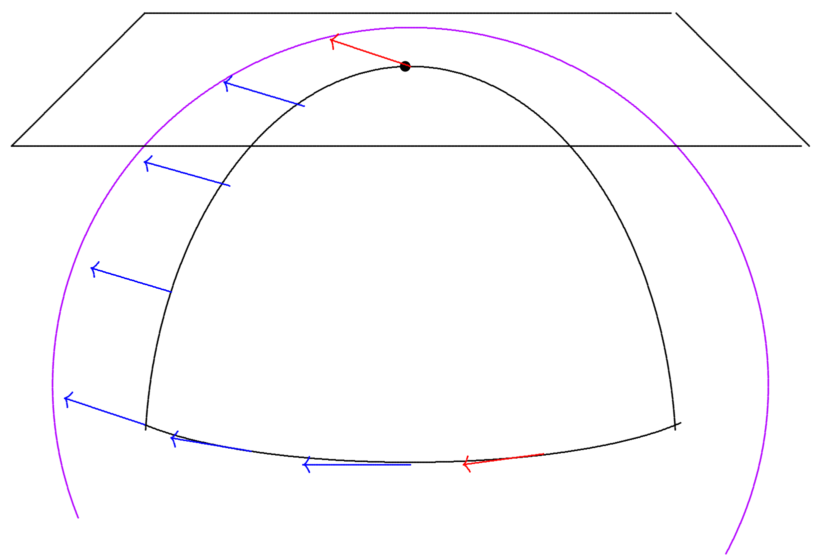

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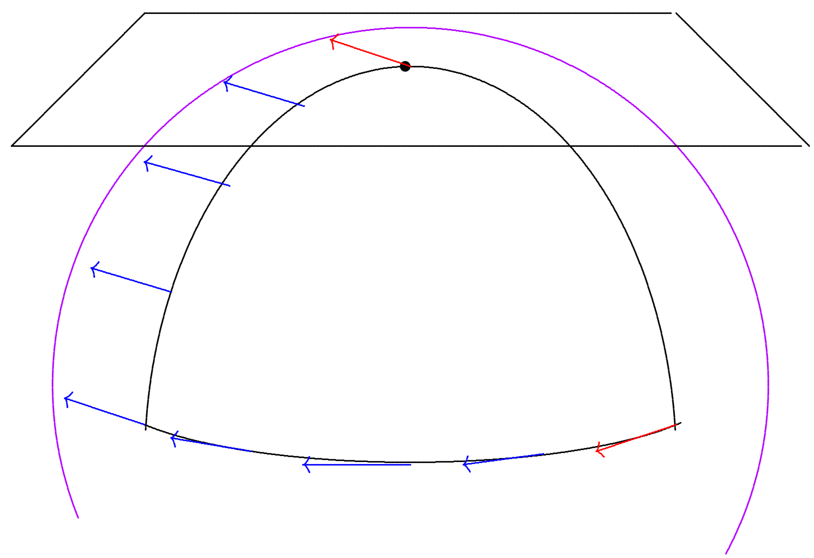

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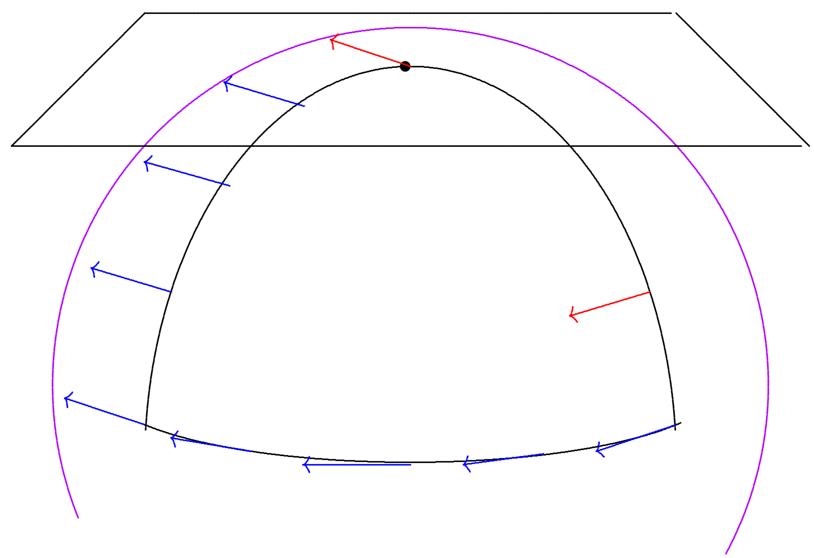

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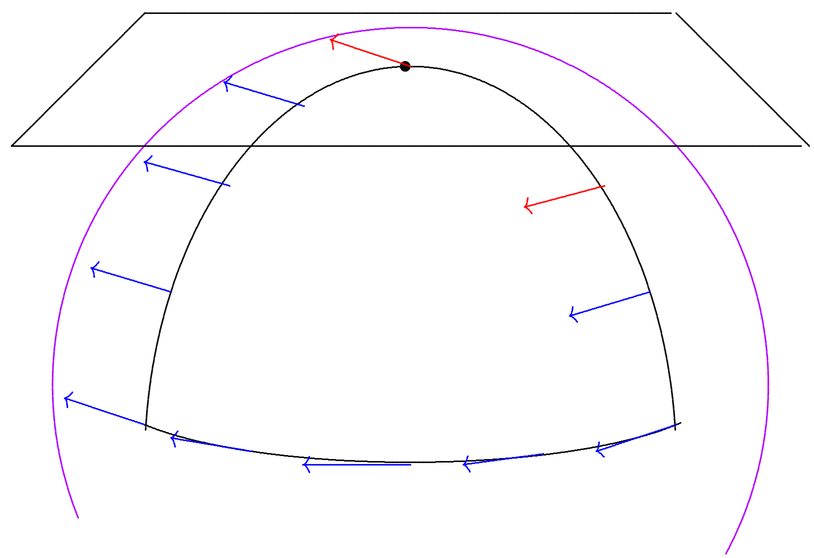

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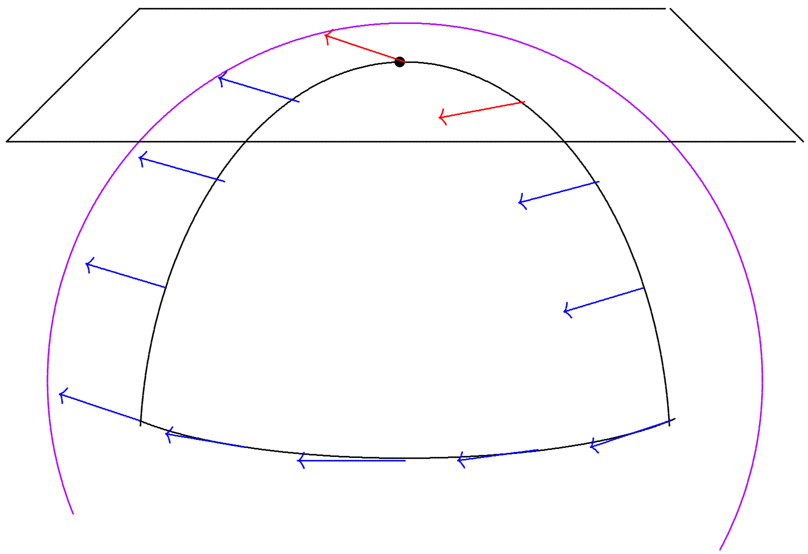

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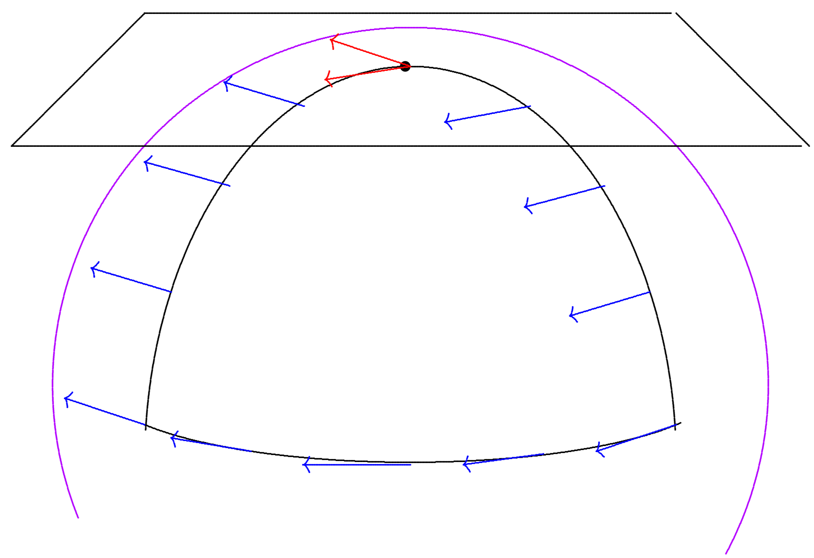

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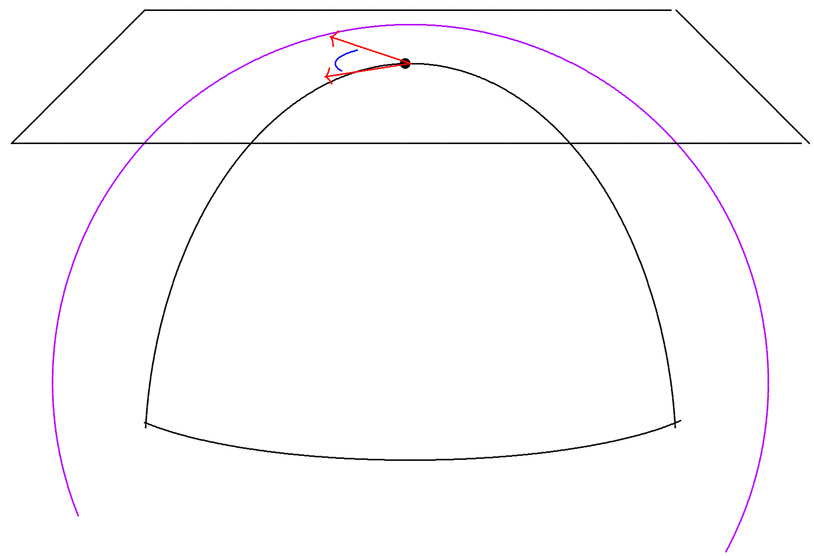
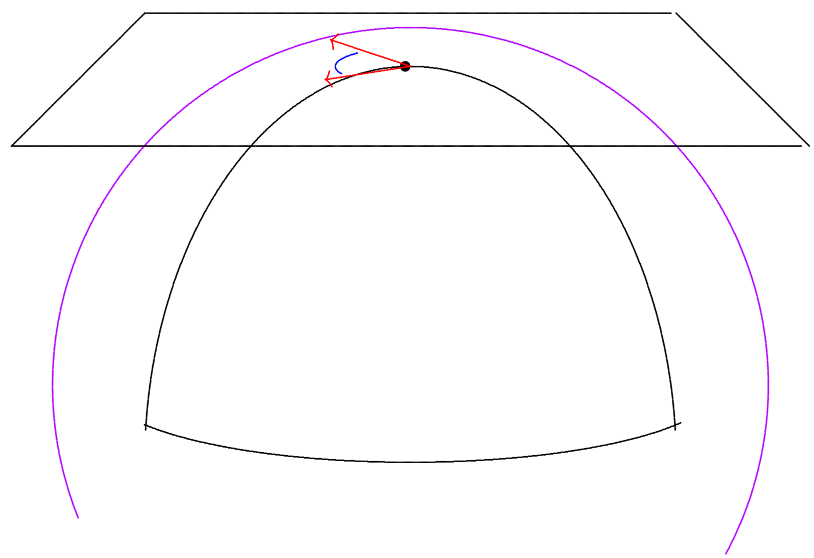


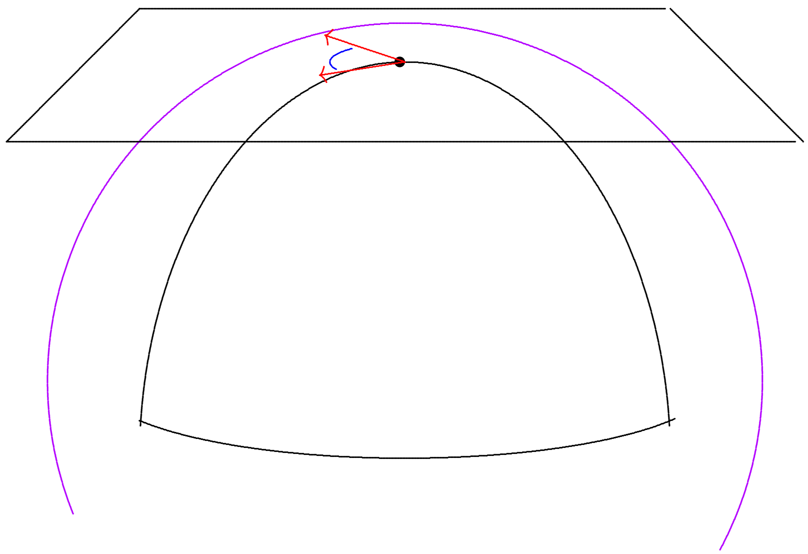
$(M^n, g)$:

holonomy $\subset \mathbf{O}(n)$



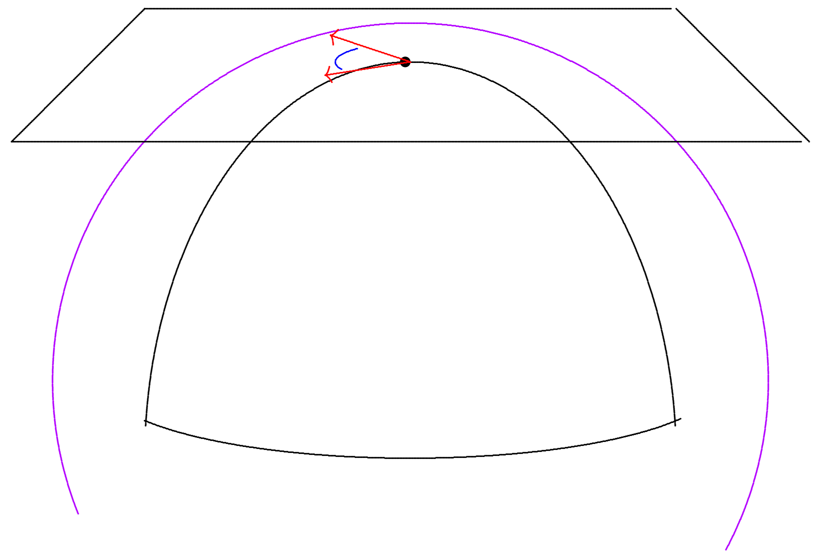
Kähler metrics:

$(M^{2m}, g)$: holonomy



Kähler metrics:

$(M^{2m}, g)$ Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{U}(m)$



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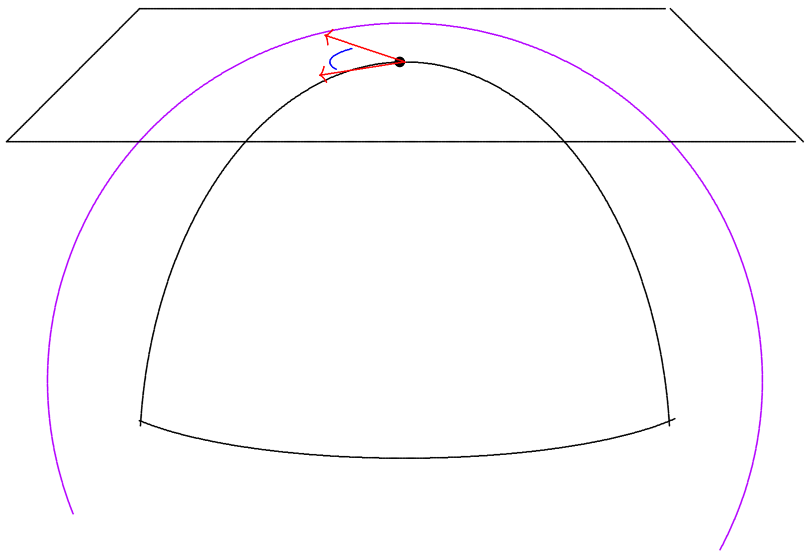

$$\mathbf{U}(m) := \mathbf{O}(2m) \cap \mathbf{GL}(m, \mathbb{C})$$

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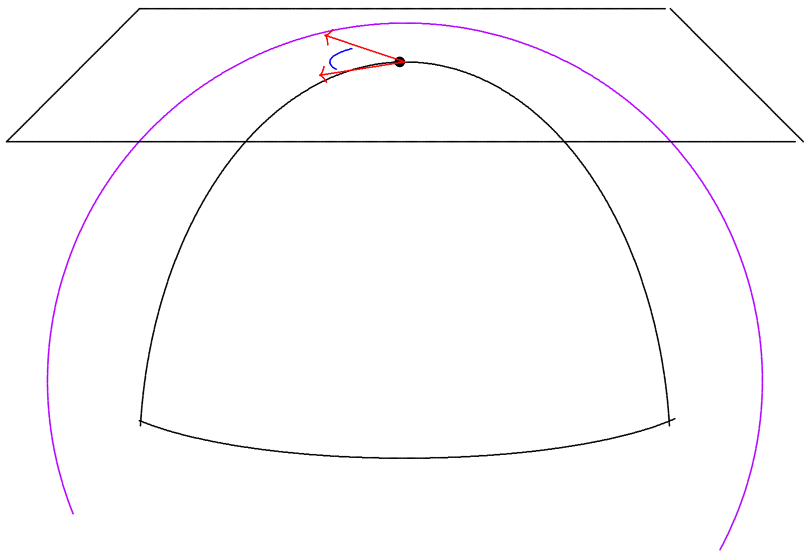

Makes tangent space a complex vector space!

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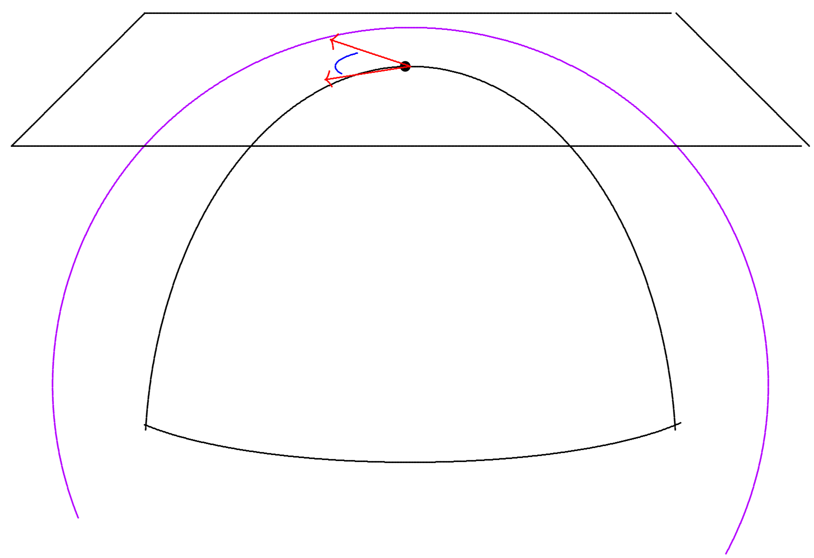

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$$J : TM \to TM, \qquad J^2 = -\text{identity}$$

"almost-complex structure"

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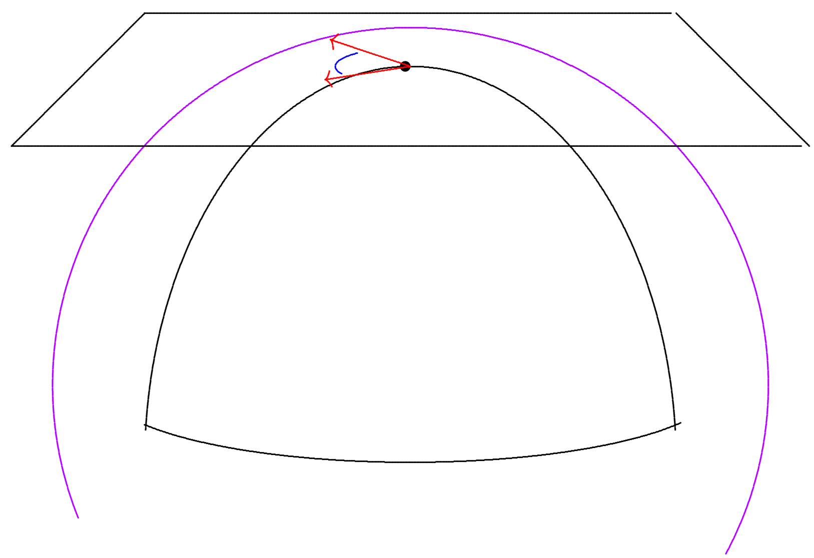

Makes tangent space a complex vector space!

Invariant under parallel transport!

Kähler metrics:

$(M^{2m}, g)$ Kähler $\iff$ holonomy $\subset \mathbf{U}(m)$

$\iff \exists$ almost complex-structure $J$ with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$d\omega = 0$$

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$$[\omega] \in H^2(M)$$

“Kähler class”

Kähler metrics:

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$\Longleftrightarrow$ $\exists$ almost complex-structure $J$ with $\nabla J = 0$ and $g(J\cdot, J\cdot) = g$.

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$$\omega = i \sum_{j,k=1}^{m} \frac{\partial^2 f}{\partial z^j \partial \bar{z}^k} \, dz^j \wedge d\bar{z}^k$$

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---

Kähler magic:

$$r = -\sum_{j,k=1}^{m} \frac{\partial^2}{\partial z^j \partial \bar{z}^k} \log \det[g_{p\bar{q}}] \left[dz^j \otimes d\bar{z}^k + d\bar{z}^k \otimes dz^j\right]$$

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Kähler magic:

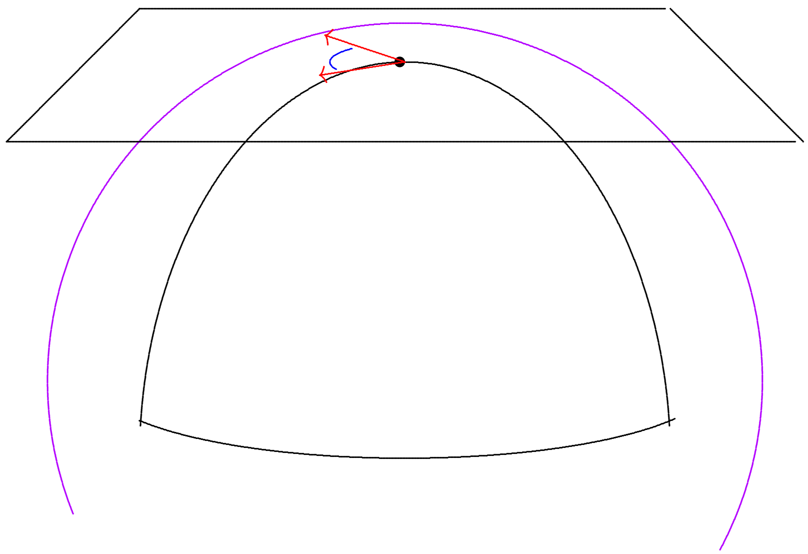
If we define the Ricci form by

$$\rho = r(J\cdot, \cdot)$$

then $i\rho$ is curvature of canonical line bundle $\Lambda^{m,0}$.

Kähler metrics:

$(M^{2m}, g)$: holonomy



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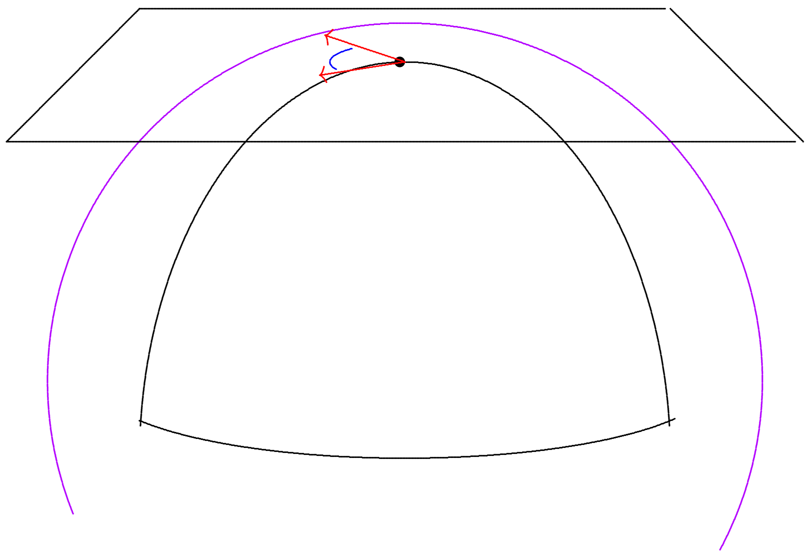

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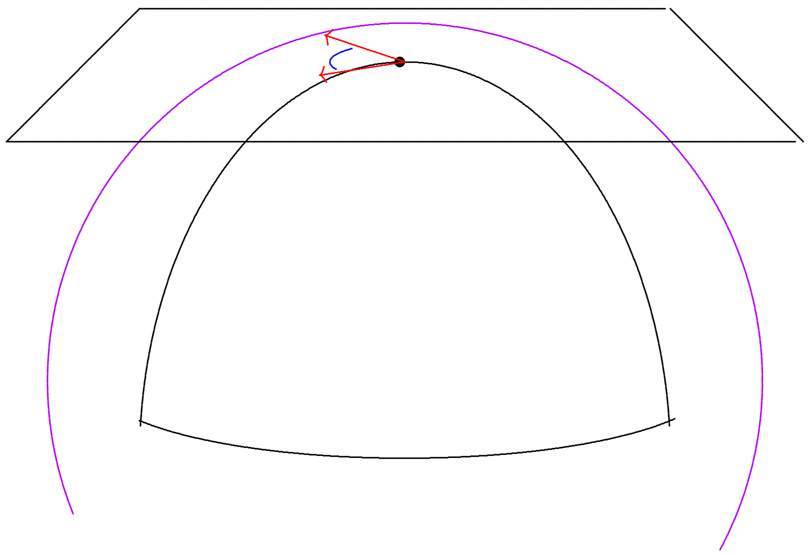

$\mathbf{SU}(m) \subset \mathbf{U}(m)$ : $\{A \mid \det A = 1\}$

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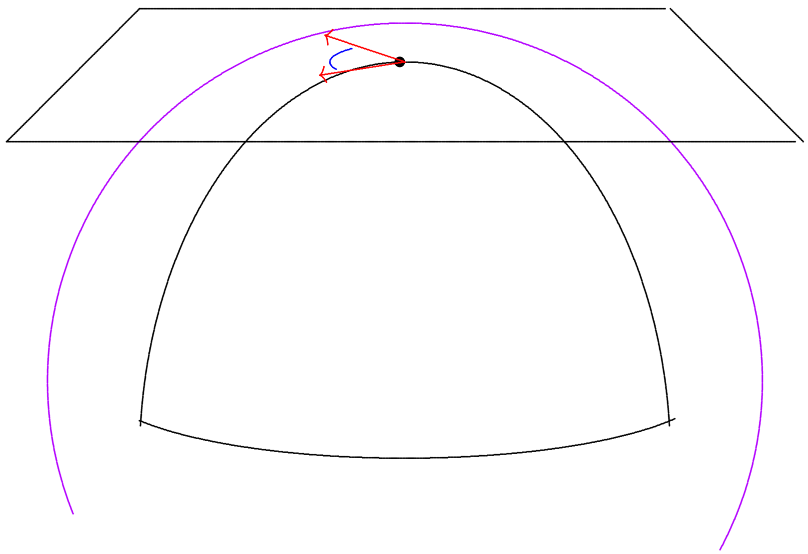

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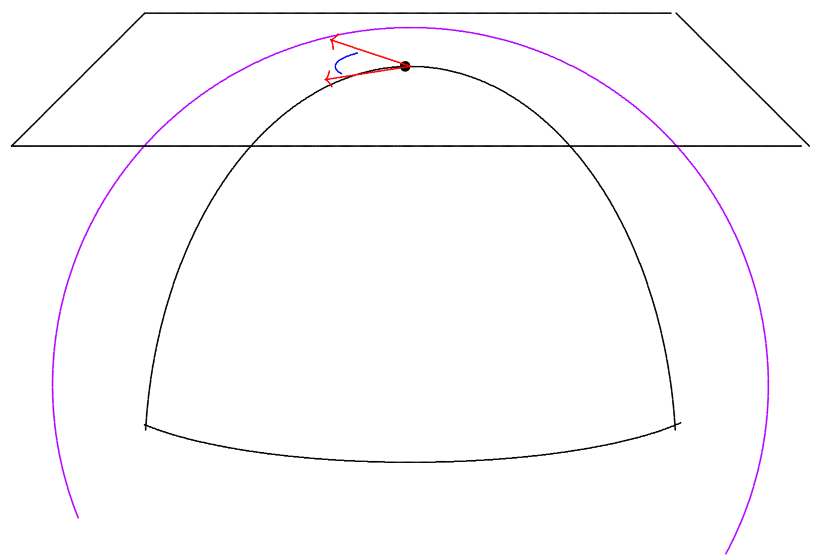
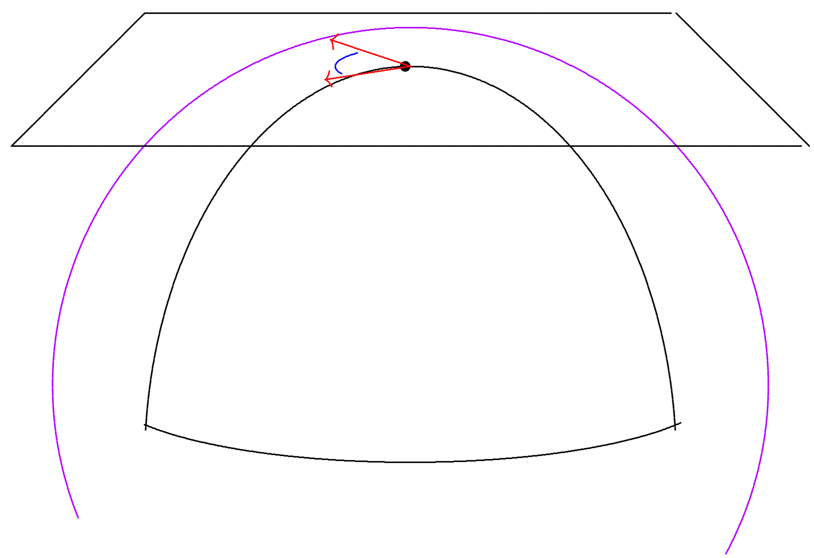


if $M$ is simply connected.

Hyper-Kähler metrics:

$(M^{4\ell}, g)$

holonomy



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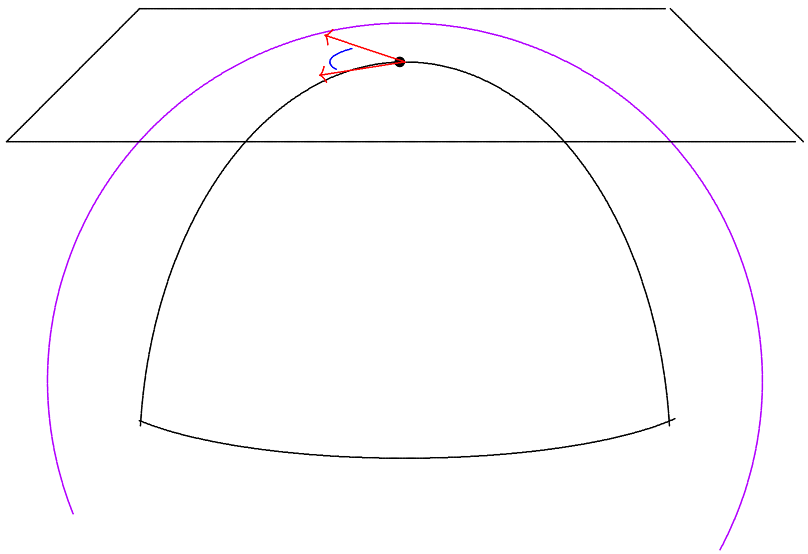

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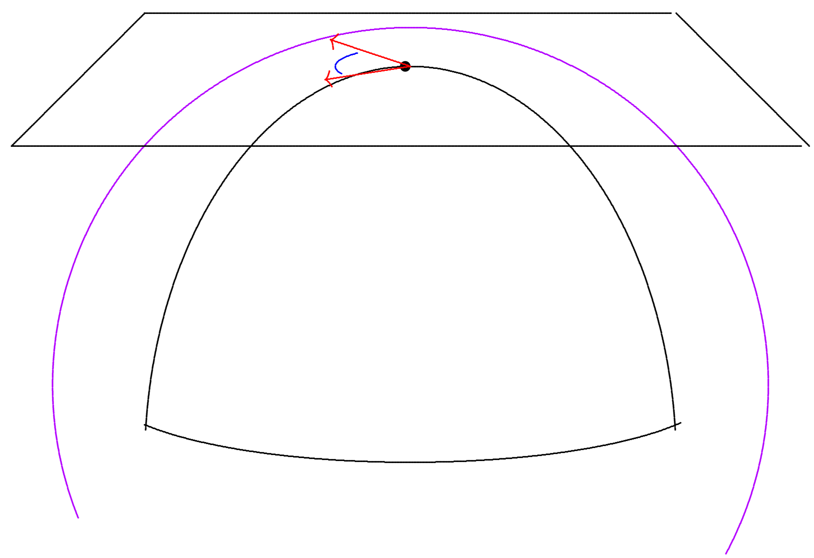

$$\mathbf{Sp}(\ell) := \mathbf{O}(4\ell) \cap \mathbf{GL}(\ell, \mathbb{H})$$

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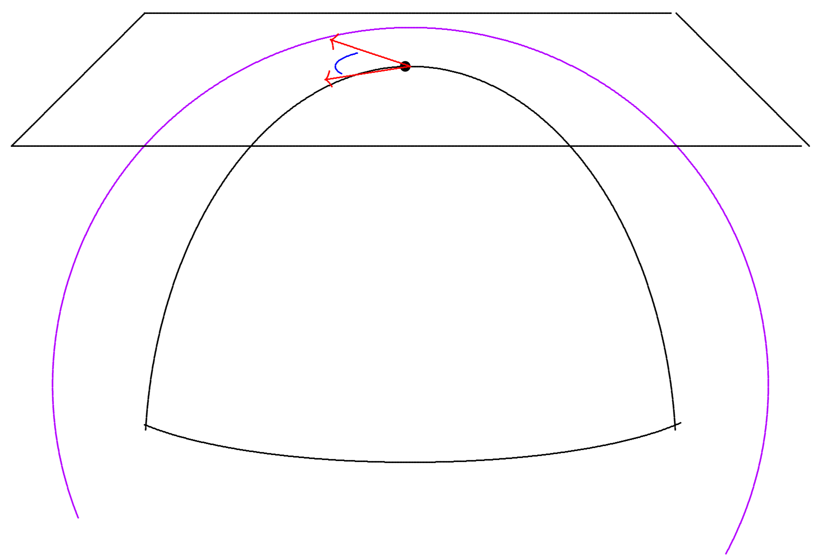

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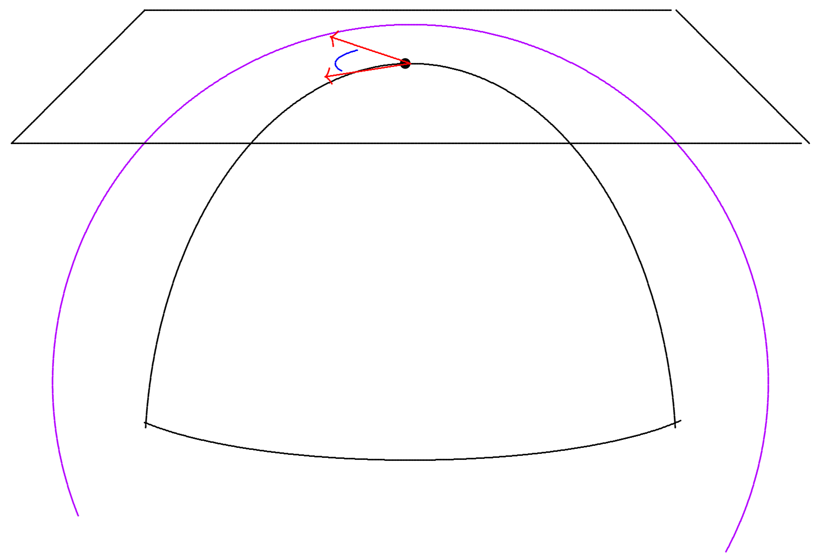

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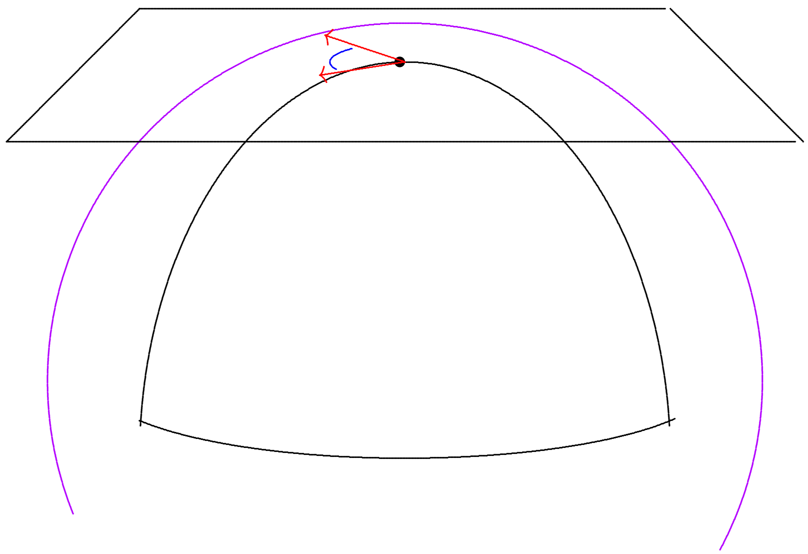

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in many ways! (For example, permute $i$, $j$, $k$...)

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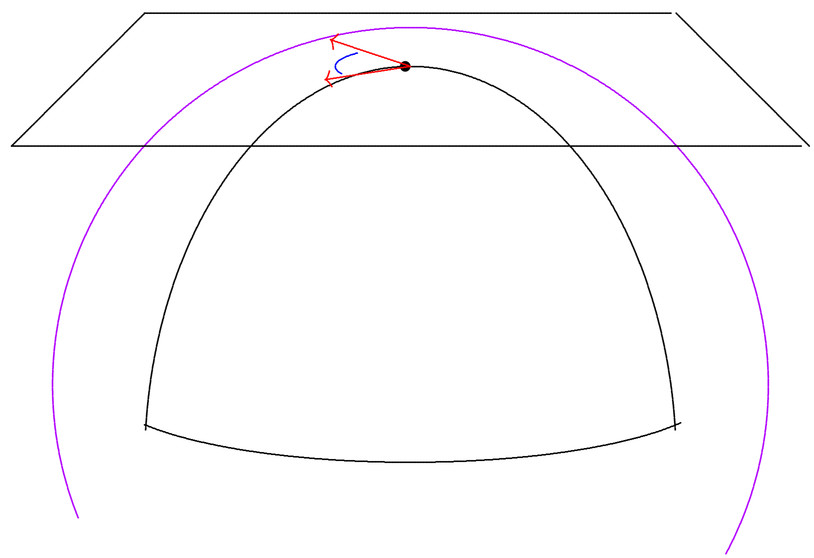

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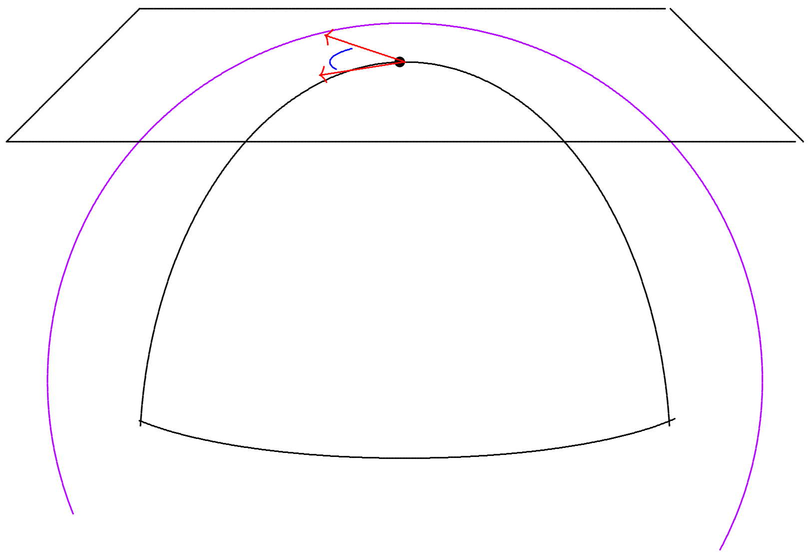
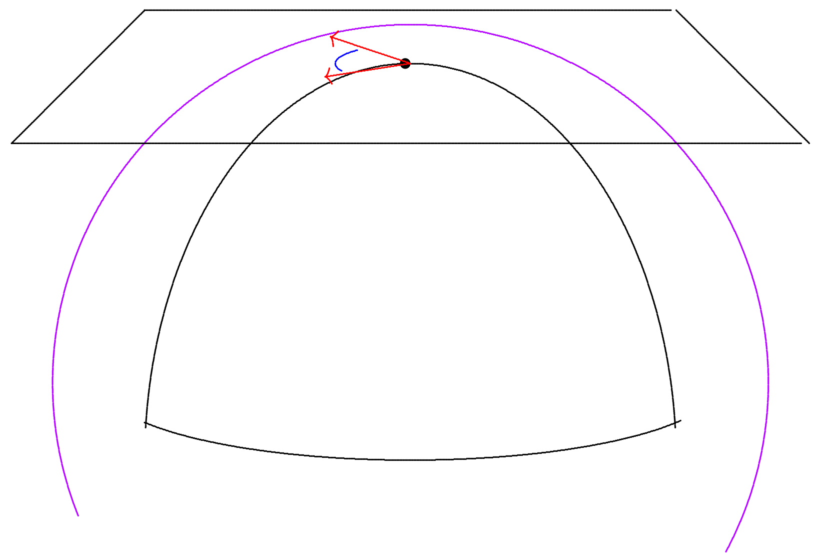


$\mathbf{Sp}(\ell) \subset \mathbf{SU}(2\ell)$

Hyper-Kähler metrics:

$(M^4, g)$ hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



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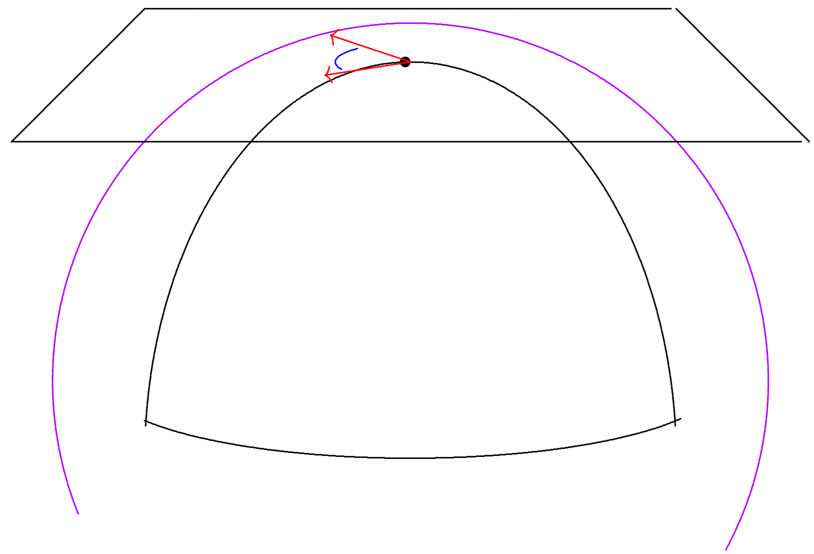

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When $(M^4, g)$ simply connected:

hyper-Kähler $\Longleftrightarrow$ Ricci-flat Kähler.

Hyper-Kähler metrics:

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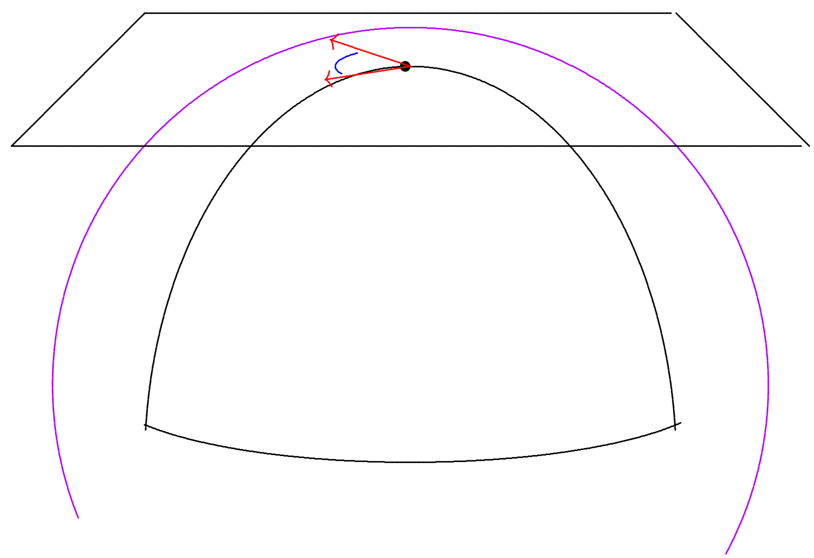

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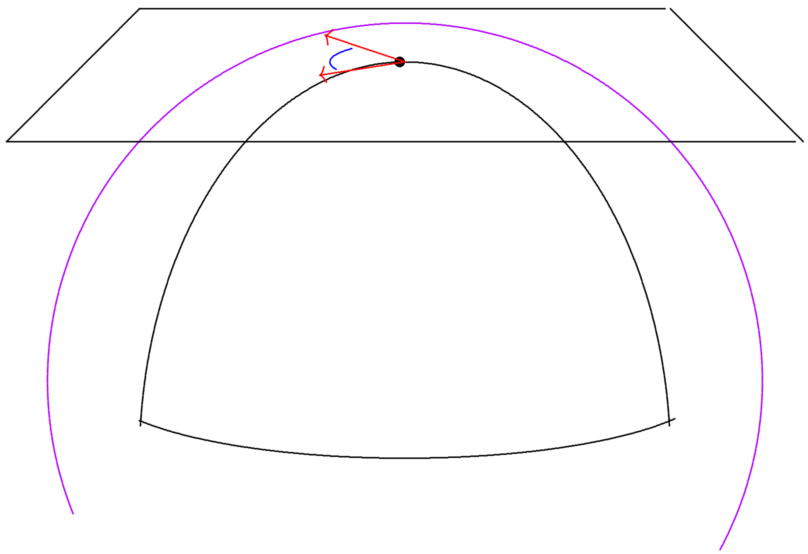

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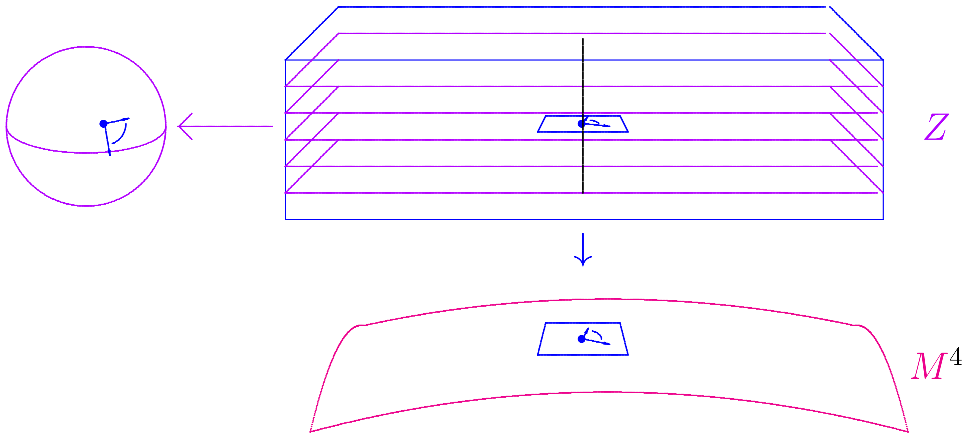
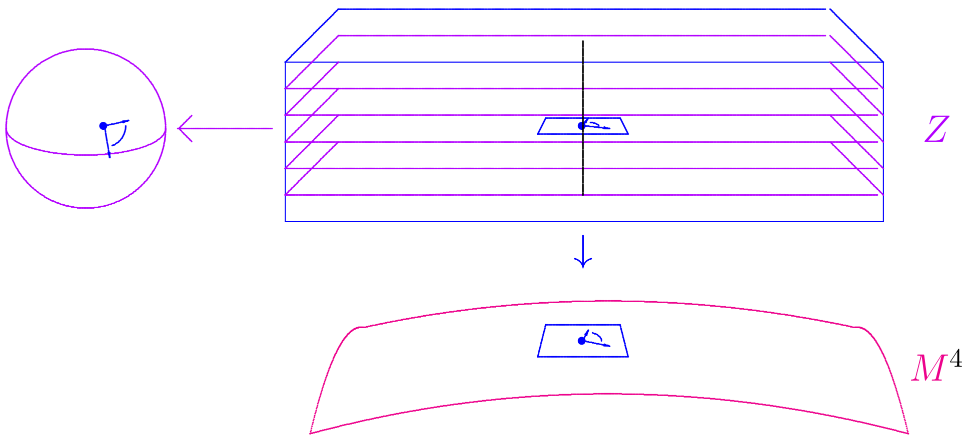
# All these complex structures can be repackaged



All these complex structures can be repackaged as

**Penrose Twistor Space** $(Z^6, J)$,



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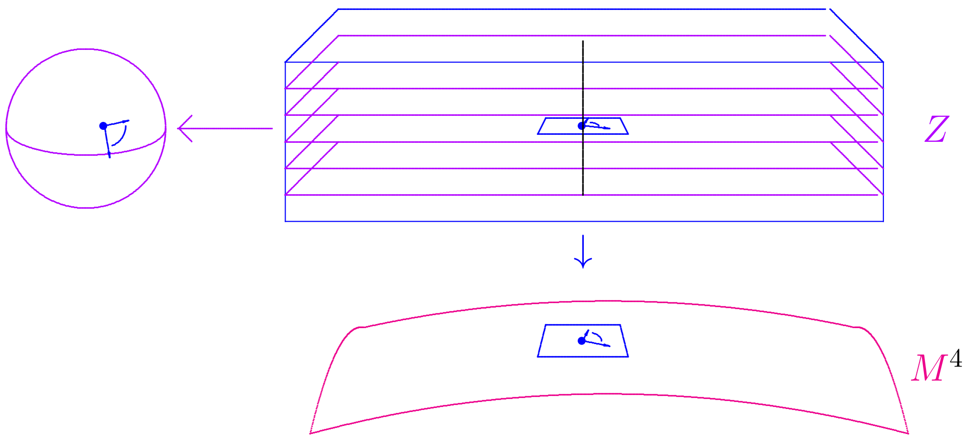

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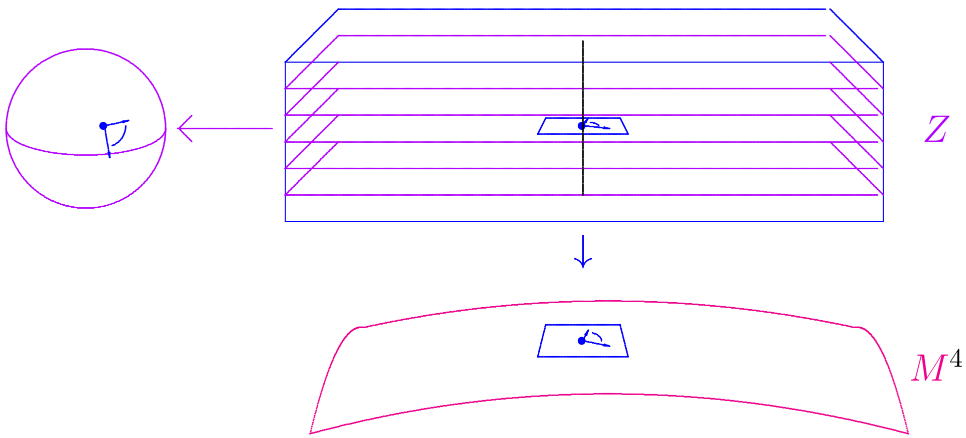


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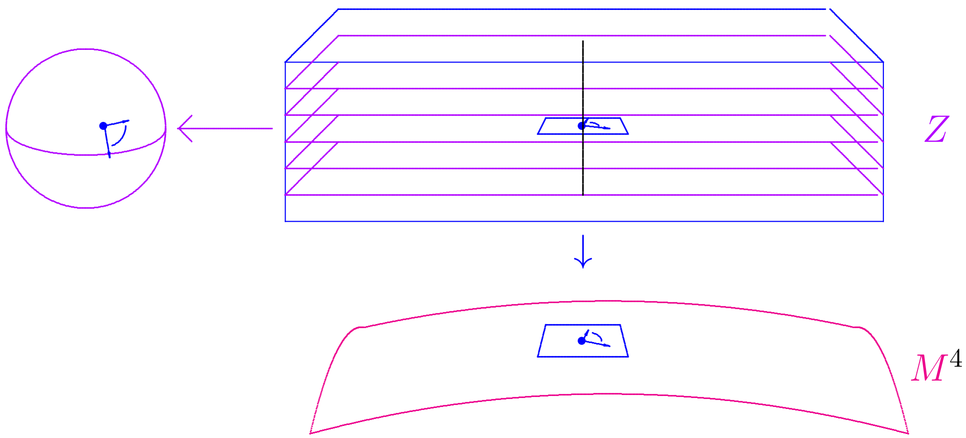


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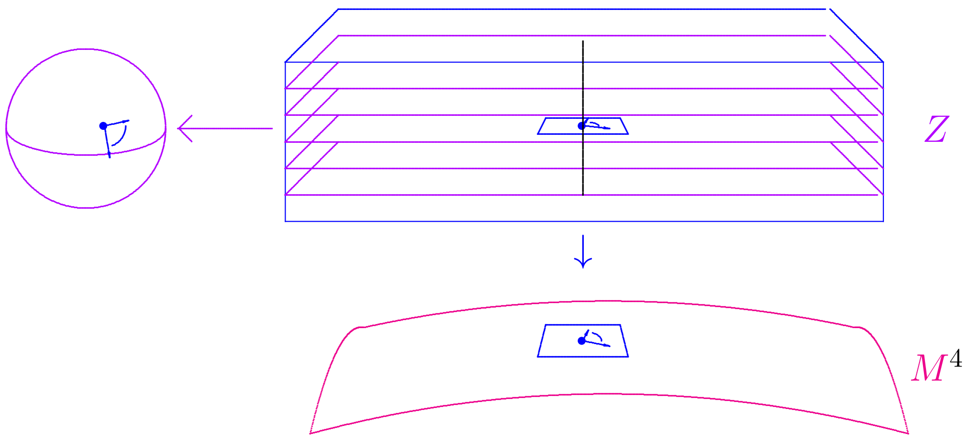


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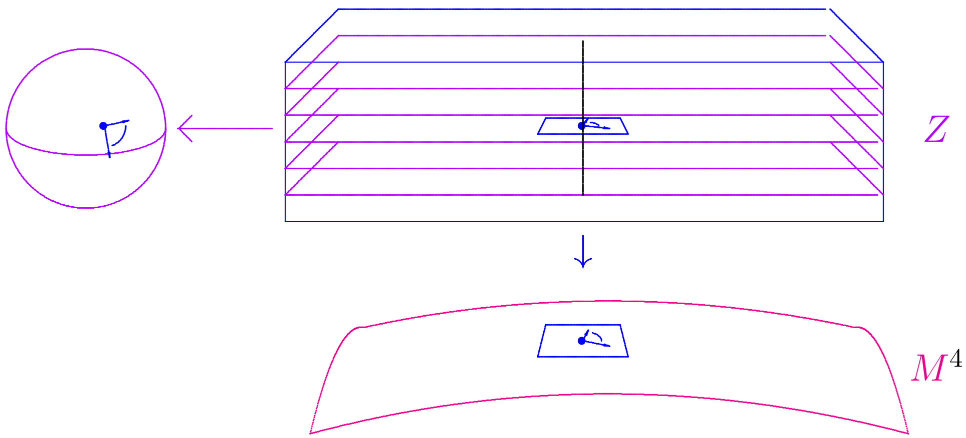


Complex structure faithfully encodes the metric.

Constructing twistor space suffices for existence.

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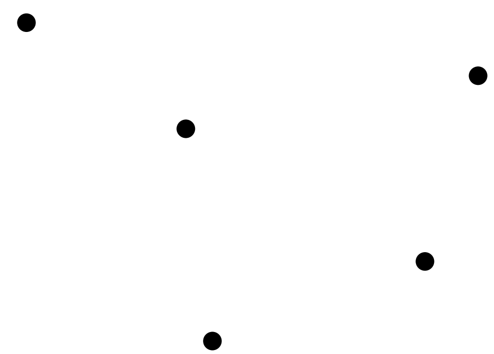

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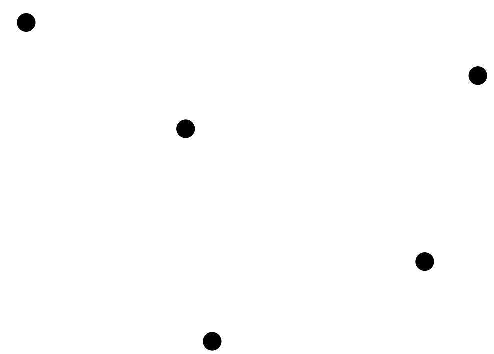

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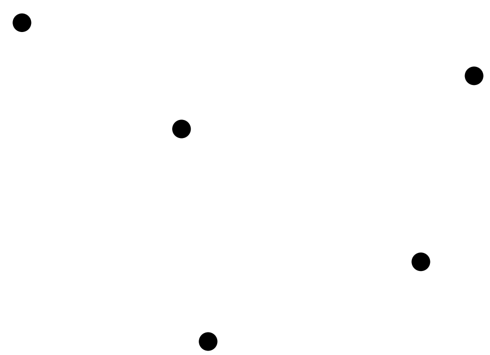

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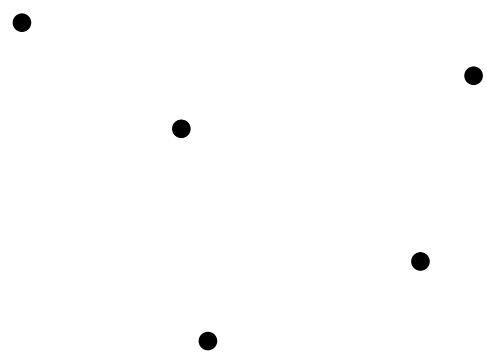

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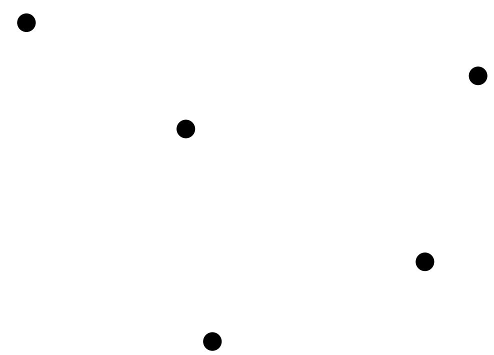

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is the twistor space of a Gibbons-Hawking metric.

**Key examples:**

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_\ell \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

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This conjecture was proved by Kronheimer, 1986.

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$$w^2 + x^2 + y^m = 0.$$

# Felix Klein's Table of Singularities (1884)

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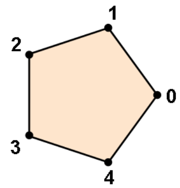


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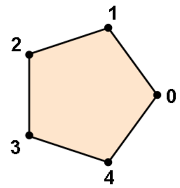


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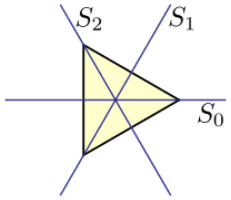
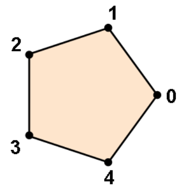
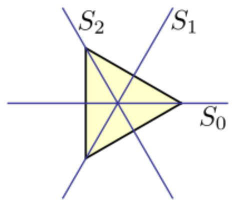



$\mathrm{Dih}_m^* \longleftrightarrow w^2 + y(x^2 + y^m) = 0$

# Felix Klein's Table of Singularities (1884)




$\mathbb{Z}_m \longleftrightarrow w^2 + x^2 + y^m = 0$

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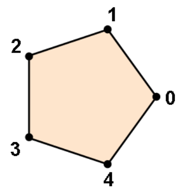


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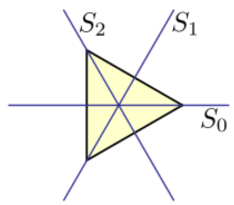


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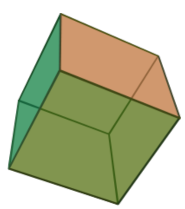

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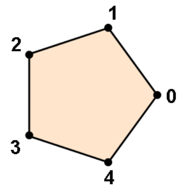


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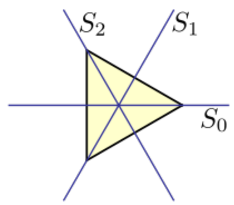


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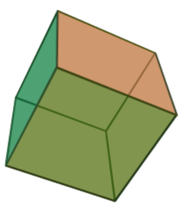

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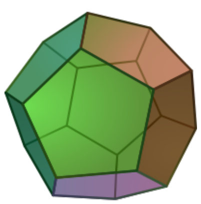

$I^* \longleftrightarrow w^2 + x^3 + y^5 = 0$

## Prototypical Klein singularity:

$$w^2 + x^2 + y^2 = 0$$

## Two ways to get rid of a singularity:

$$w^2 + x^2 + y^2 = 0$$

## Two ways to get rid of a singularity:

- Smooth it, by deformation:

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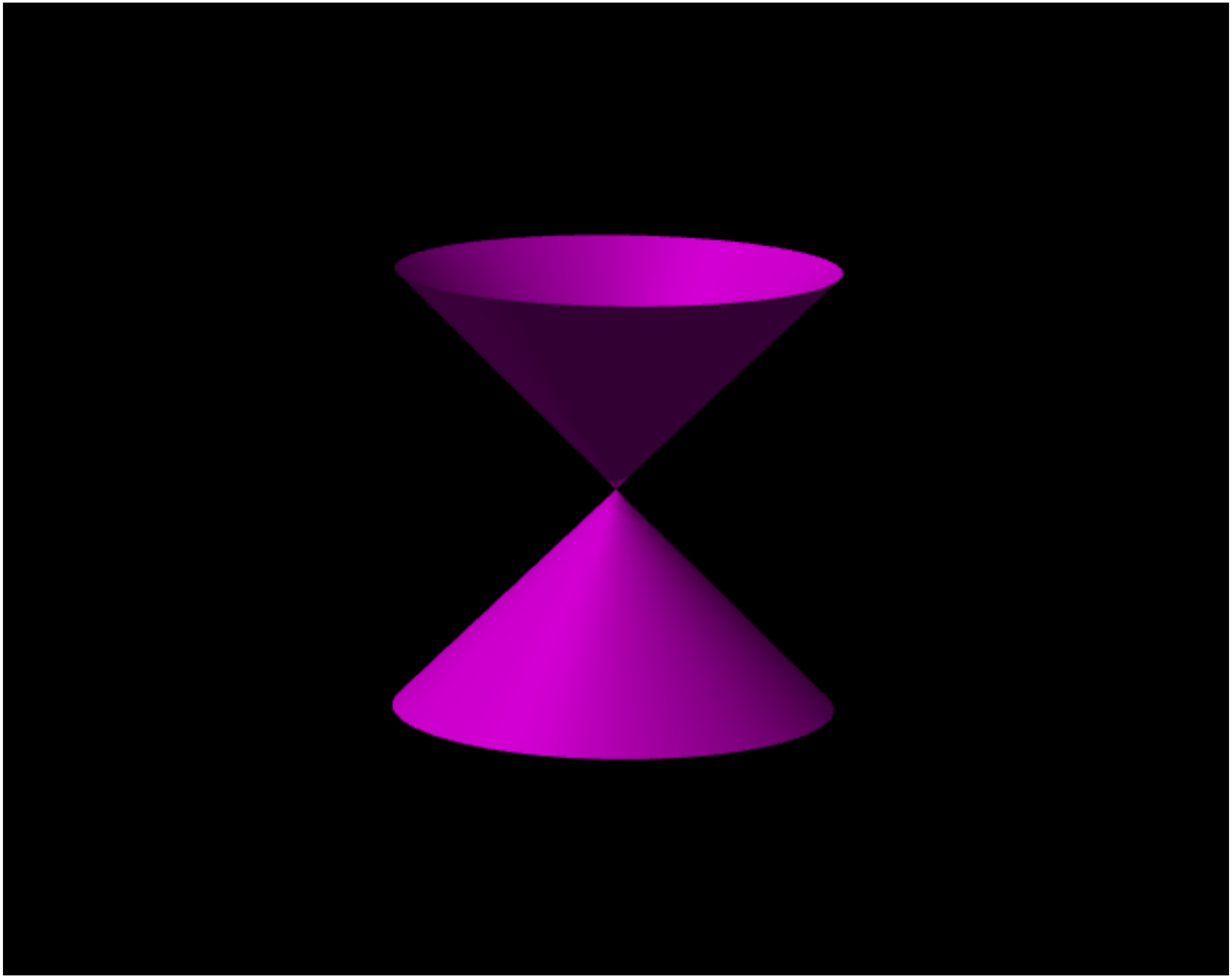

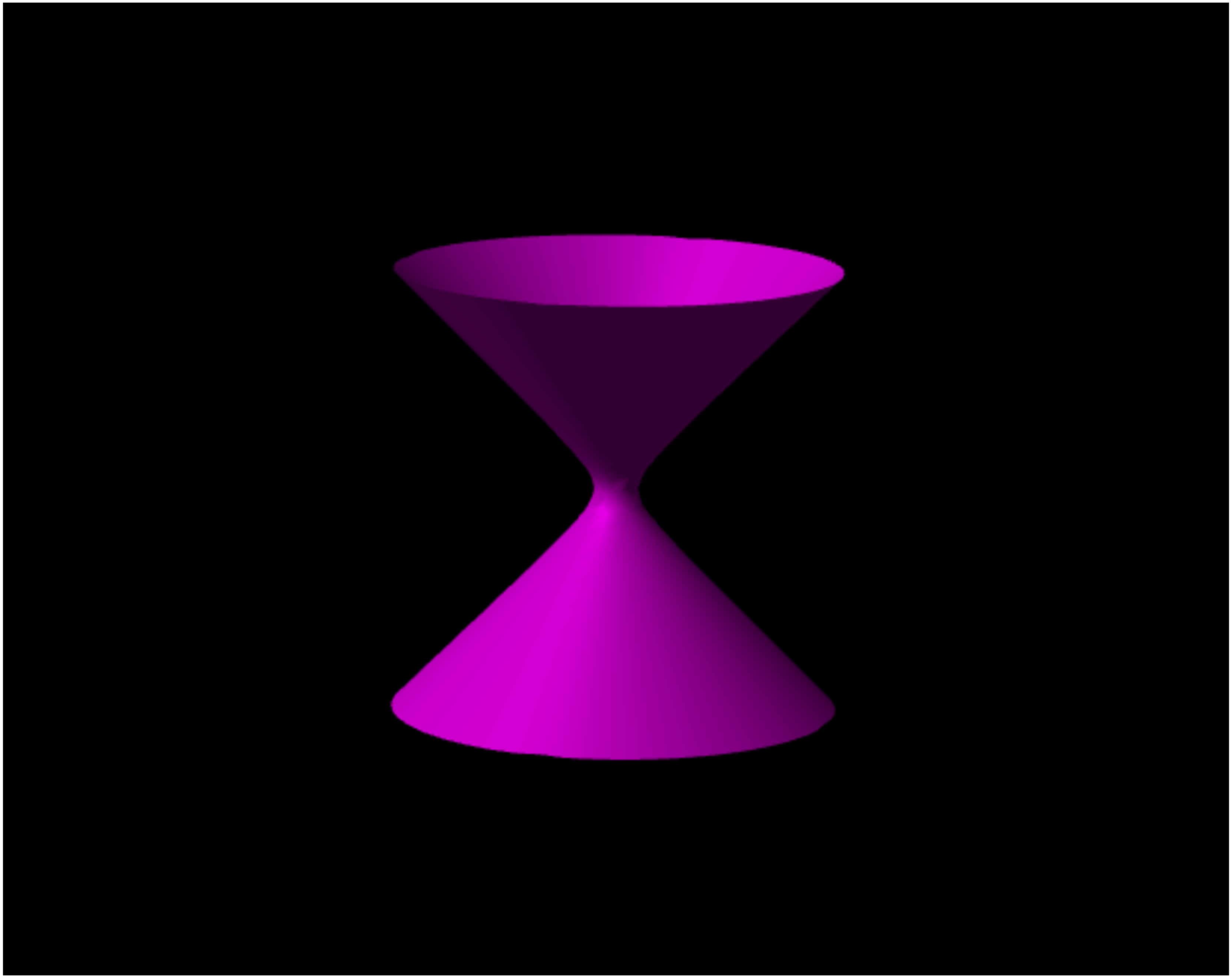

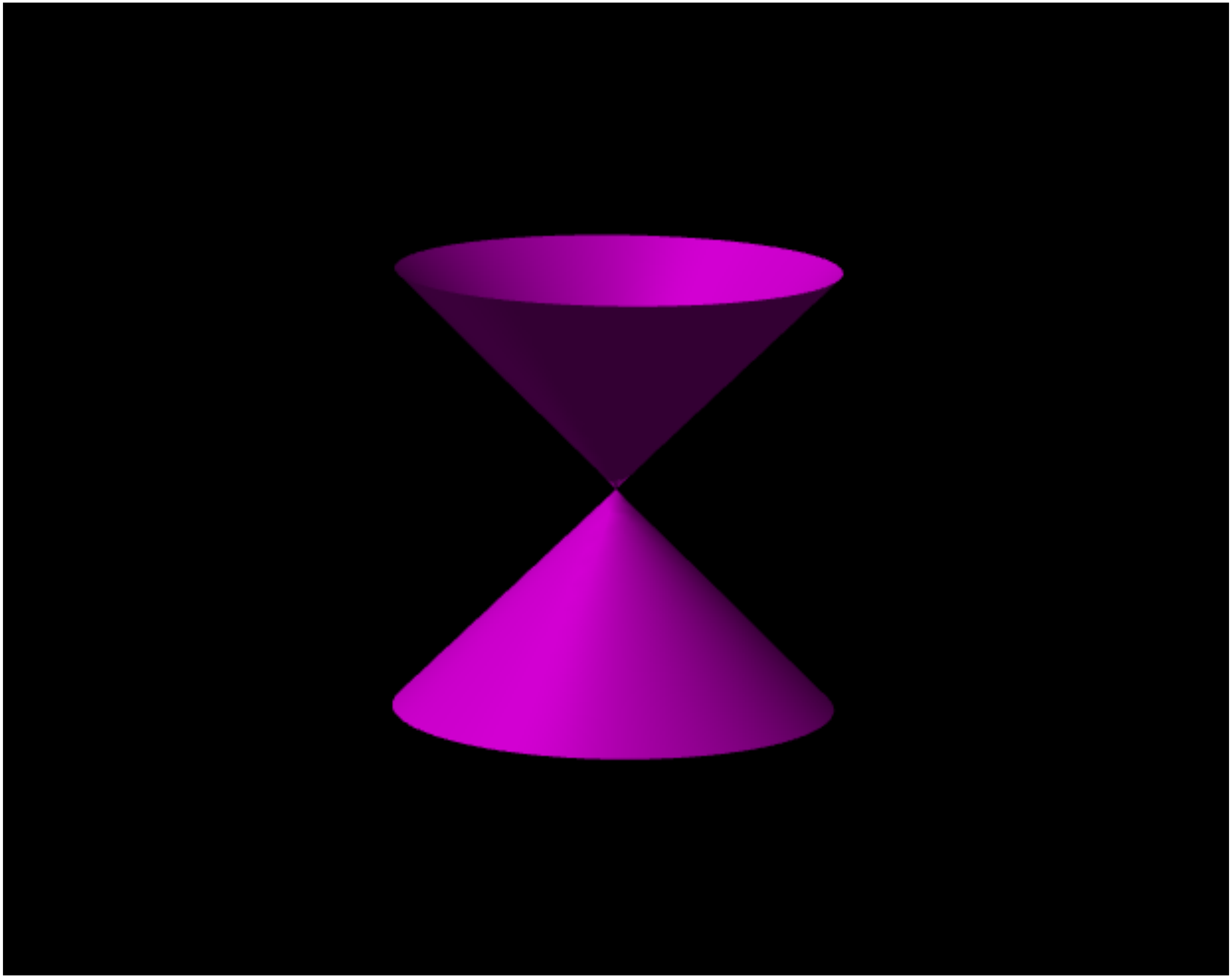

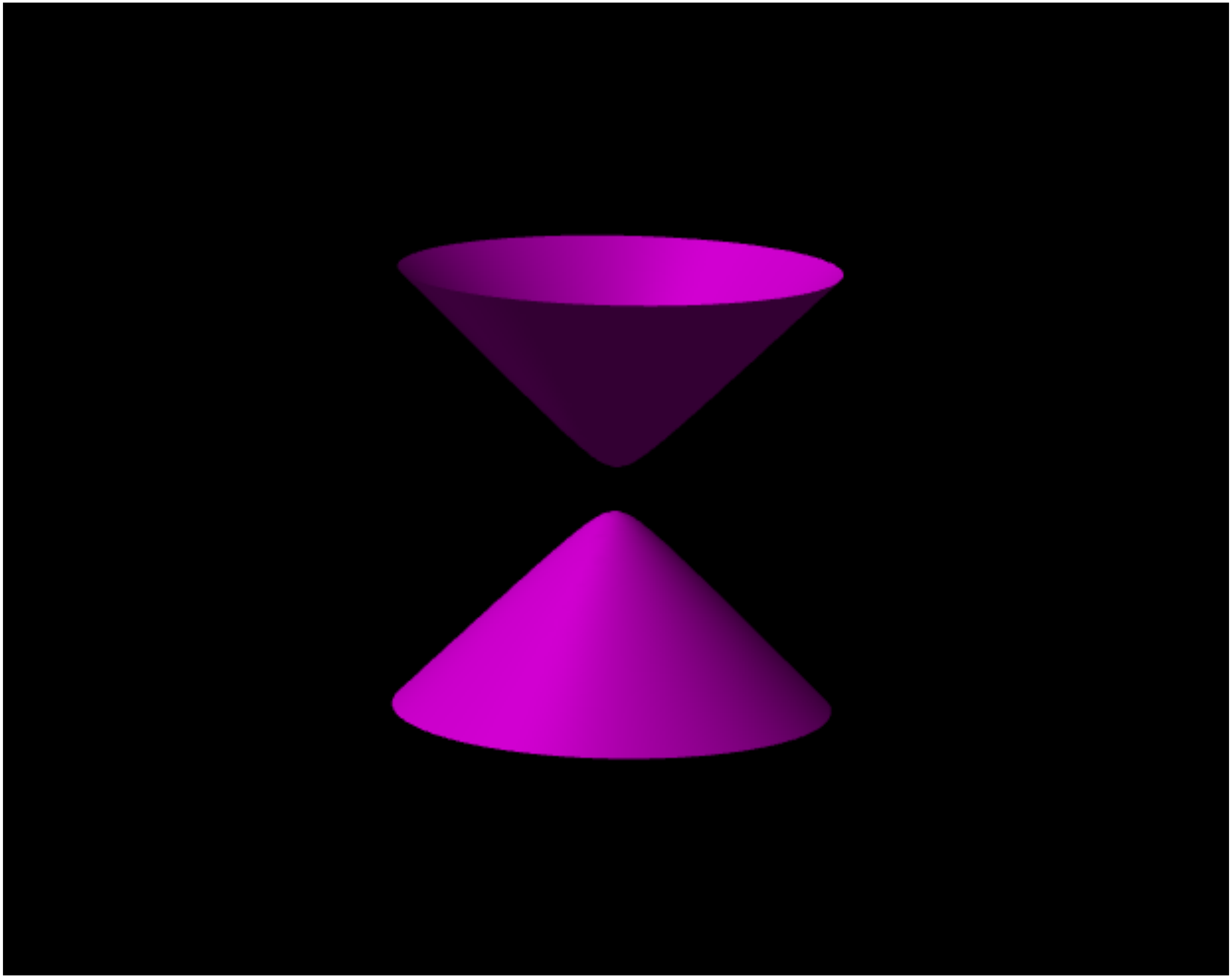

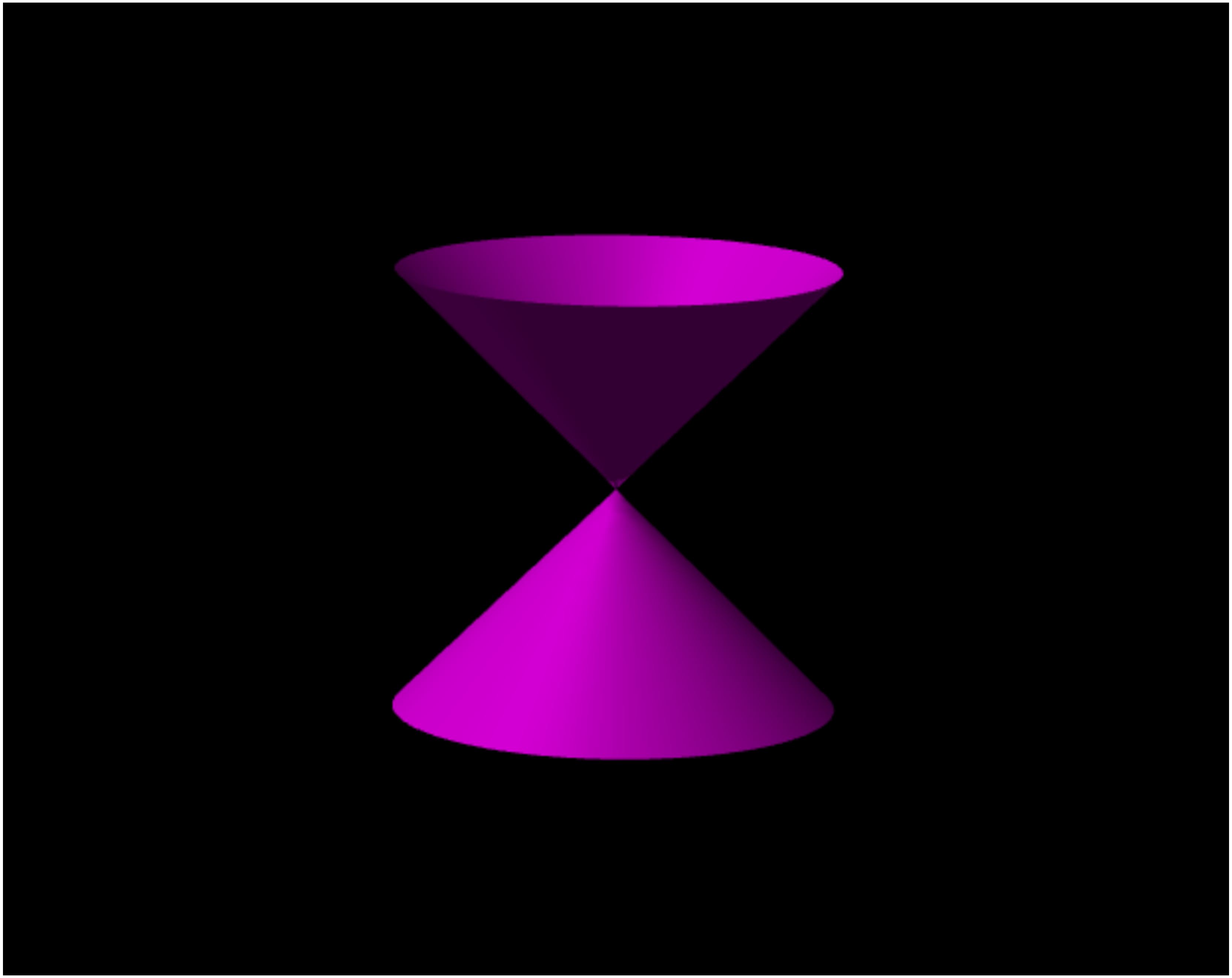

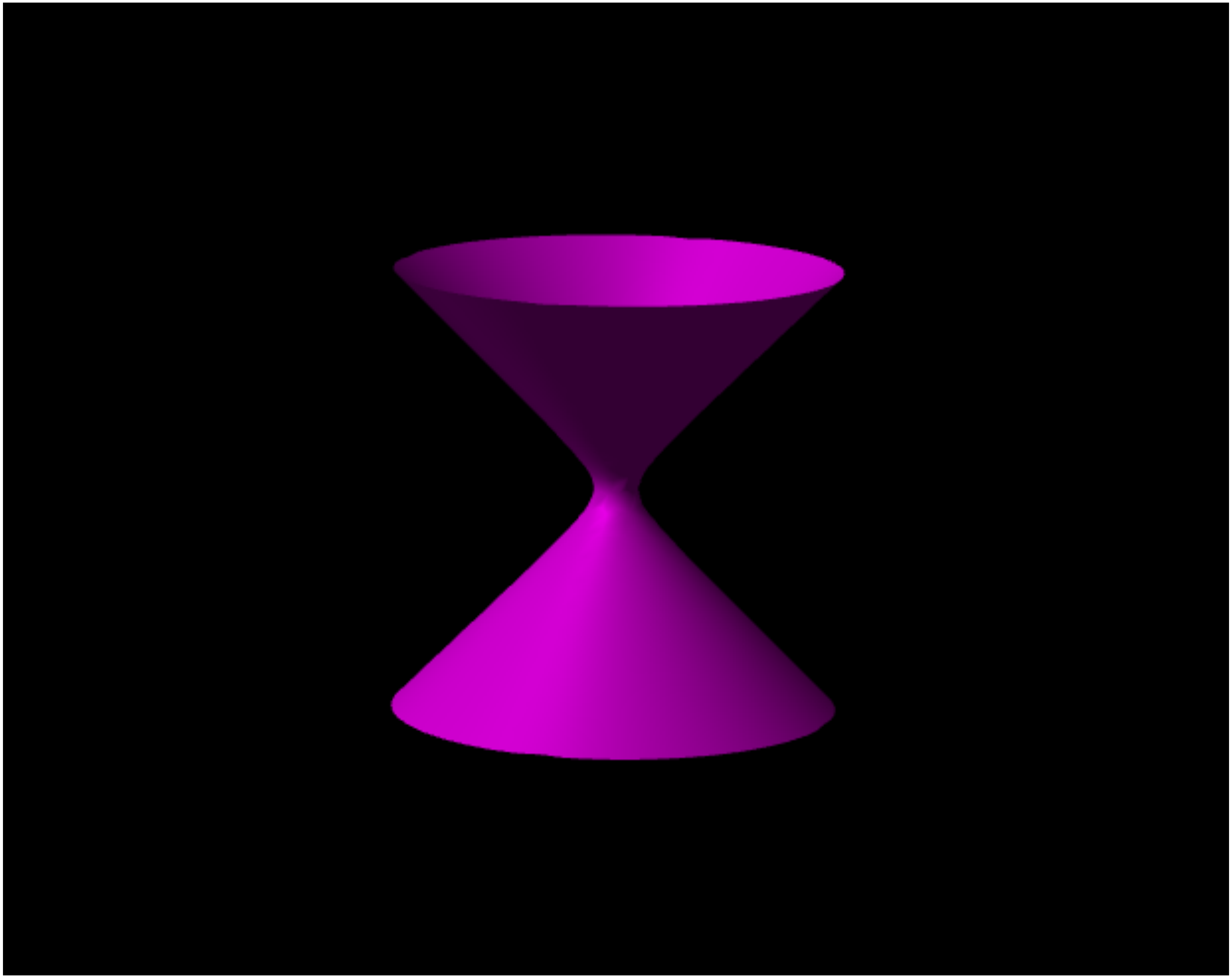

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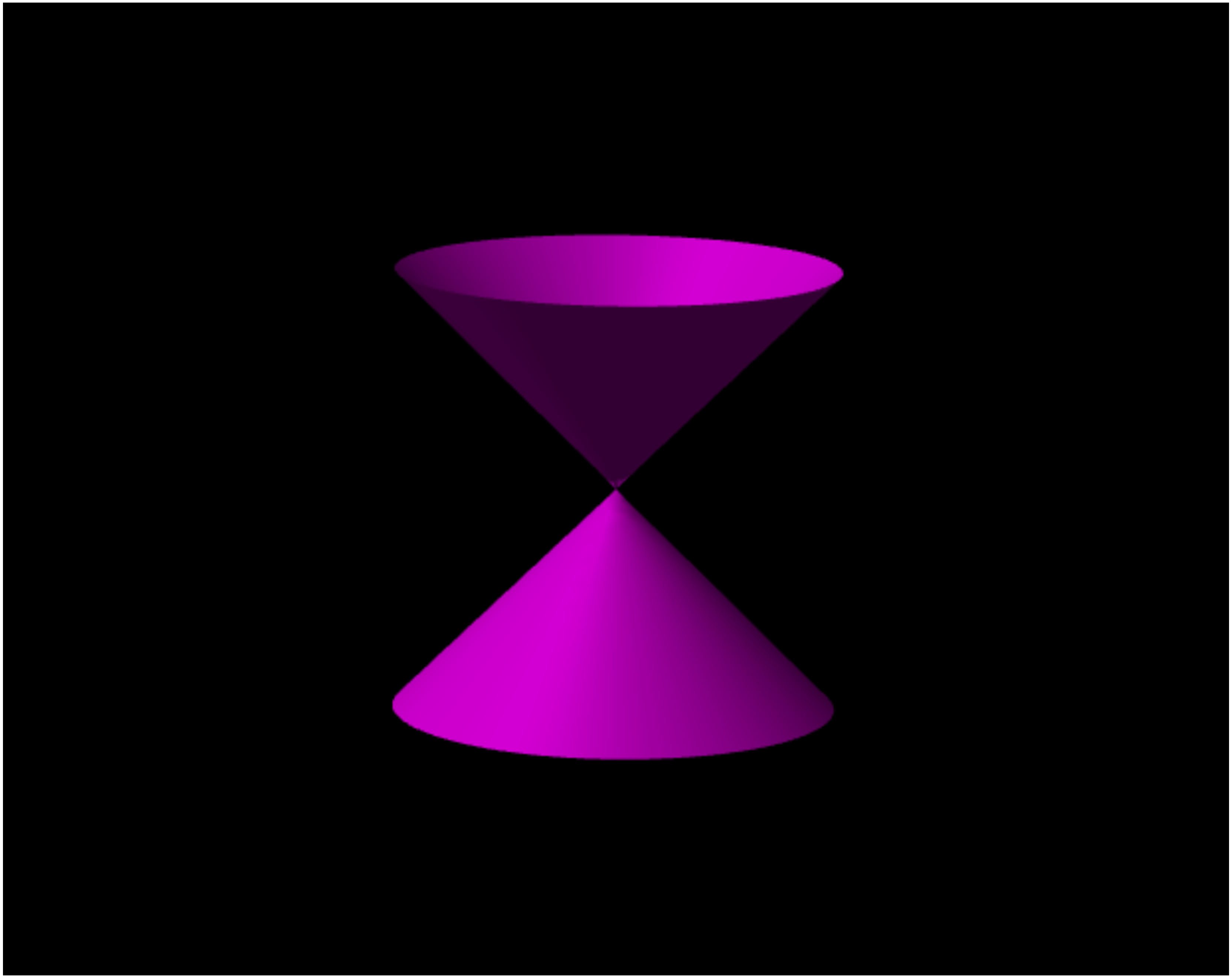
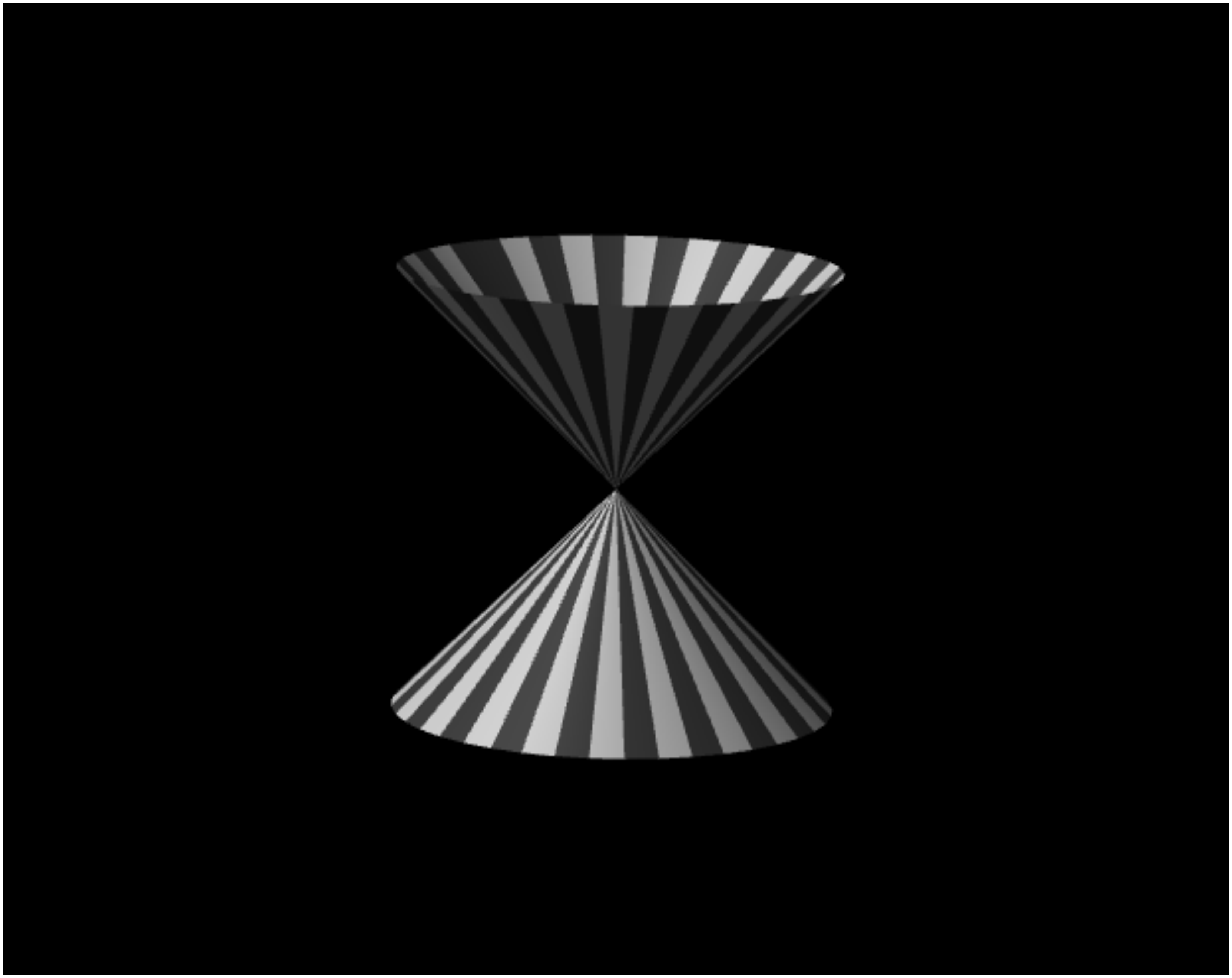
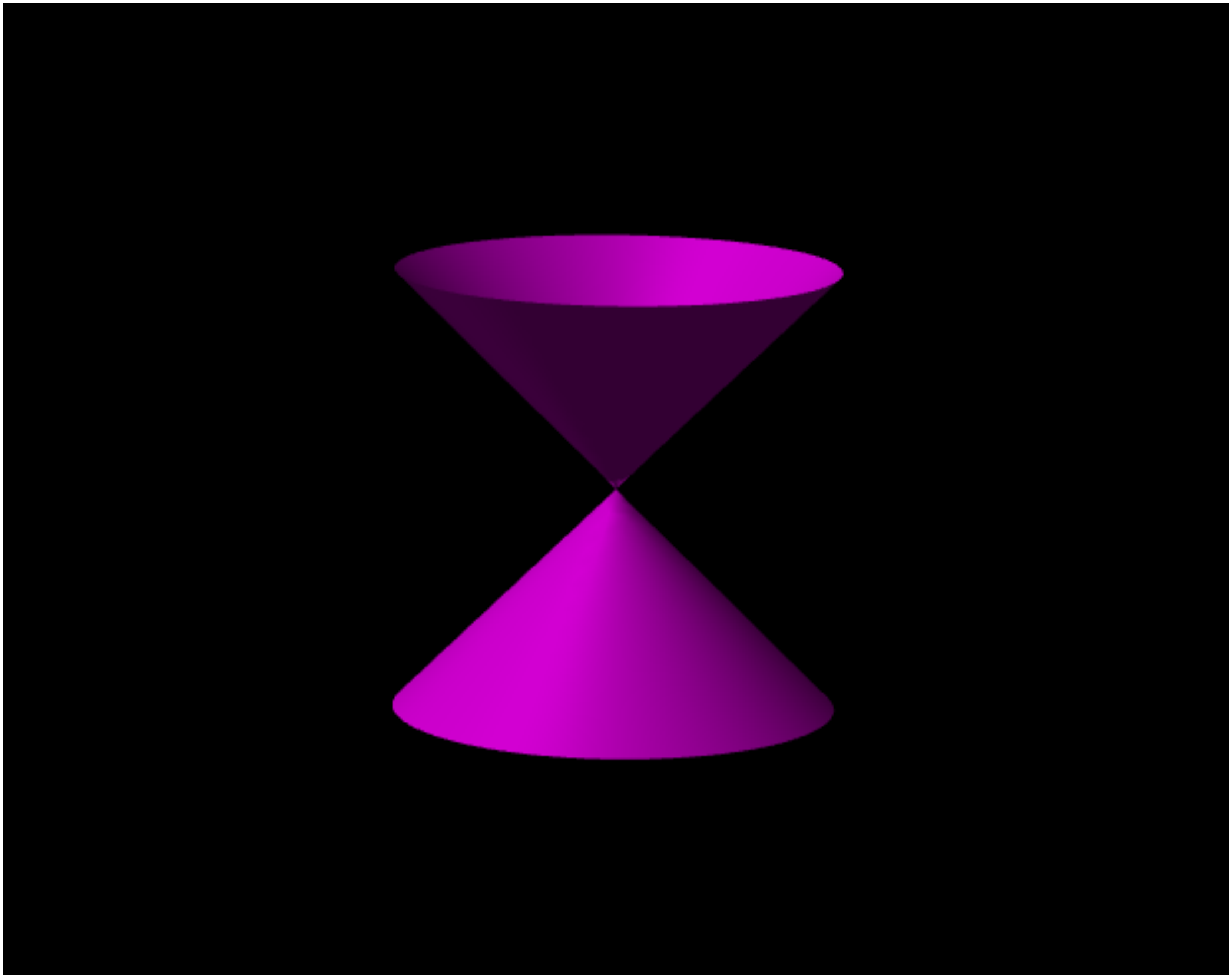
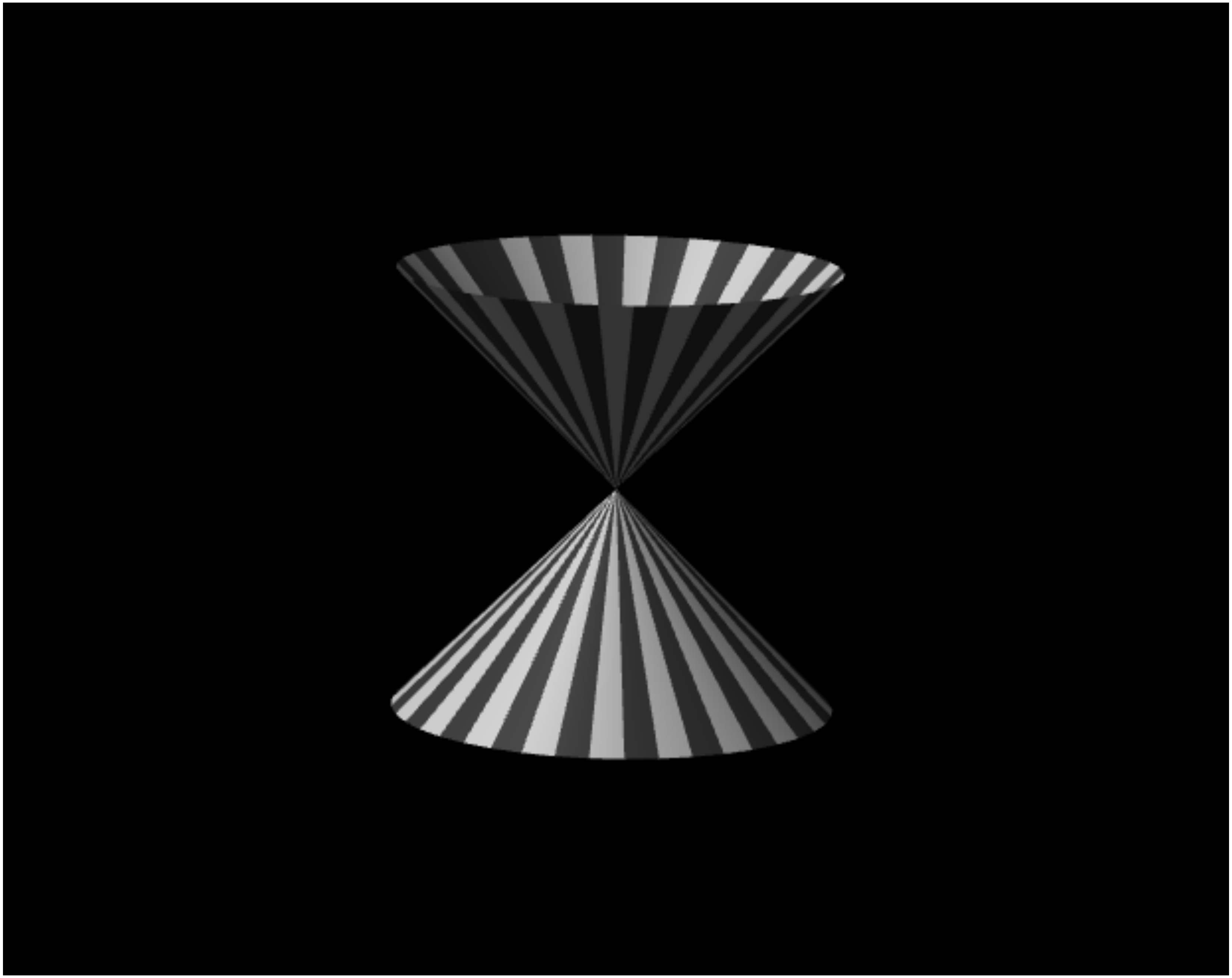
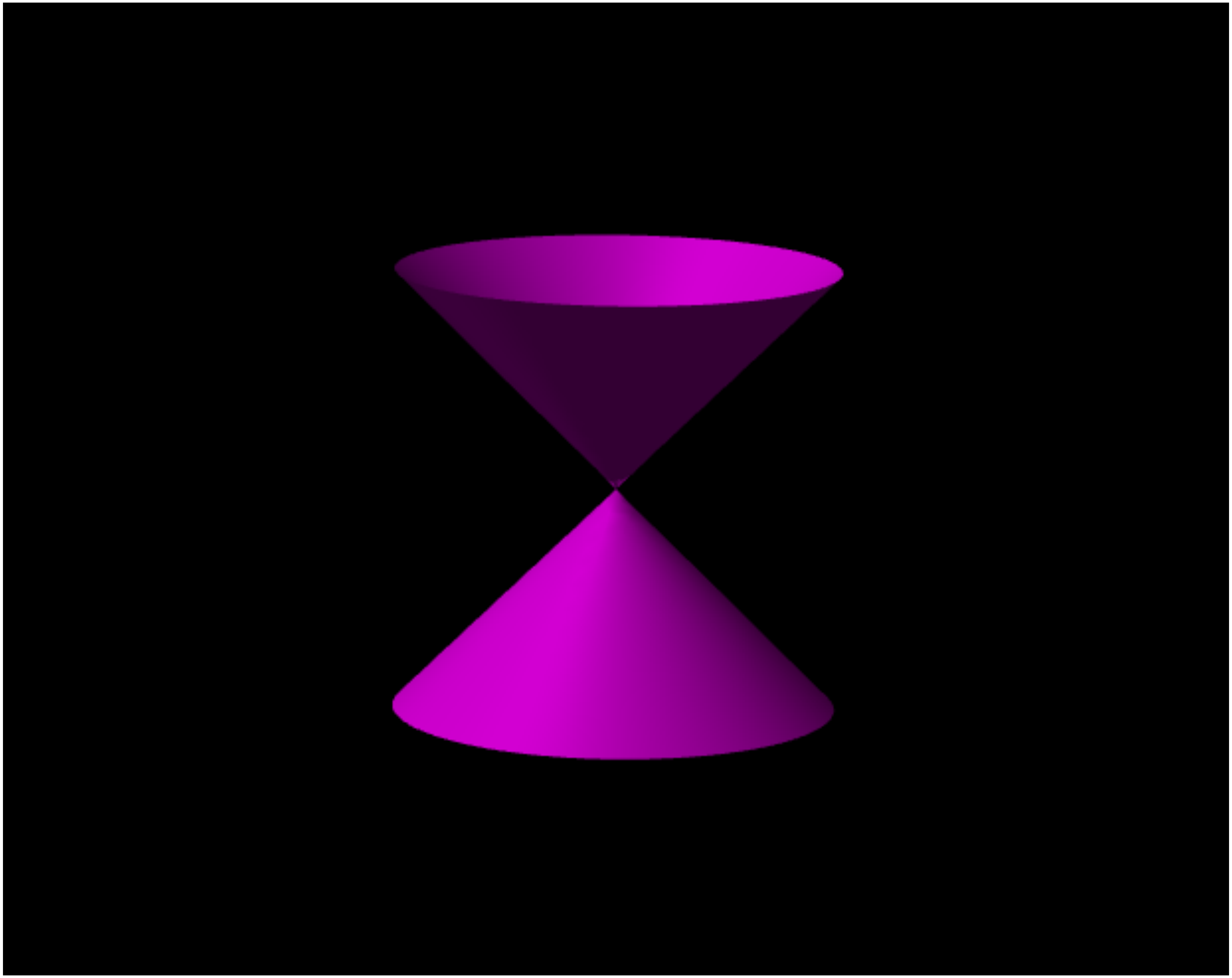
$$\begin{array}{ccc} \mathcal{O}(-2) & \rightarrow & \mathcal{O}(-1) \\ \downarrow & & \downarrow \\ \mathbb{CP}_1 & \hookrightarrow & \mathbb{CP}_2 \end{array}$$

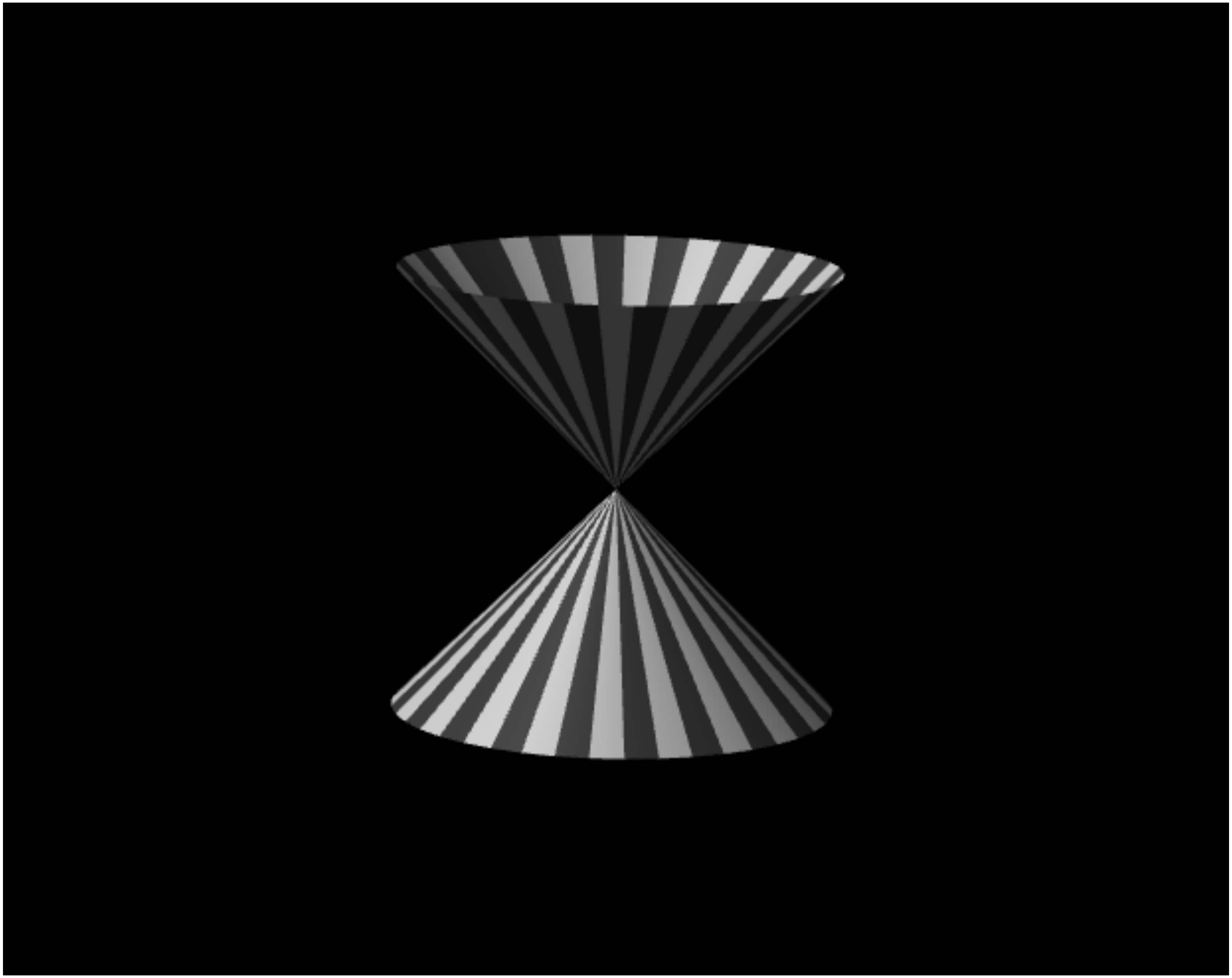












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Gorenstein singularities. Crepant Resolutions.

# Resolutions of Klein Singularities:

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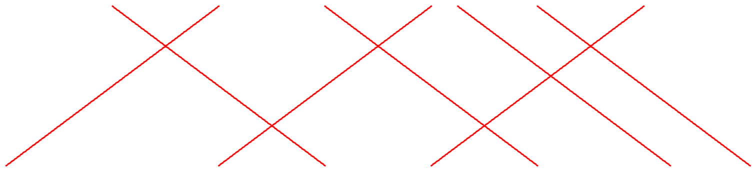

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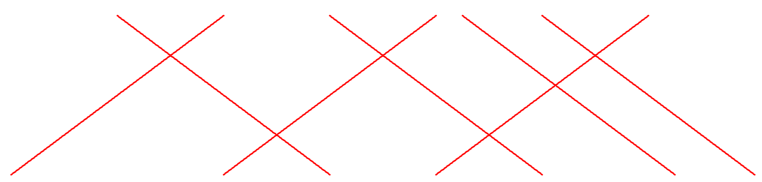

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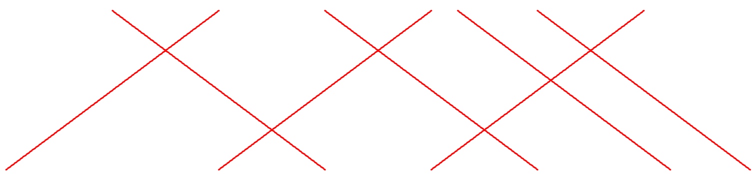

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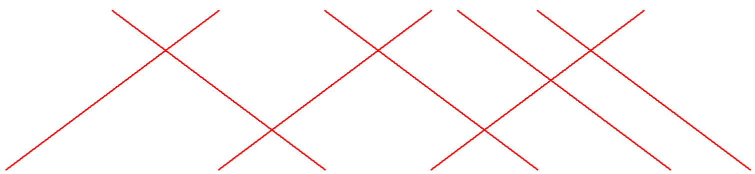

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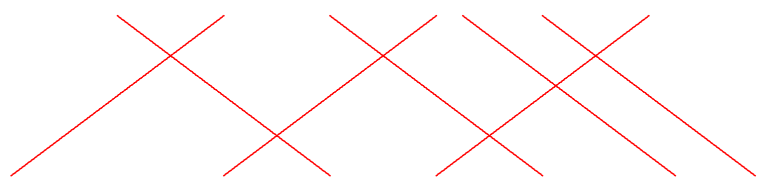
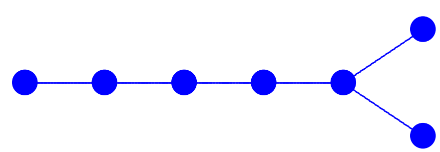


Intersection pattern dual to Dynkin diagram!



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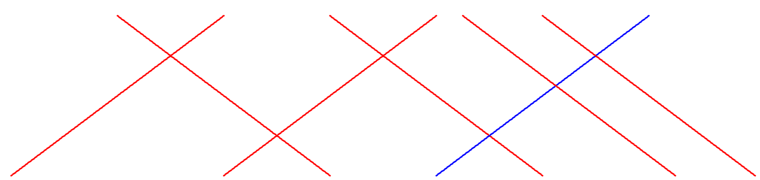

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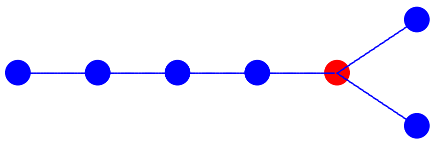

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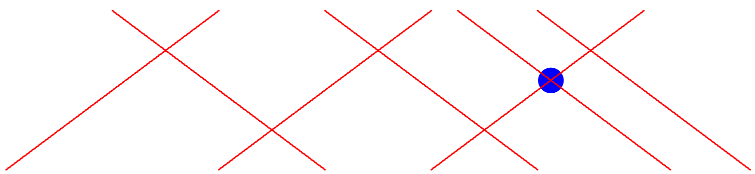

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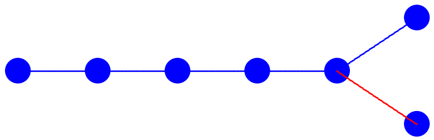

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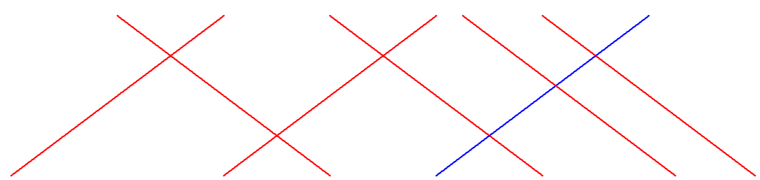

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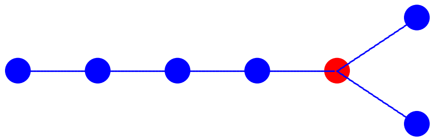

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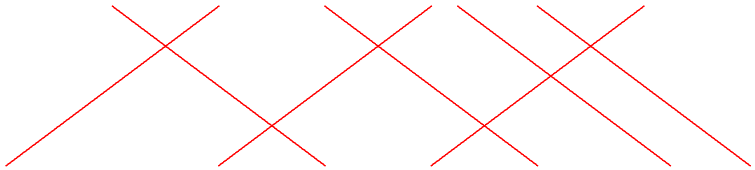

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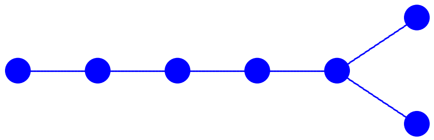
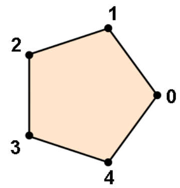
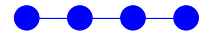
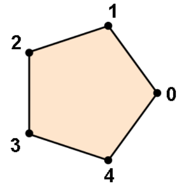





$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$



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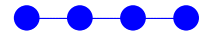
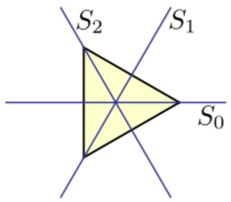
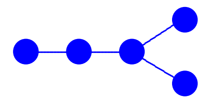
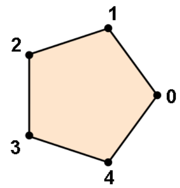





$$\mathrm{Dih}^*_{k-2} \longleftrightarrow D_k$$



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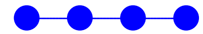

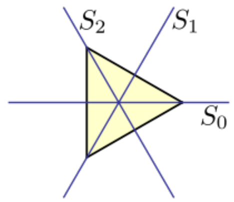


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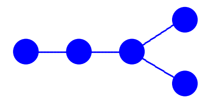
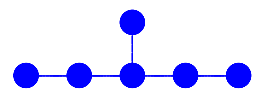
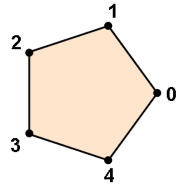


$$T^* \longleftrightarrow E_6$$



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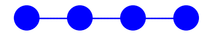

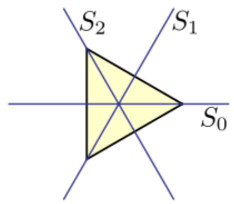


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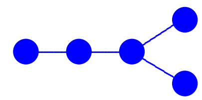
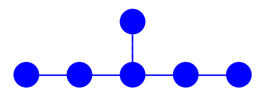
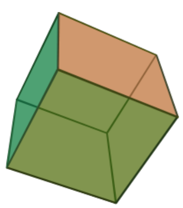
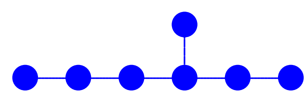


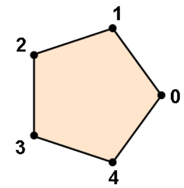
$$T^* \longleftrightarrow E_6$$



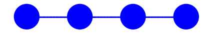


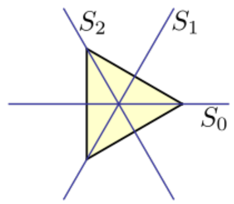
$$O^* \longleftrightarrow E_7$$



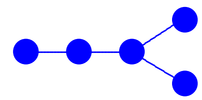



$$\mathbb{Z}_{k+1} \longleftrightarrow A_k$$

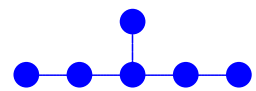


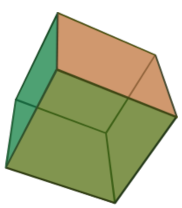



$$\mathrm{Dih}^*_{k-2} \longleftrightarrow D_k$$

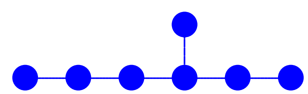


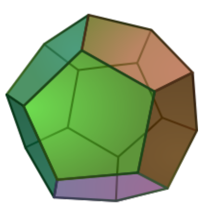
$$T^* \longleftrightarrow E_6$$



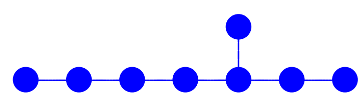


$$O^* \longleftrightarrow E_7$$





$$I^* \longleftrightarrow E_8$$



## Key examples:

Term ALE coined by Gibbons & Hawking, 1979.

They wrote down various explicit Ricci-flat ALE 4-manifolds they called gravitational instantons.

Their examples have just one end, with

$$\Gamma \cong \mathbb{Z}_\ell \subset \mathbf{SU}(2) \subset \mathbf{O}(4).$$

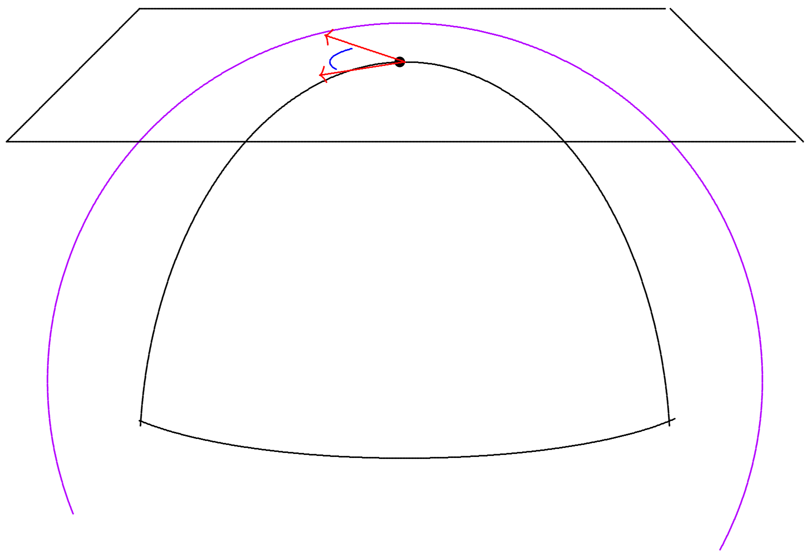
The G-H metrics are hyper-Kähler, and were soon independently rediscovered by Hitchin.

Hitchin conjectured that similar metrics would exist for each finite $\Gamma \subset \mathbf{SU}(2)$.

Proved by Kronheimer, who also showed (1989) this gives complete classification of ALE hyper-Kählers.

Hyper-Kähler metrics:

$(M^4, g)$ hyper-Kähler $\Longleftrightarrow$ holonomy $\subset \mathbf{Sp}(1)$



$$\mathbf{Sp}(1) = \mathbf{SU}(2)$$

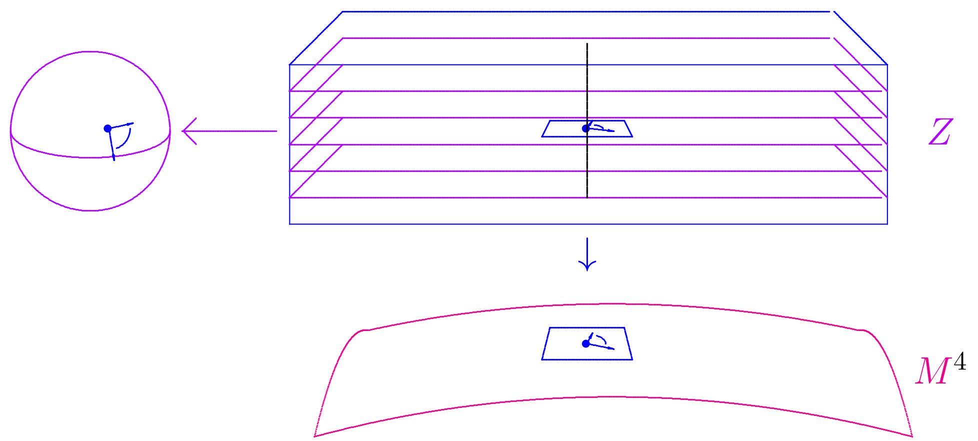
Ricci-flat and Kähler,

for many different complex structures!

All these complex structures can be repackaged as

**Penrose Twistor Space** $(Z^6, J)$,

which is a complex 3-manifold.



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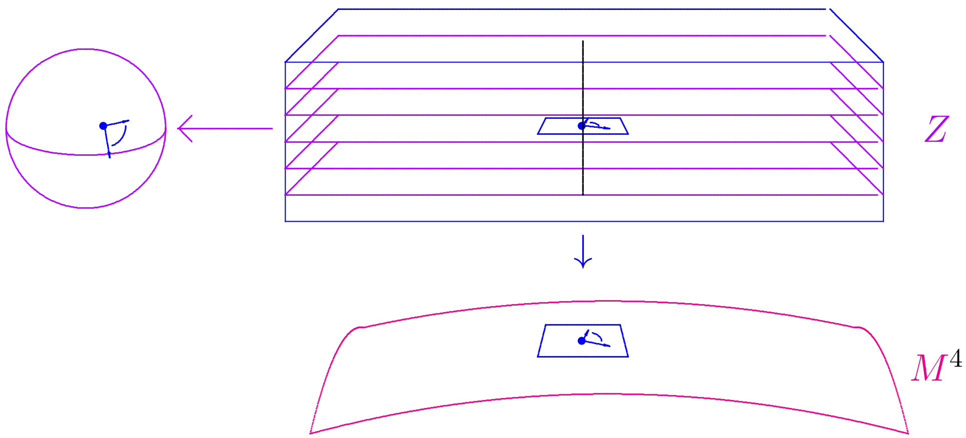
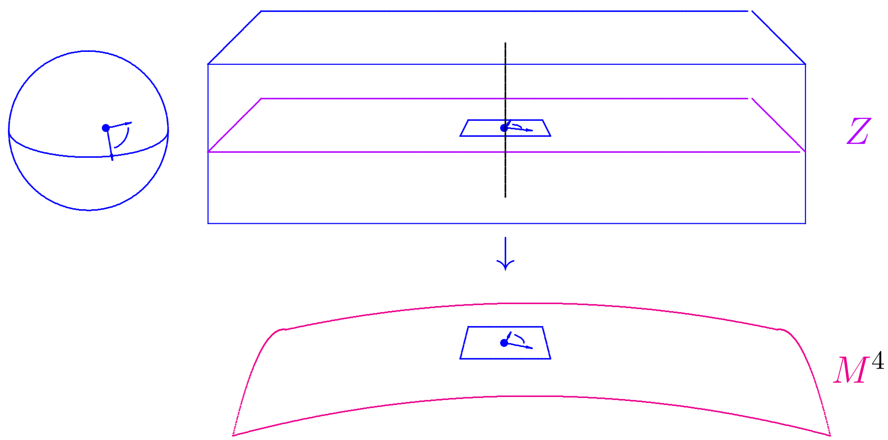



But similar for scalar-flat Kähler surfaces $(M^4, g, J)$!

Any scalar-flat Kähler surface $(M^4, g, J)$ has a

**Penrose Twistor Space** $(Z^6, J)$,

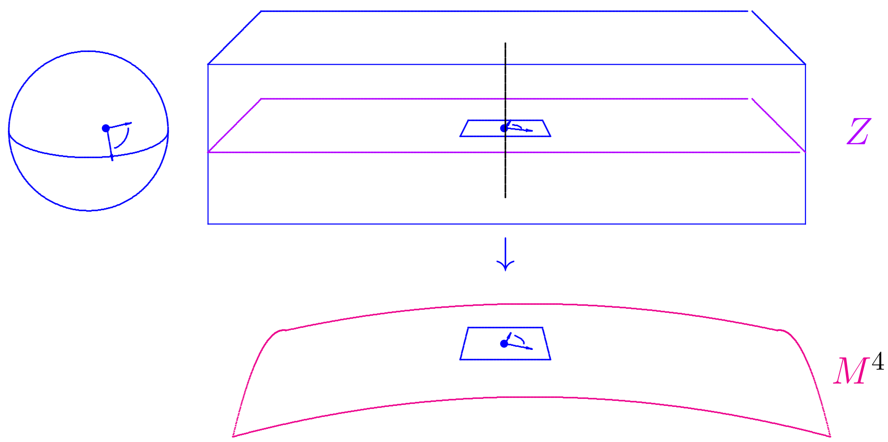
which is once again a complex 3-manifold.



Any scalar-flat Kähler surface $(M^4, g, J)$ has a

**Penrose Twistor Space** $(Z^6, J)$,

which is once again a complex 3-manifold.

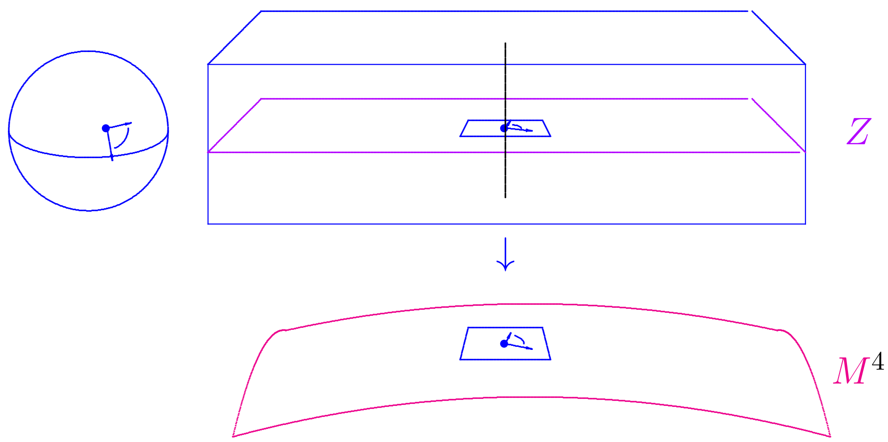



Integrability condition for twistor space: $W_+ \equiv 0$.

Any scalar-flat Kähler surface $(M^4, g, J)$ has a

**Penrose Twistor Space** $(Z^6, J)$,

which is once again a complex 3-manifold.



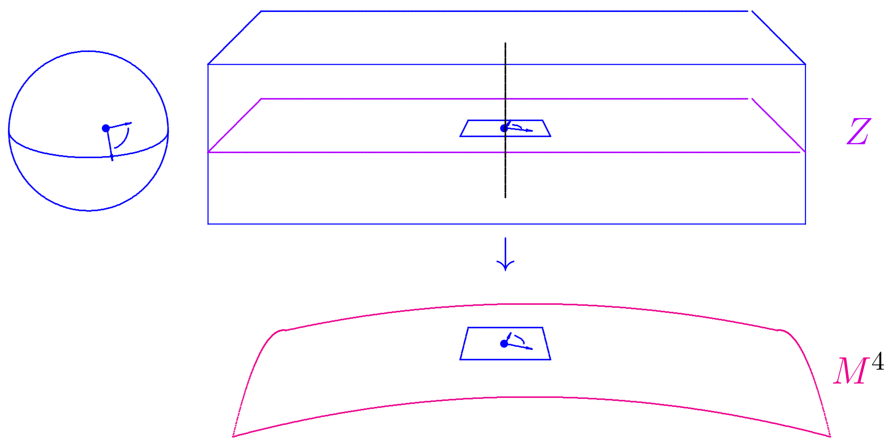

Integrability condition for twistor space: $W_+ \equiv 0$.

For Kähler surfaces, $|W_+|^2 = s^2/24$.

Any scalar-flat Kähler surface $(M^4, g, J)$ has a

**Penrose Twistor Space** $(Z^6, J)$,

which is once again a complex 3-manifold.




Integrability condition for twistor space: $W_+ \equiv 0$.

For Kähler surfaces, integrable $\Longleftrightarrow$ scalar-flat!

Any scalar-flat Kähler surface $(M^4, g, J)$ has a

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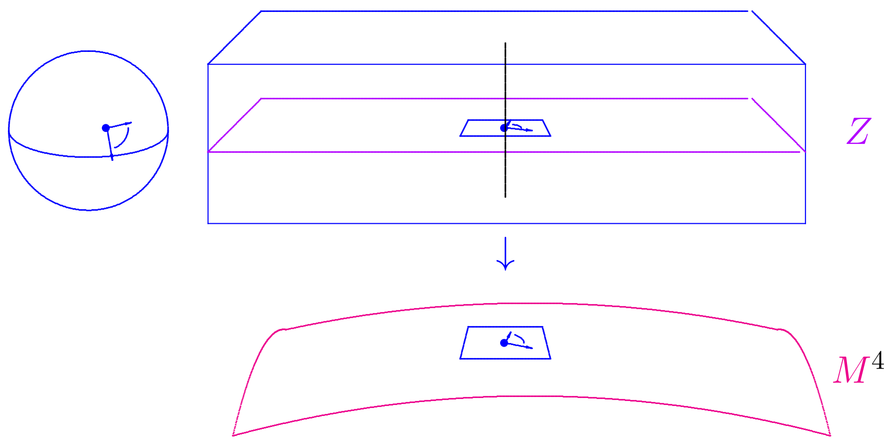
which is once again a complex 3-manifold.




Integrability condition for twistor space: $W_+ \equiv 0$.

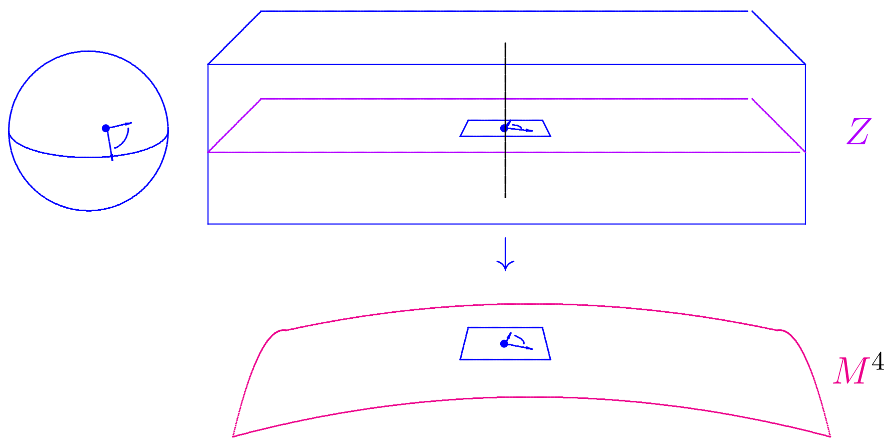
For Kähler surfaces, integrable $\Longleftrightarrow$ scalar-flat!

Leads to constructions of explicit examples.

Any scalar-flat Kähler surface $(M^4, g, J)$ has a

**Penrose Twistor Space** $(Z^6, J)$,

which is once again a complex 3-manifold.




Integrability condition for twistor space: $W_+ \equiv 0$.

For Kähler surfaces, integrable $\Longleftrightarrow$ scalar-flat!

Many simple examples are AE or ALE.

## Some AE Scalar-Flat Kähler Surfaces:

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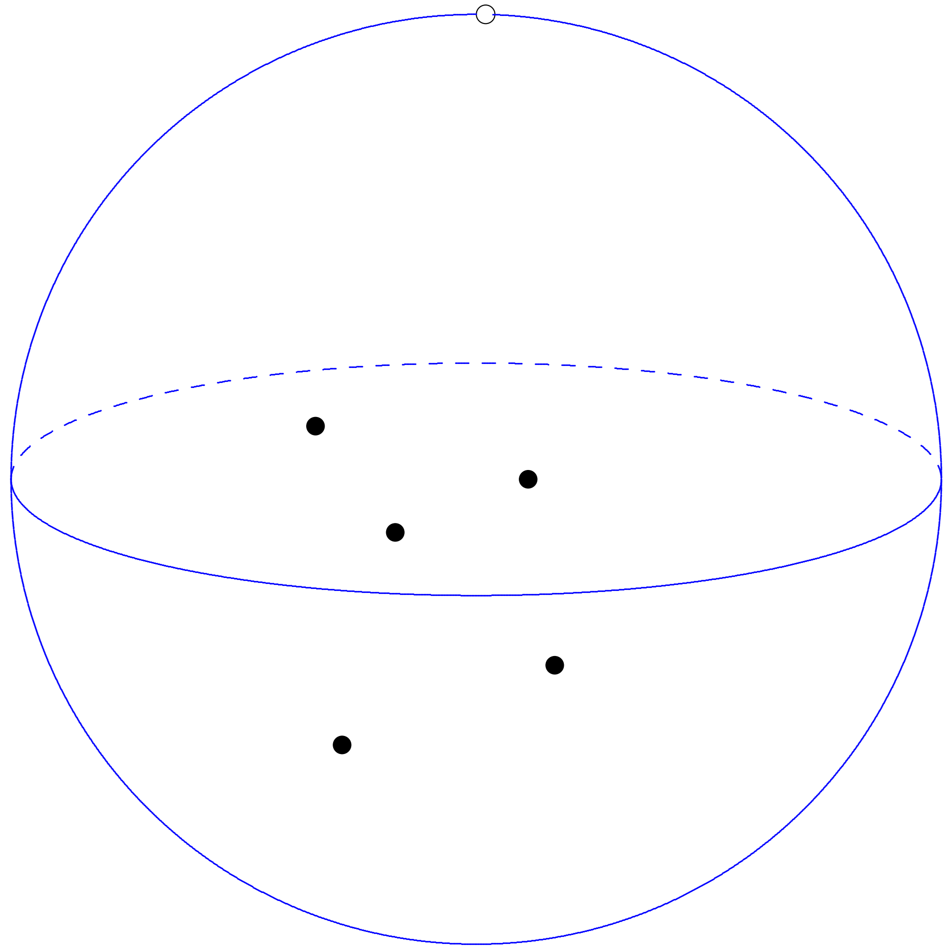
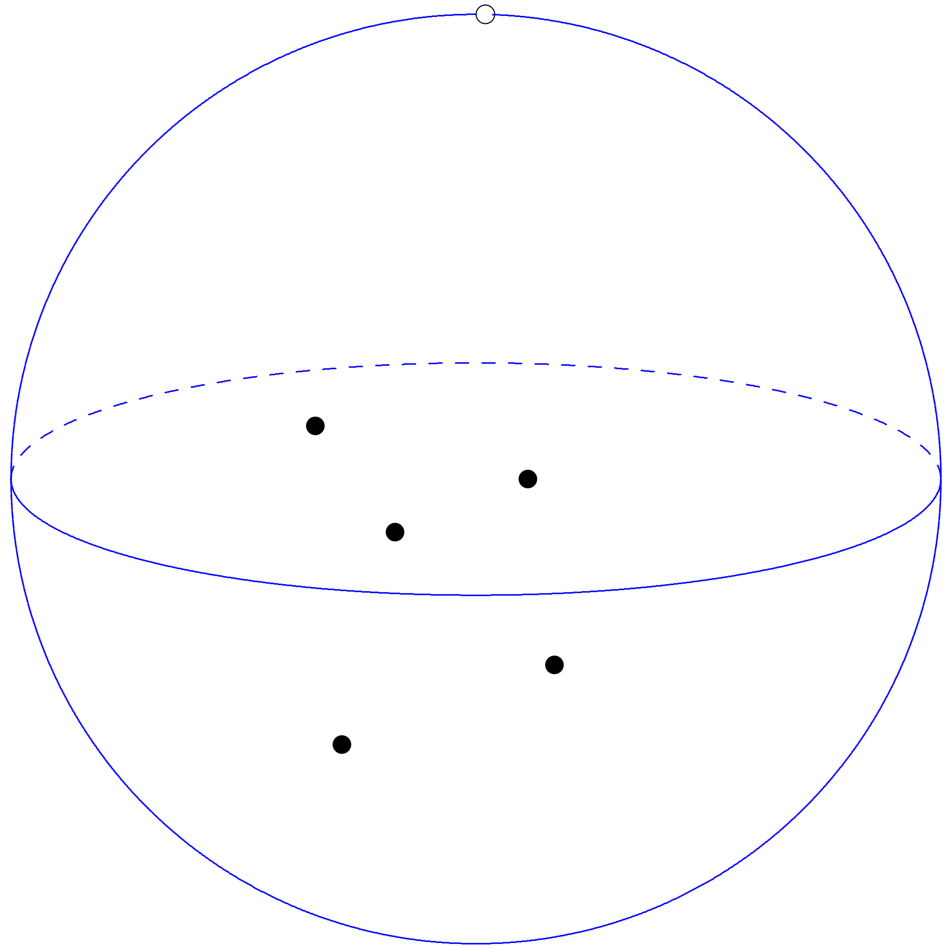
**(L '91)**

# Some AE Scalar-Flat Kähler Surfaces:

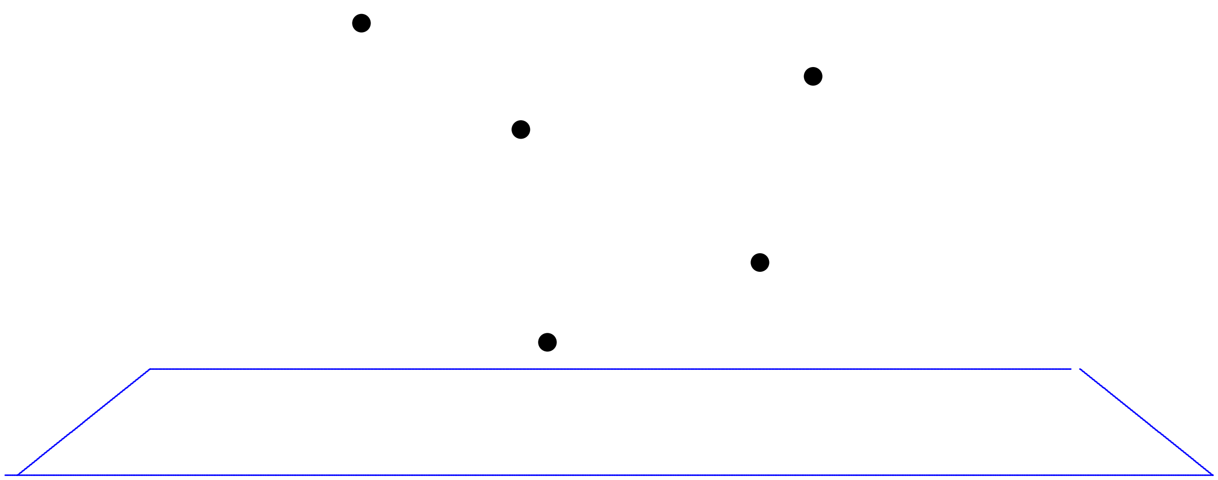


## Some AE Scalar-Flat Kähler Surfaces:



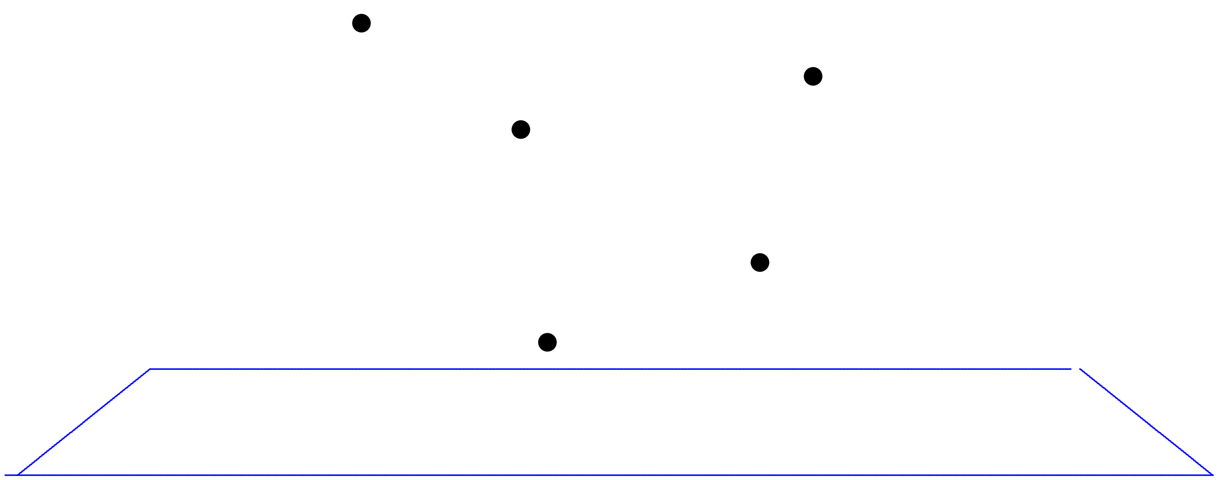
Data: $k$ points in $\mathcal{H}^3$ and one point at infinity.

## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$ = upper half-space model.

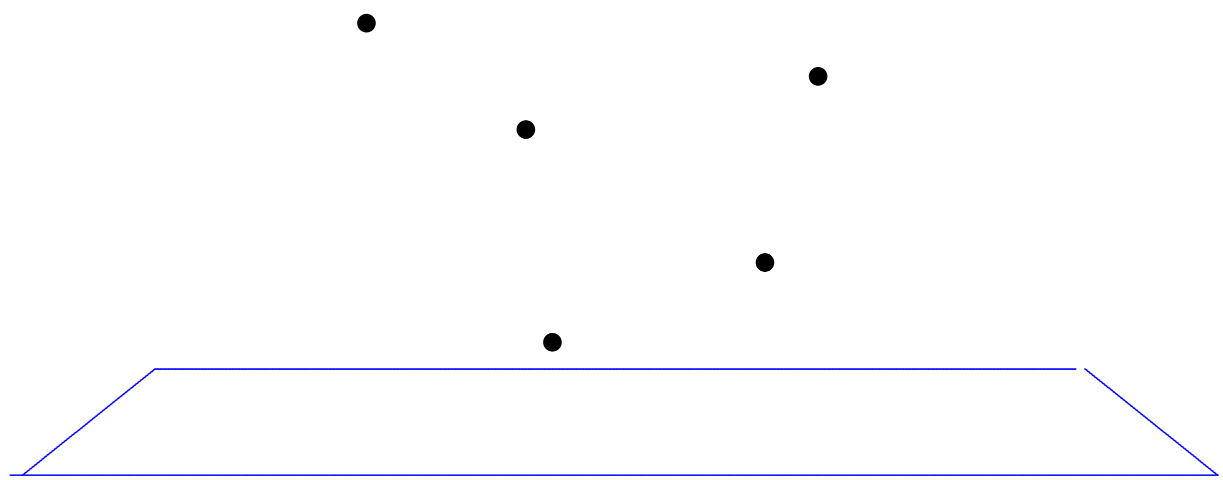
## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$V = 1 + \sum_{j=1}^{k} G_j$$

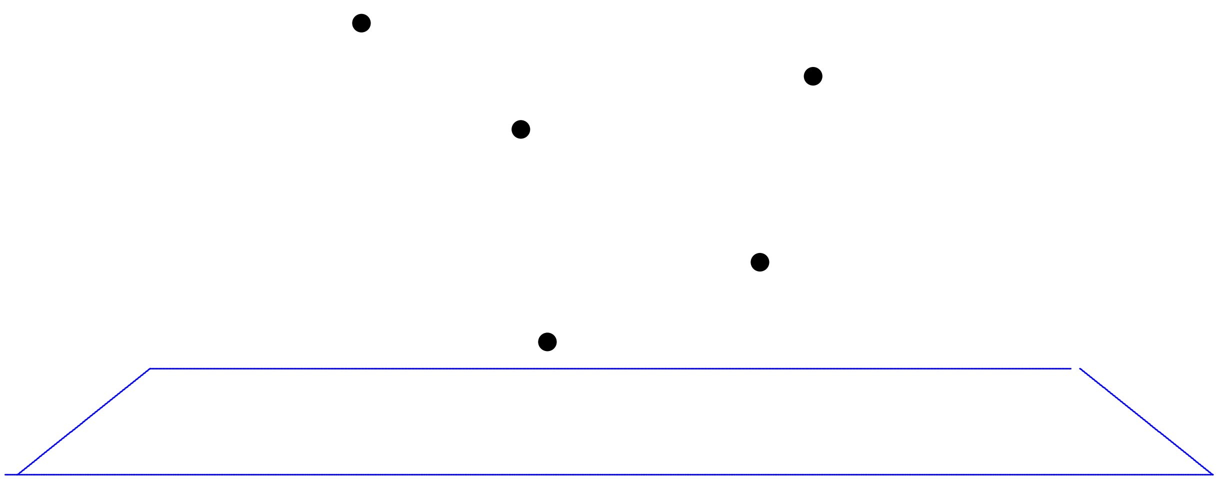
## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$V = 1 + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

## Some AE Scalar-Flat Kähler Surfaces:



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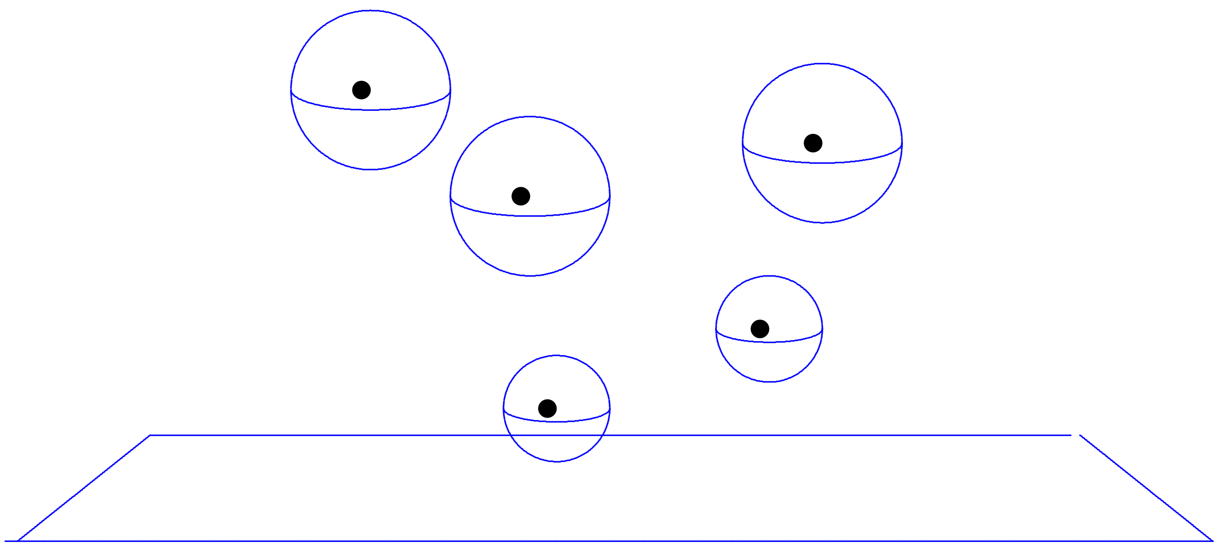
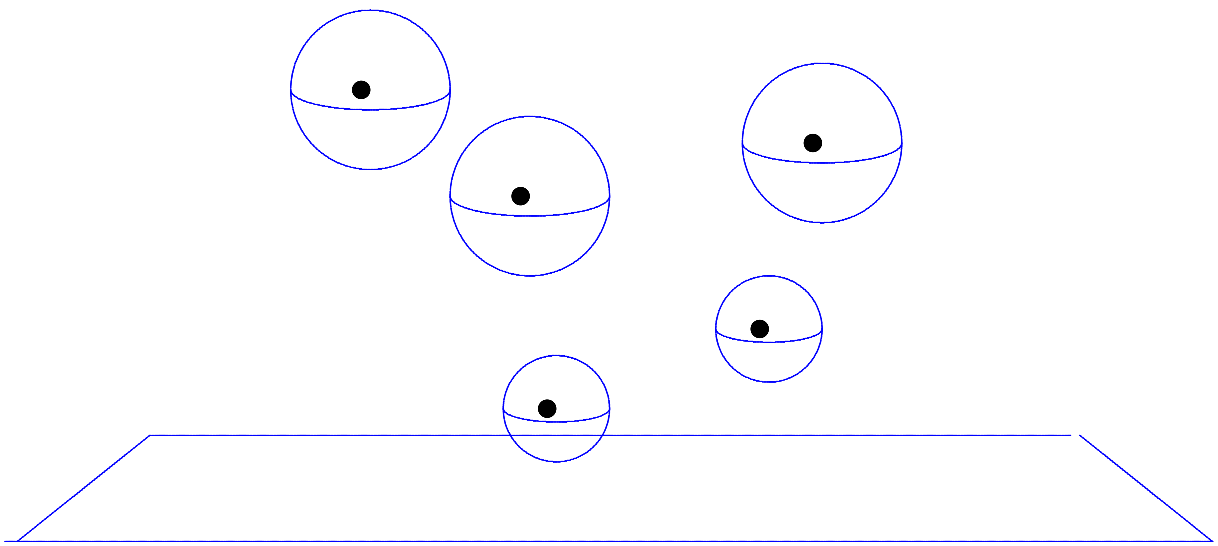


Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$F = \star dV$ curvature $\theta$ on $P \to \mathcal{H}^3 - \{\text{pts}\}$.

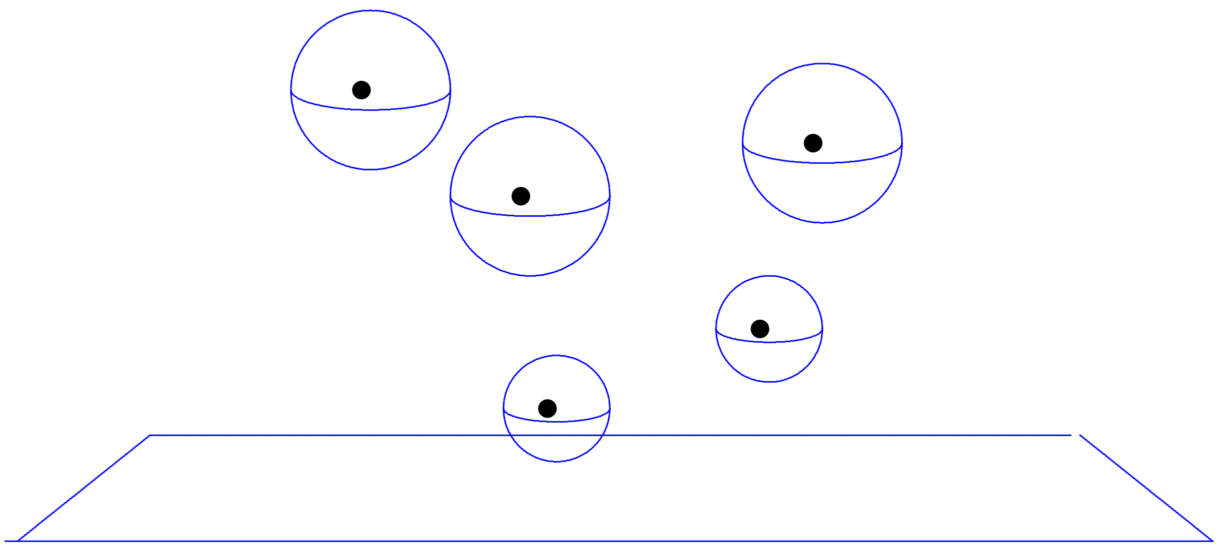
## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( V h + V^{-1} \theta^2 \right)$$

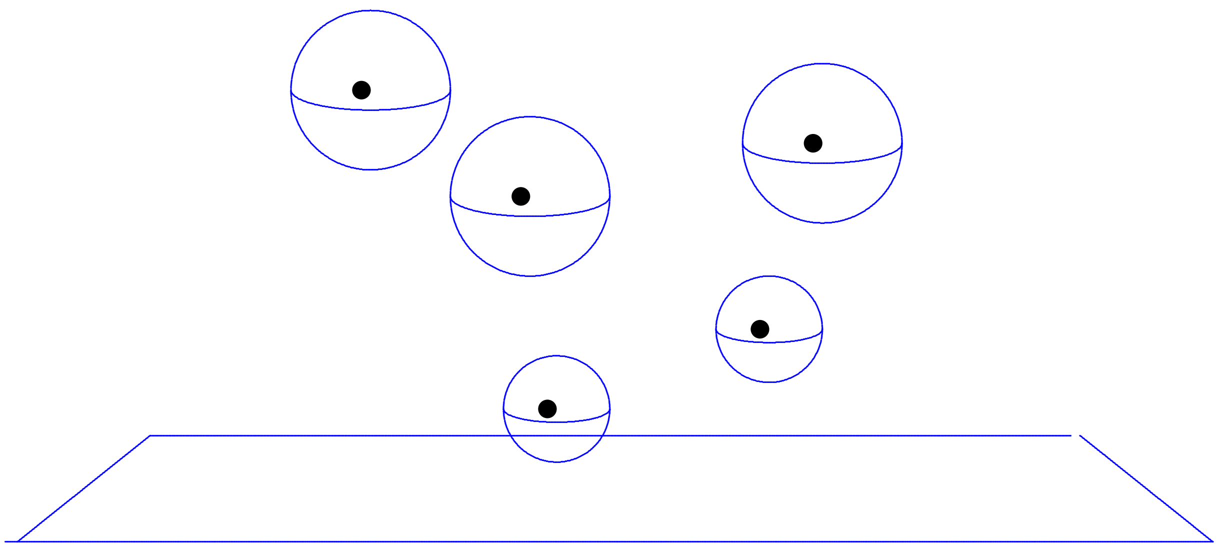
# Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( V \frac{dx^2 + dy^2 + dz^2}{z^2} + V^{-1}\theta^2 \right)$$

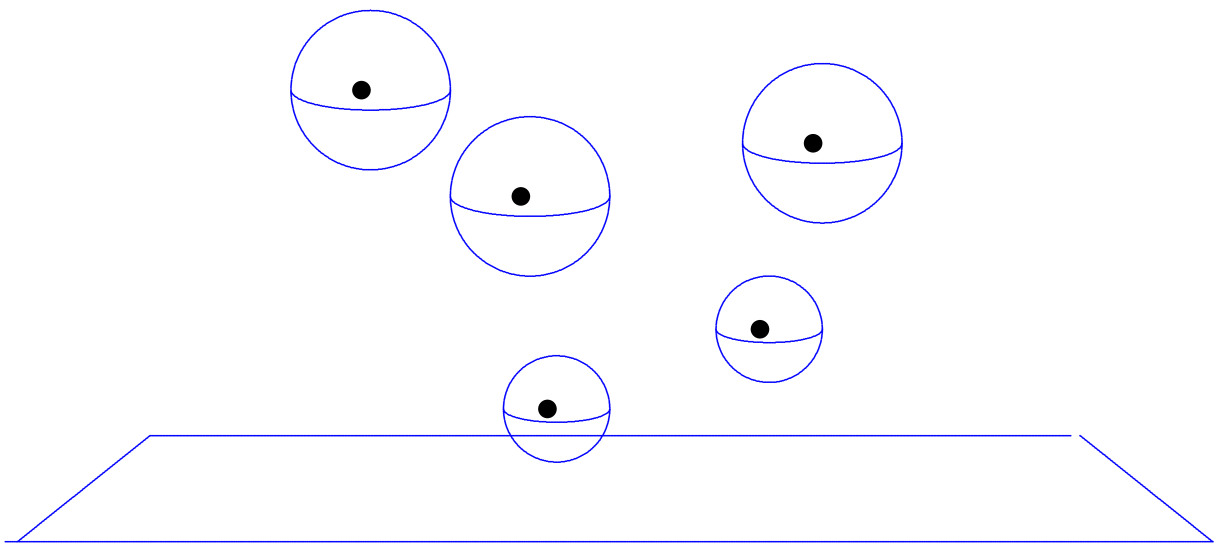
# Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = V(dx^2 + dy^2 + dz^2) + z^2 V^{-1} \theta^2$$

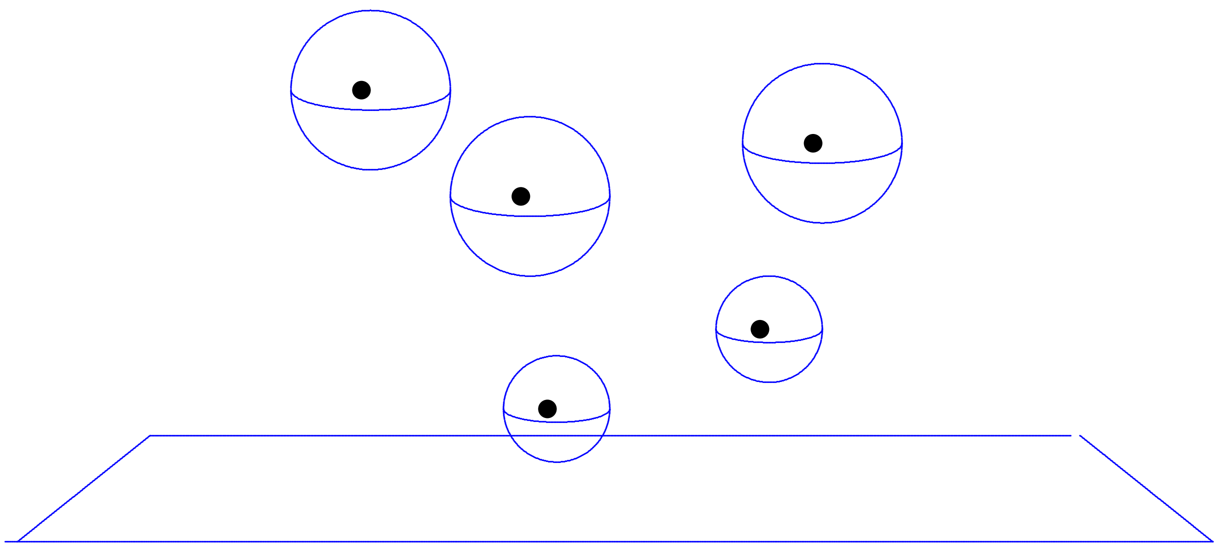
## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = z^2\left(Vh + V^{-1}\theta^2\right)$$

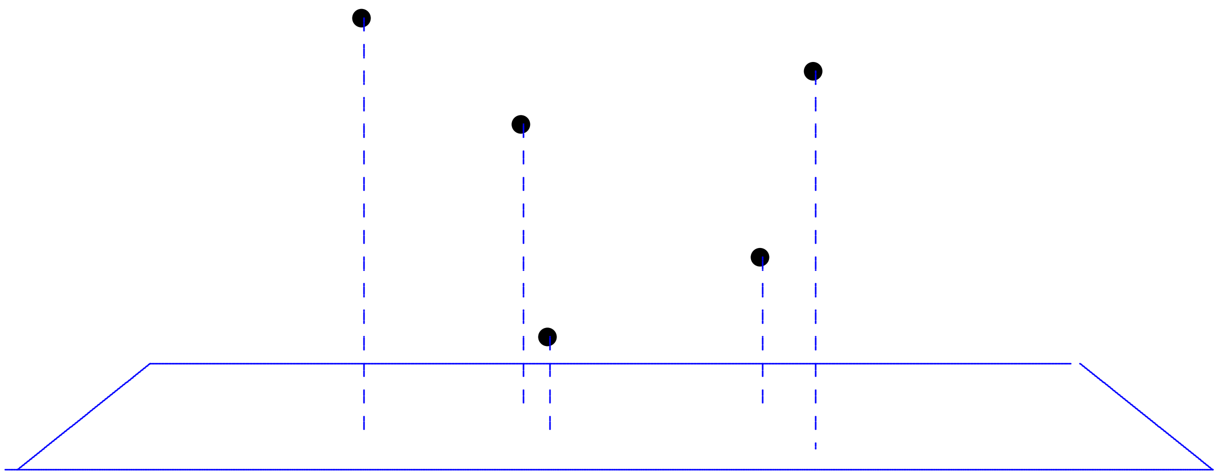
## Some AE Scalar-Flat Kähler Surfaces:



Riemannian completion is AE scalar-flat Kähler.

$$g = z^2 \left( Vh + V^{-1}\theta^2 \right)$$

## Some AE Scalar-Flat Kähler Surfaces:



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$$g = z^2 \left( V h + V^{-1} \theta^2 \right)$$

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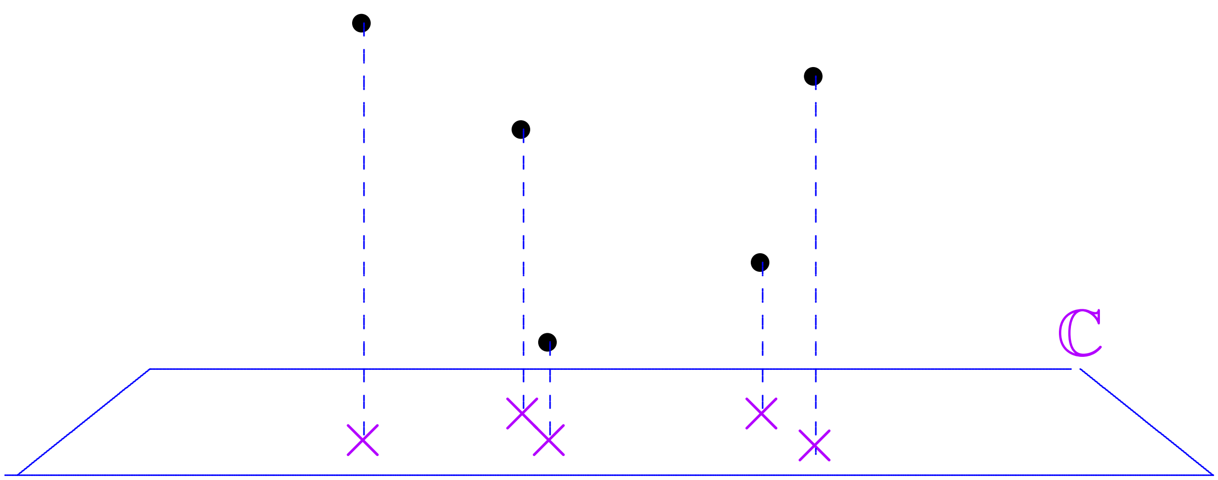


Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

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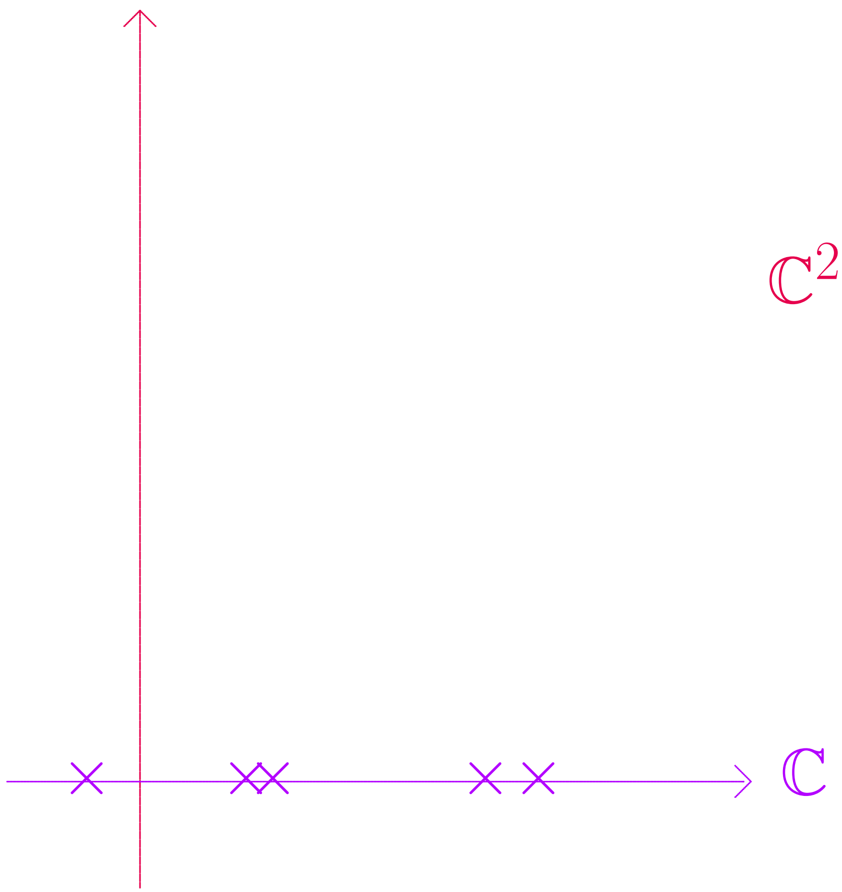
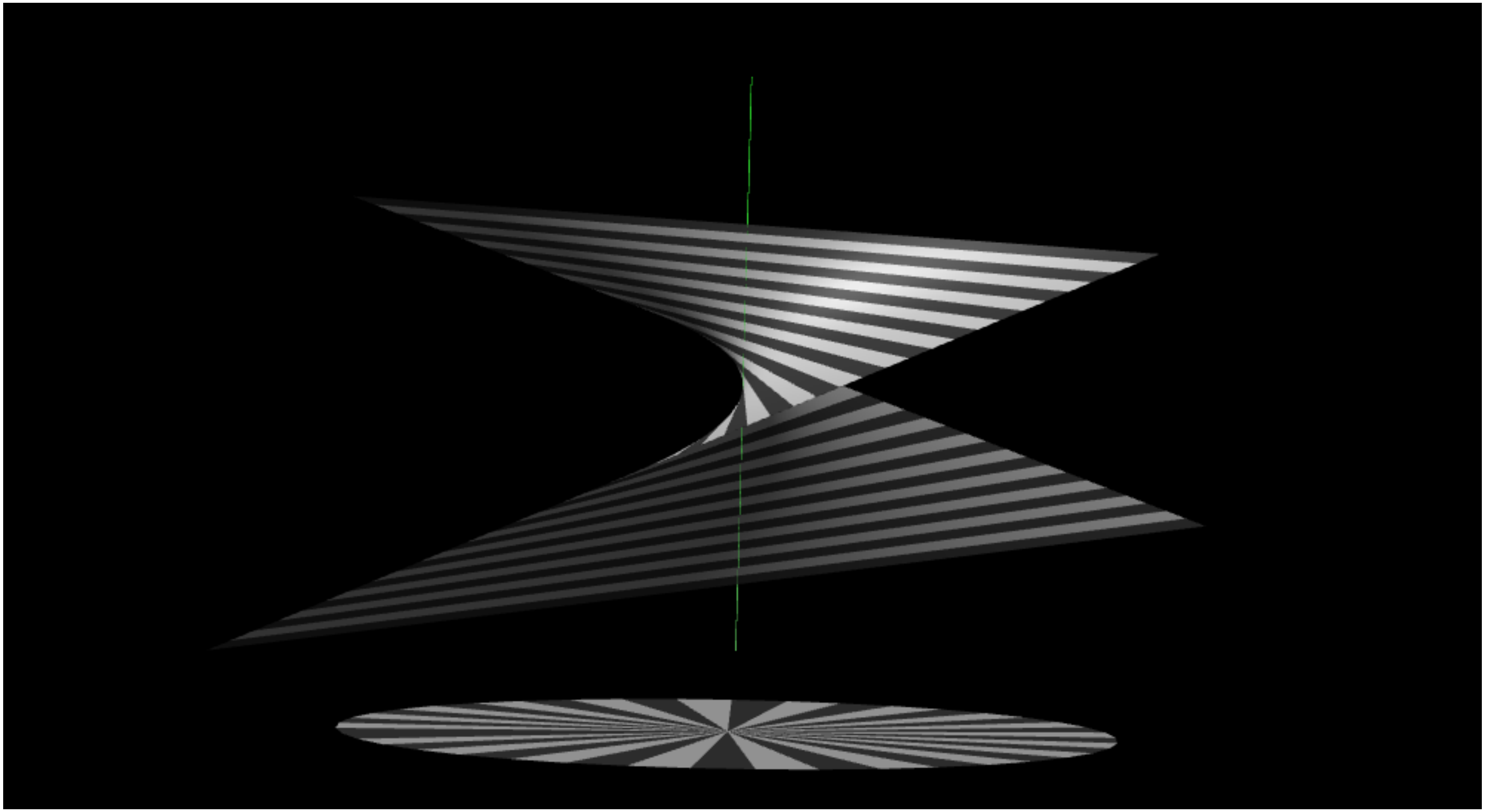
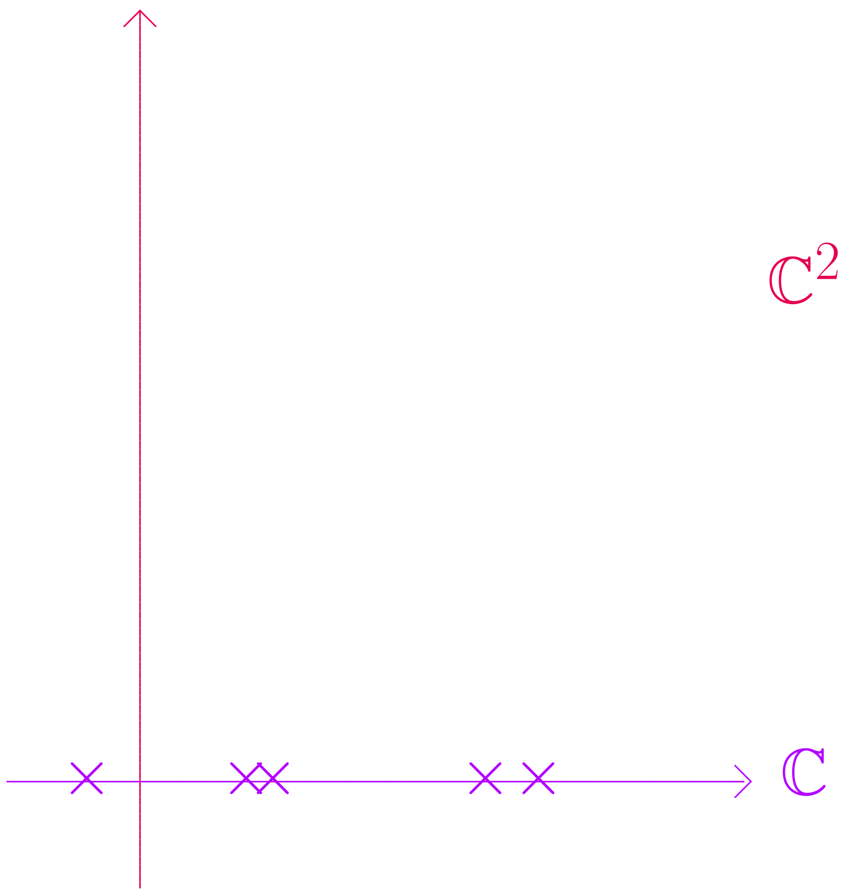



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( V h + V^{-1} \theta^2 \right)$$



**Some AE Scalar-Flat Kähler Surfaces:**




Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( Vh + V^{-1}\theta^2 \right)$$

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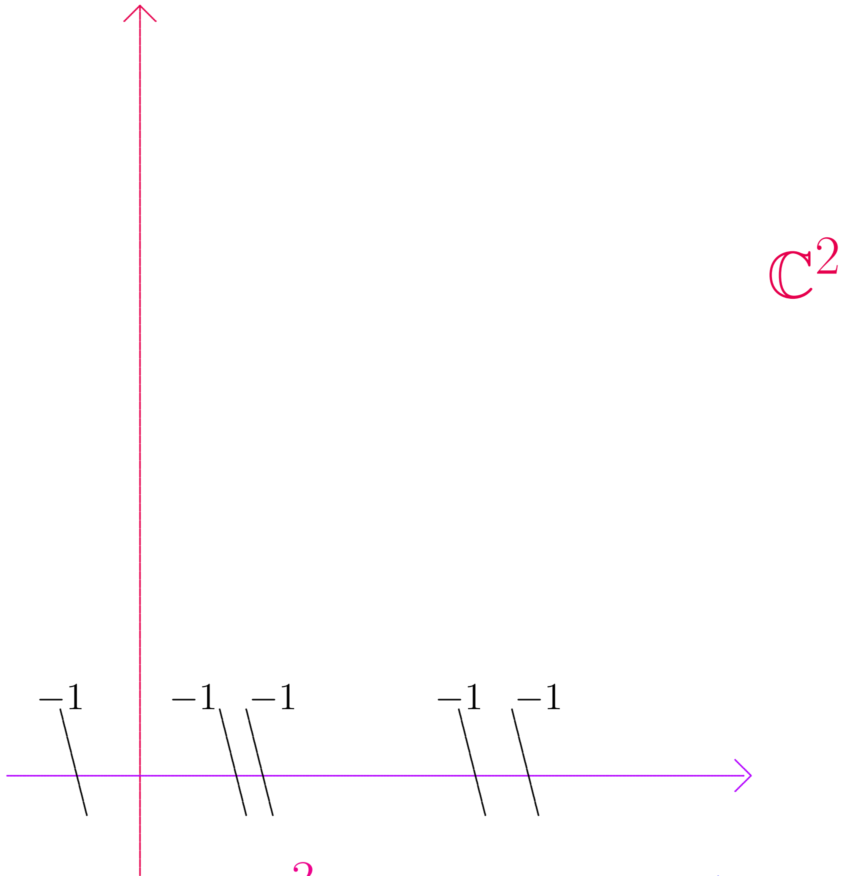


Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( Vh + V^{-1}\theta^2 \right)$$

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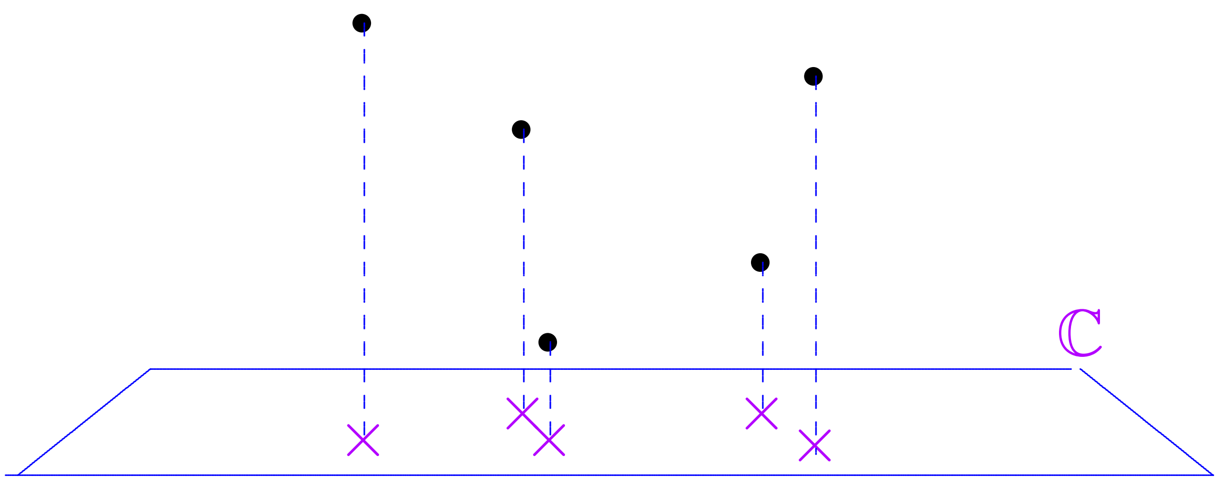


Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( V h + V^{-1} \theta^2 \right)$$

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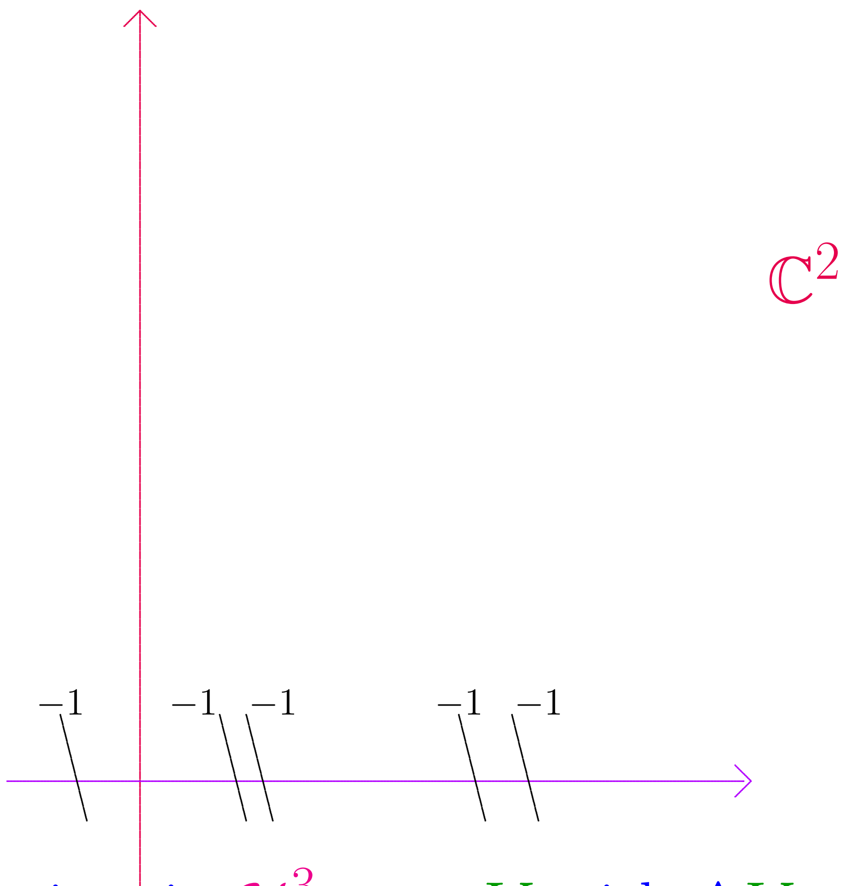



Data: $k$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = z^2 \left( Vh + V^{-1}\theta^2 \right)$$

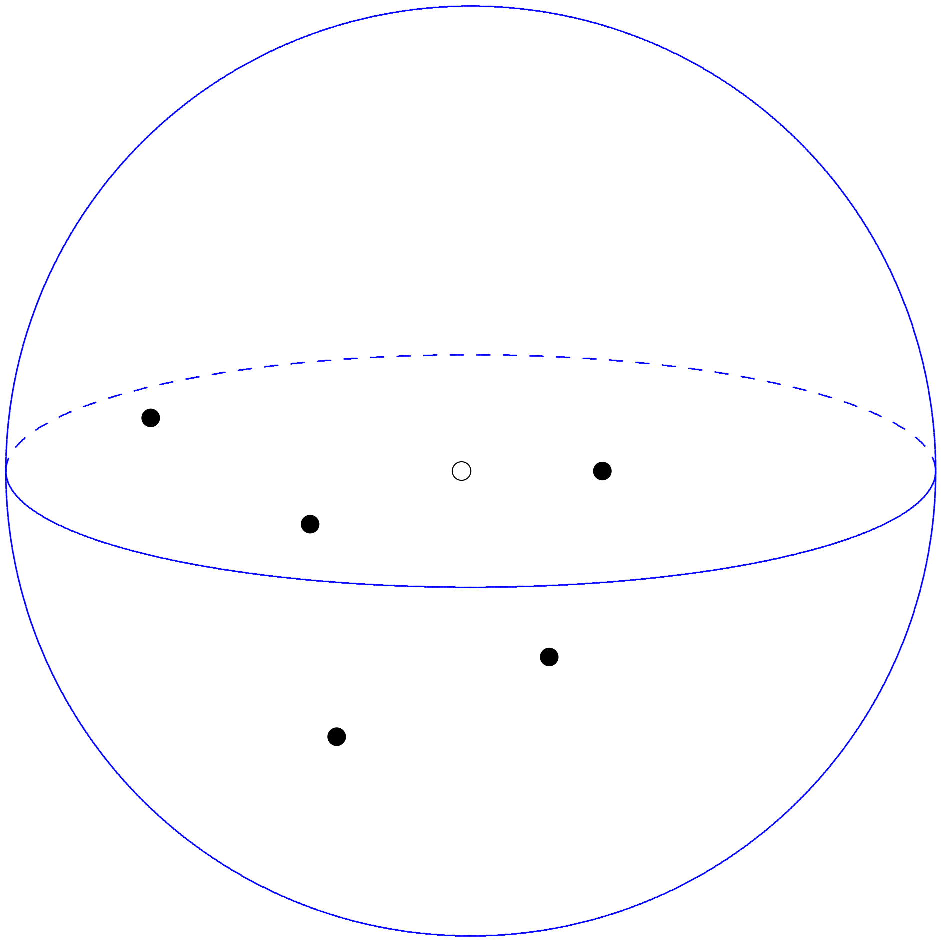
# Some ALE Scalar-Flat Kähler Surfaces:

## Some ALE Scalar-Flat Kähler Surfaces:

**(L '91)**

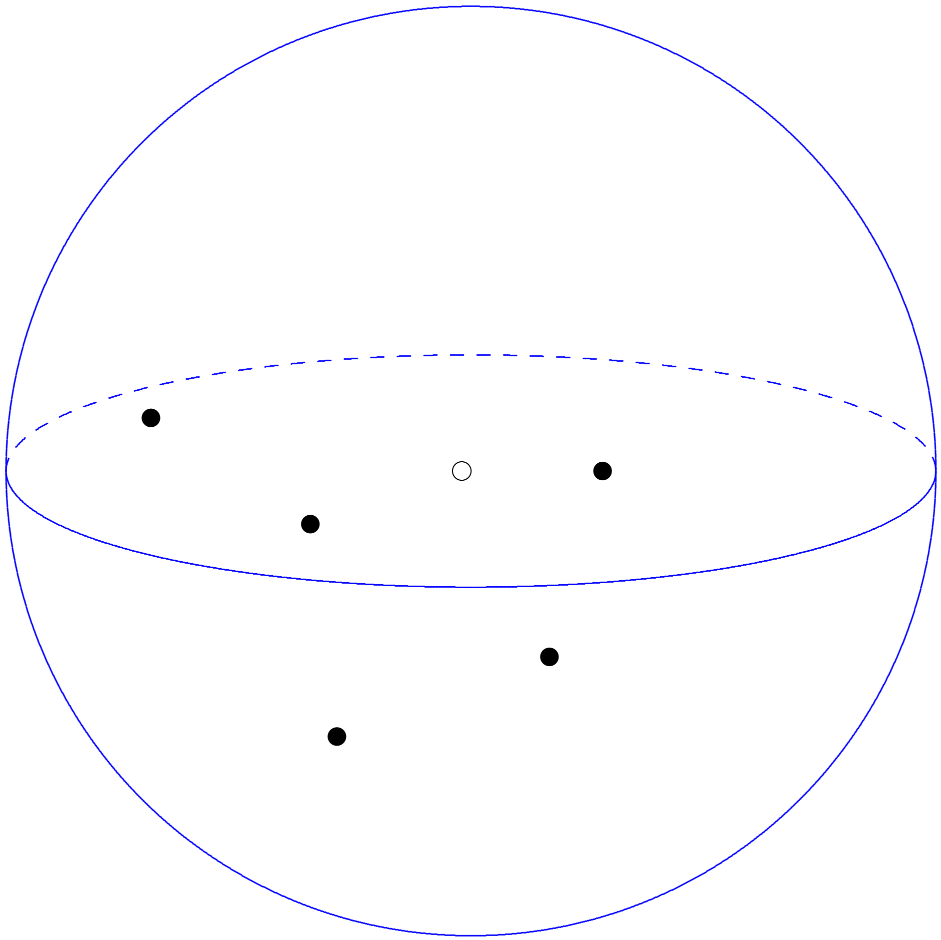
## Some ALE Scalar-Flat Kähler Surfaces:

# Some ALE Scalar-Flat Kähler Surfaces:



Data: $k+1$ points in $\mathcal{H}^3$.

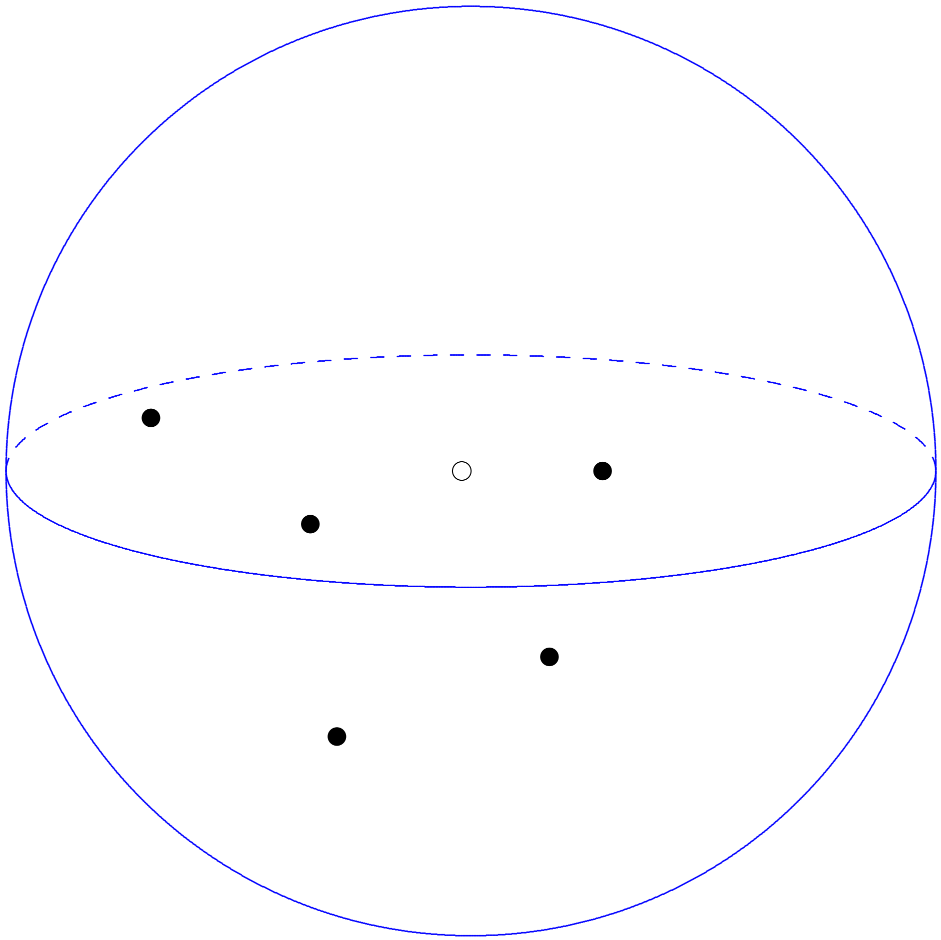
## Some ALE Scalar-Flat Kähler Surfaces:



Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$V = 1 + \ell G_0 + \sum_{j=1}^{k} G_j$$

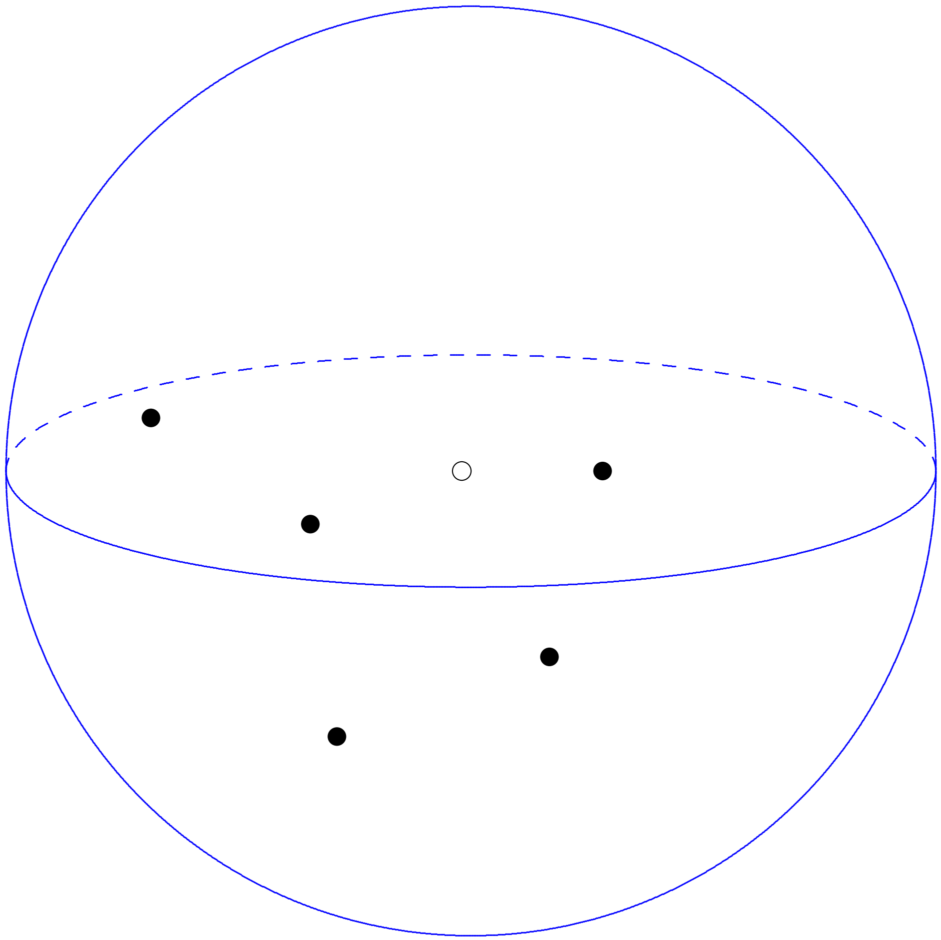
# Some ALE Scalar-Flat Kähler Surfaces:



Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$V = 1 + \frac{\ell}{e^{2\varrho_0} - 1} + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

## Some ALE Scalar-Flat Kähler Surfaces:



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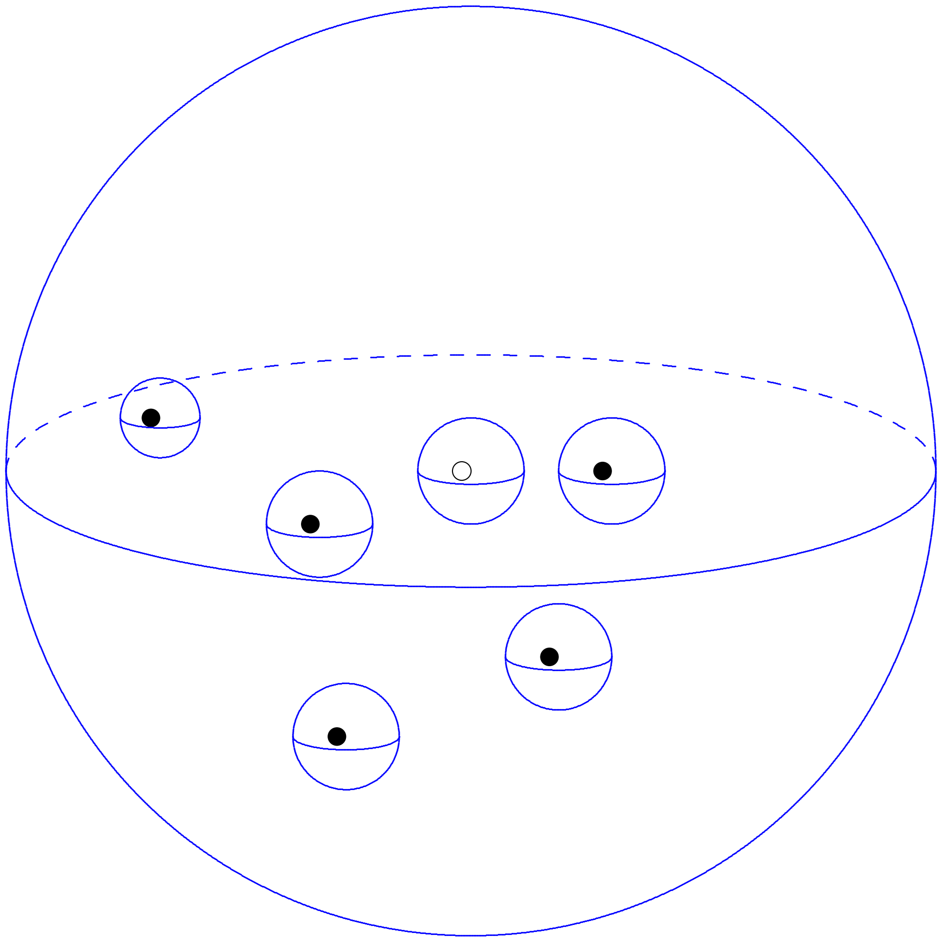
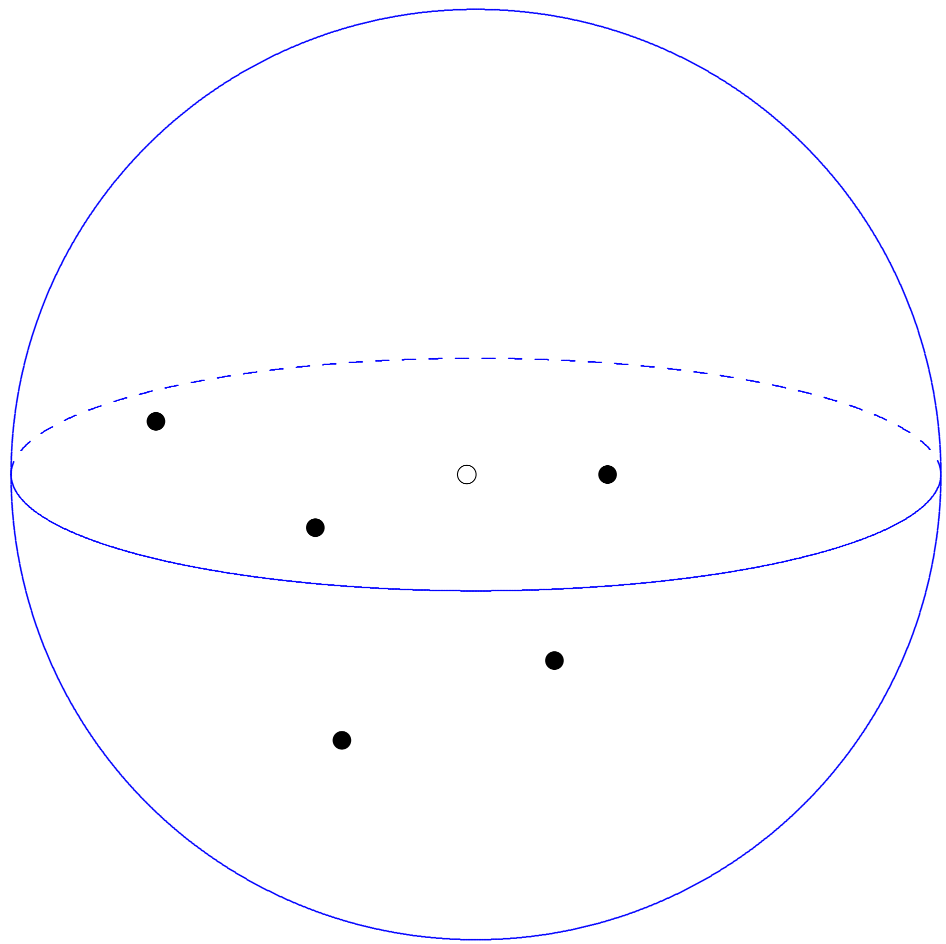


Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow V$ with $\Delta V = 0$

$F = \star dV$ curvature $\theta$ on $P \to \mathcal{H}^3 - \{\text{pts}\}$.

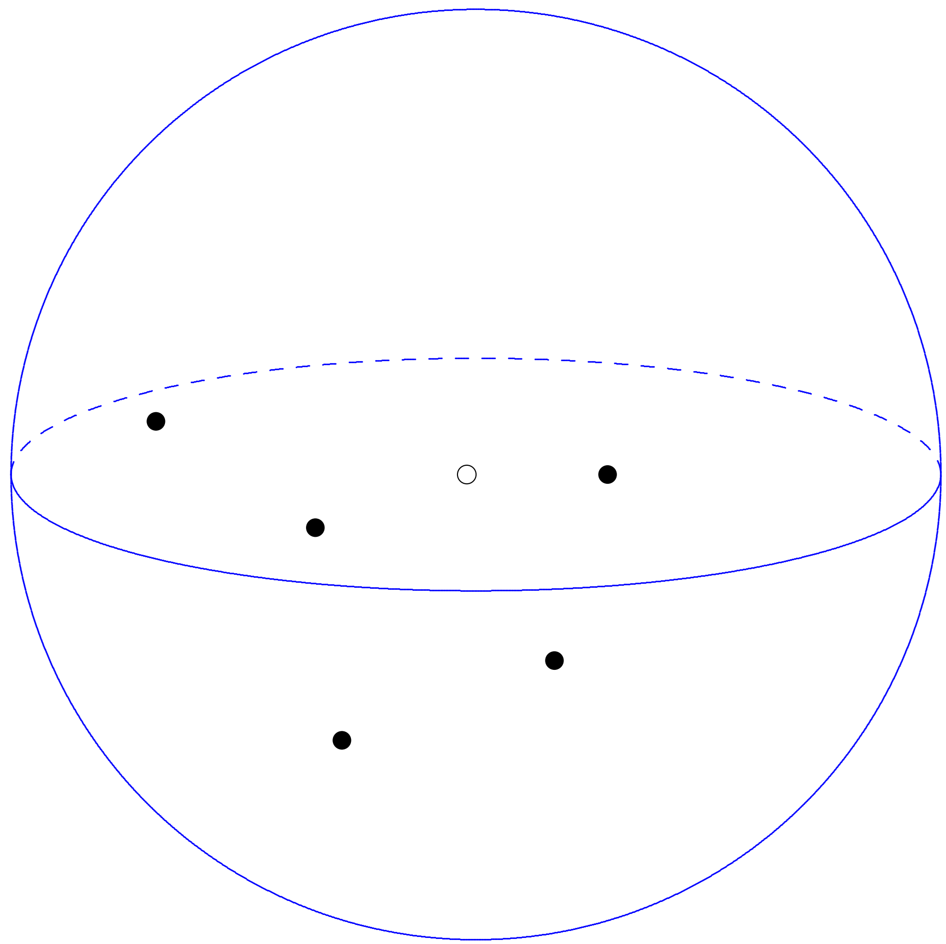
# Some ALE Scalar-Flat Kähler Surfaces:



Data: $k+1$ points in $\mathcal{H}^3$. $\Longrightarrow$ $V$ with $\Delta V = 0$

$$g = \frac{1}{4\sinh^2 \varrho_0}\left(Vh + V^{-1}\theta^2\right)$$

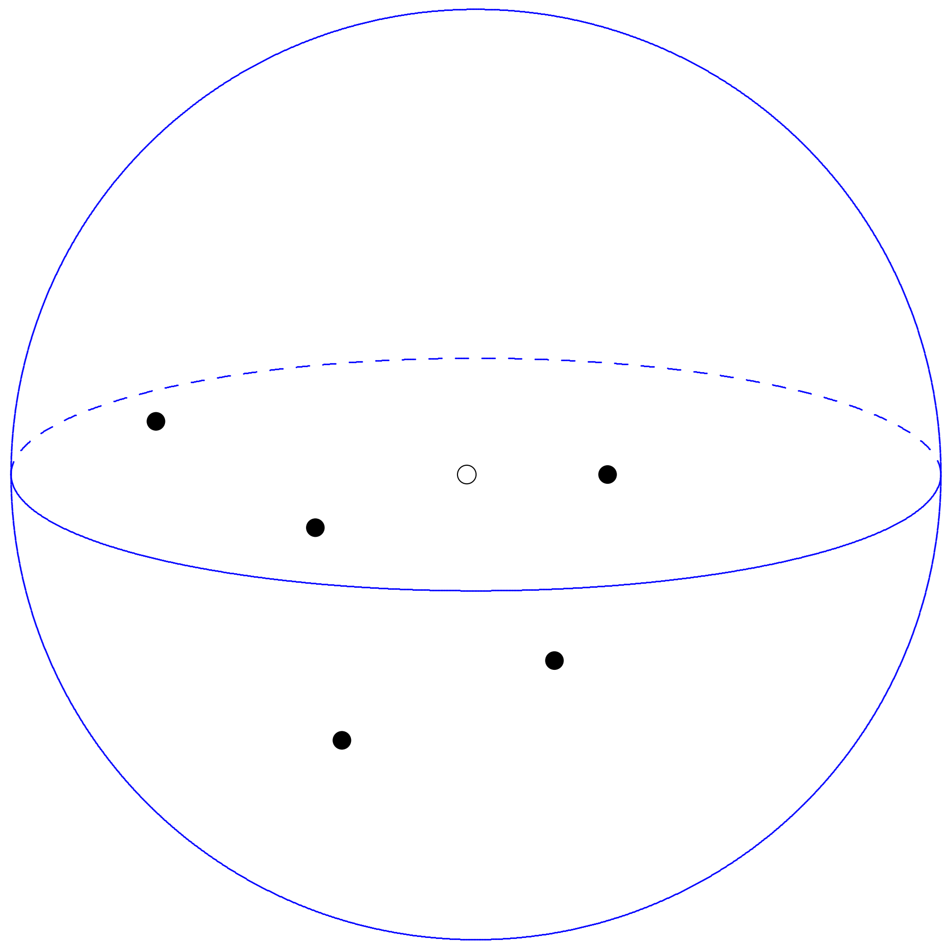
## Some ALE Scalar-Flat Kähler Surfaces:



Riemannian completion is ALE scalar-flat Kähler.

$$g = \frac{1}{4\sinh^2 \varrho_0}\left(Vh + V^{-1}\theta^2\right)$$

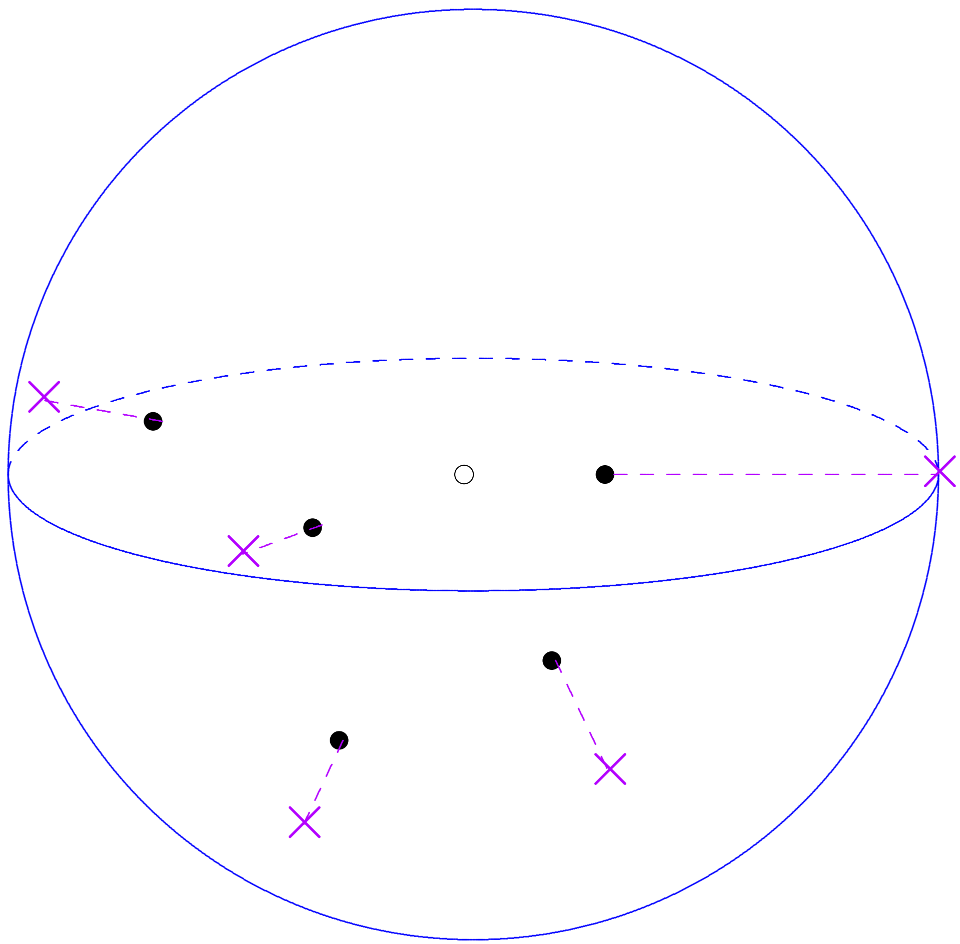
## Some ALE Scalar-Flat Kähler Surfaces:



Riemannian completion is ALE with $\Gamma = \mathbb{Z}_\ell$.

$$V = 1 + \frac{\ell}{e^{2\varrho_0} - 1} + \sum_{j=1}^{k} \frac{1}{e^{2\varrho_j} - 1}$$

## Some ALE Scalar-Flat Kähler Surfaces:



Riemannian completion is ALE with $\Gamma = \mathbb{Z}_\ell$.

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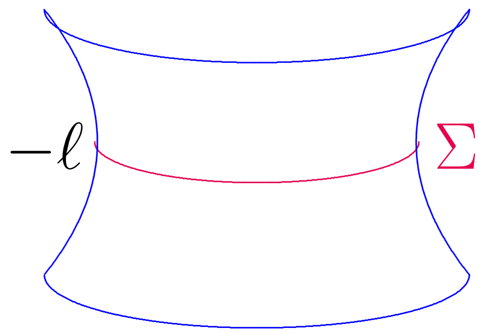
# Some ALE Scalar-Flat Kähler Surfaces:

Blow up of Chern-class $-\ell$ line bundle over $\mathbb{CP}_1$ at $k$ points on zero section $\Sigma$.

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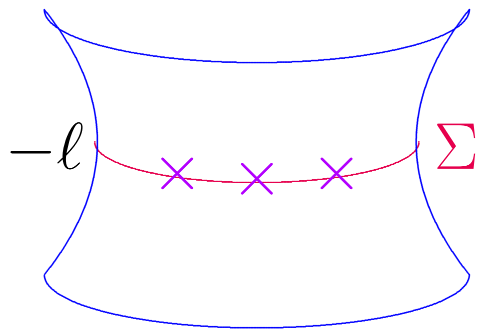


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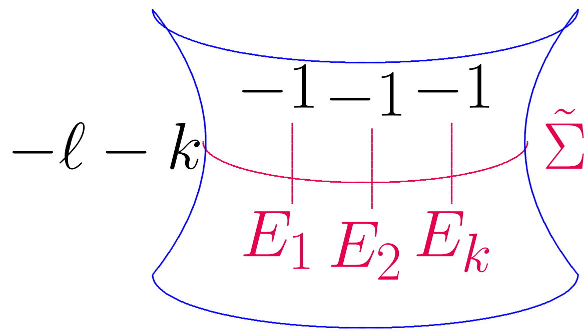


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Any scalar-flat Kähler surface $(M^4, g, J)$ has a

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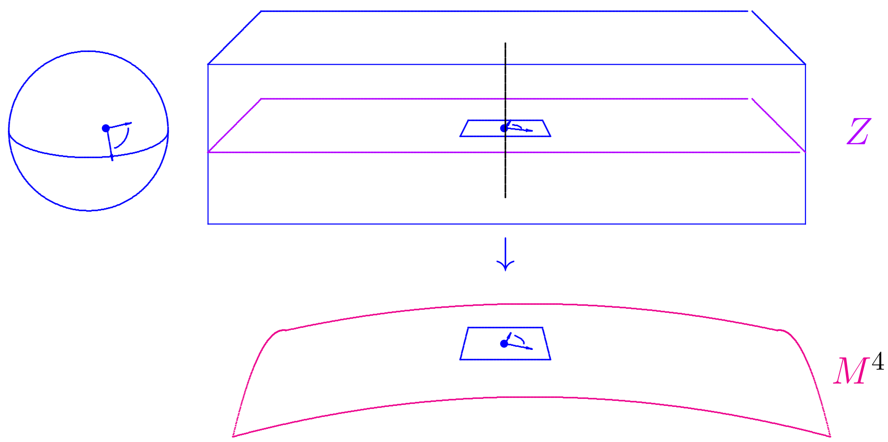

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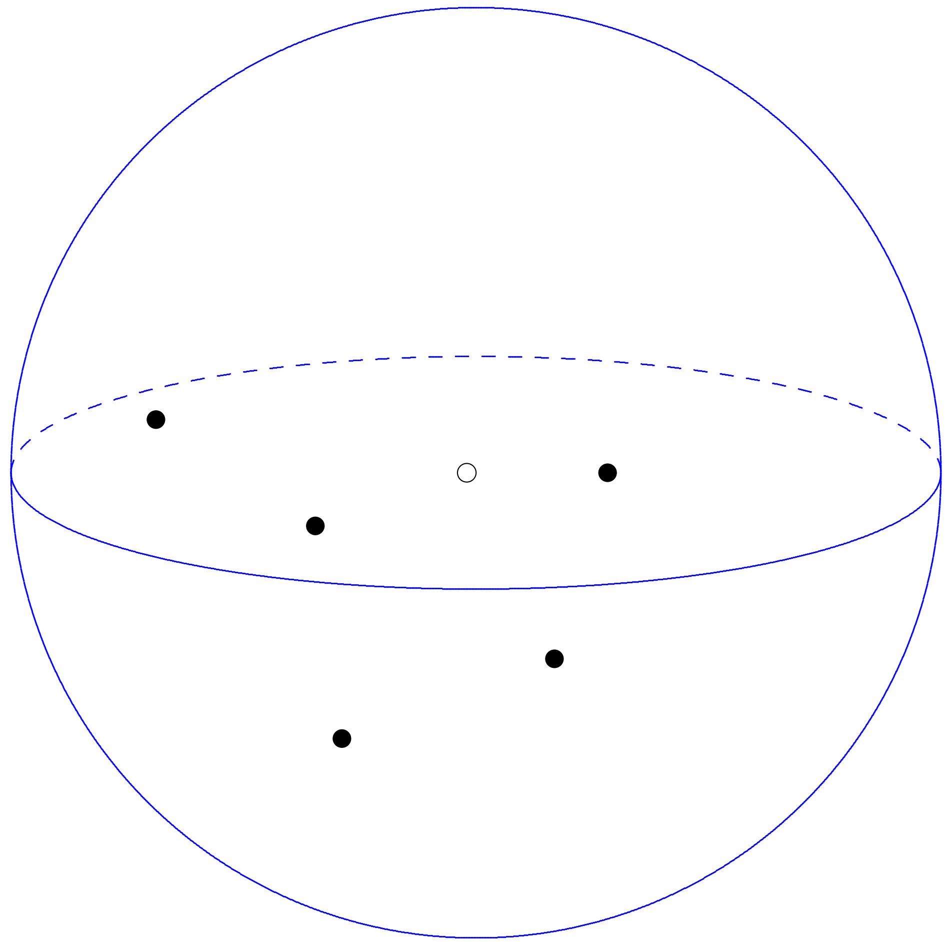

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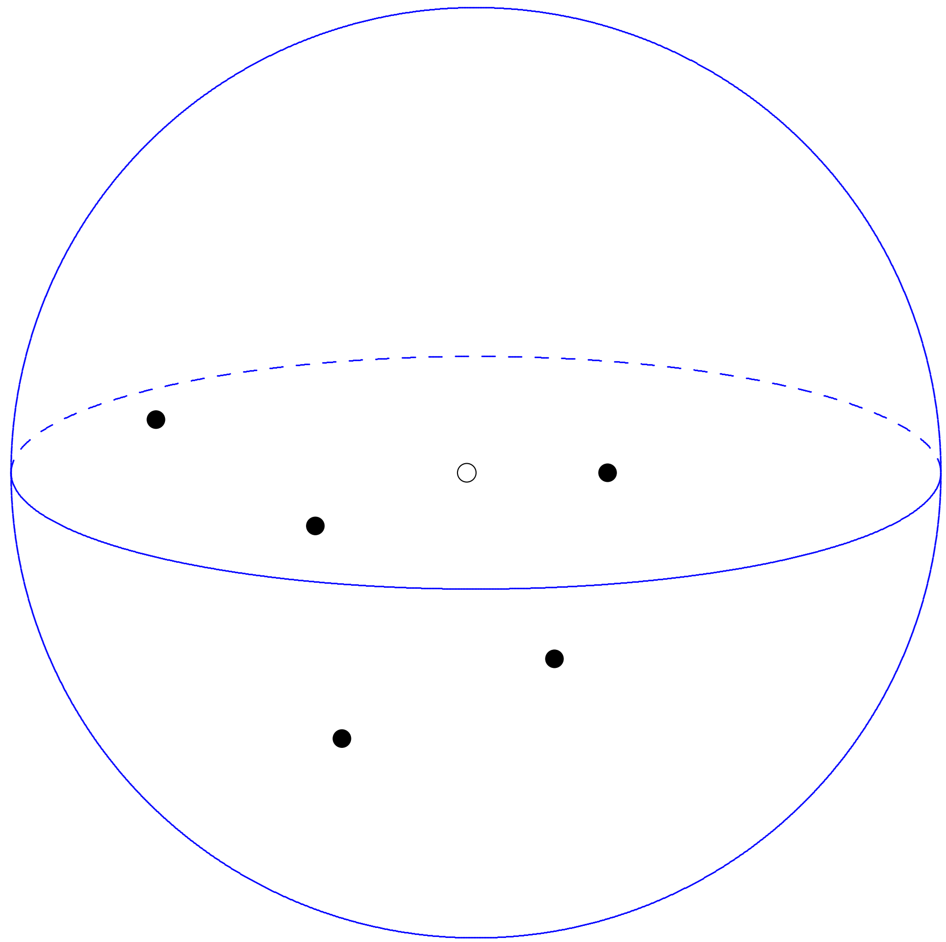

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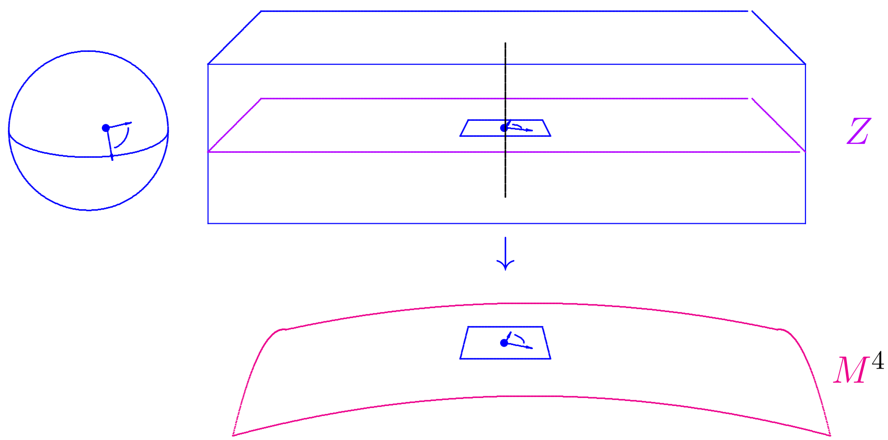

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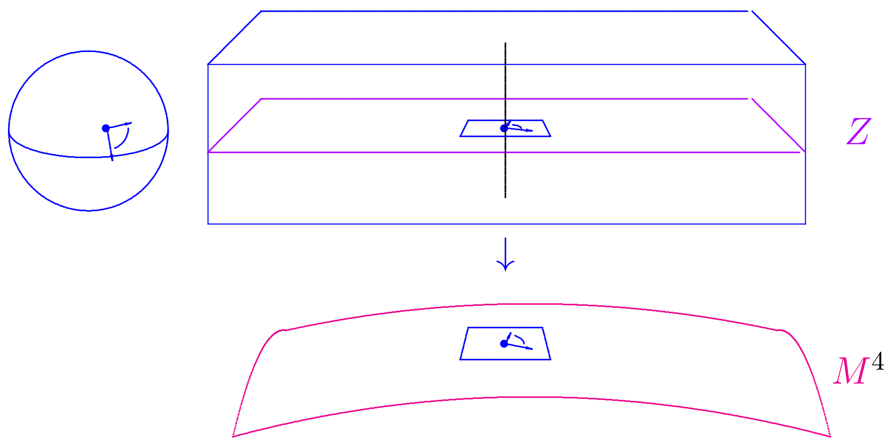


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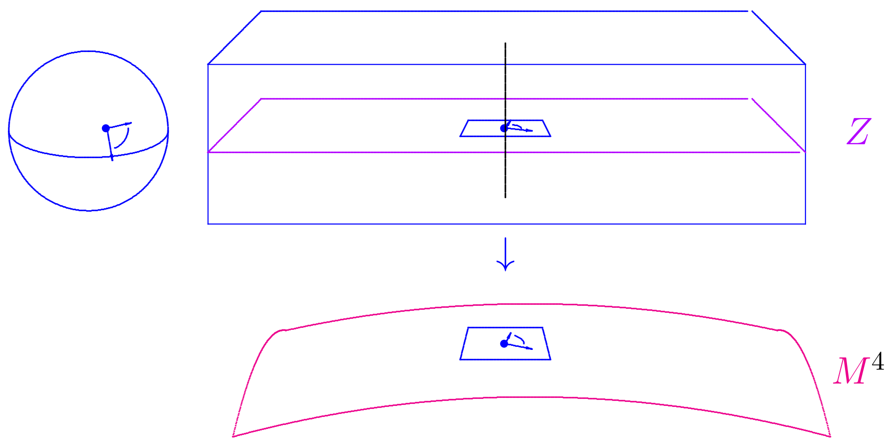


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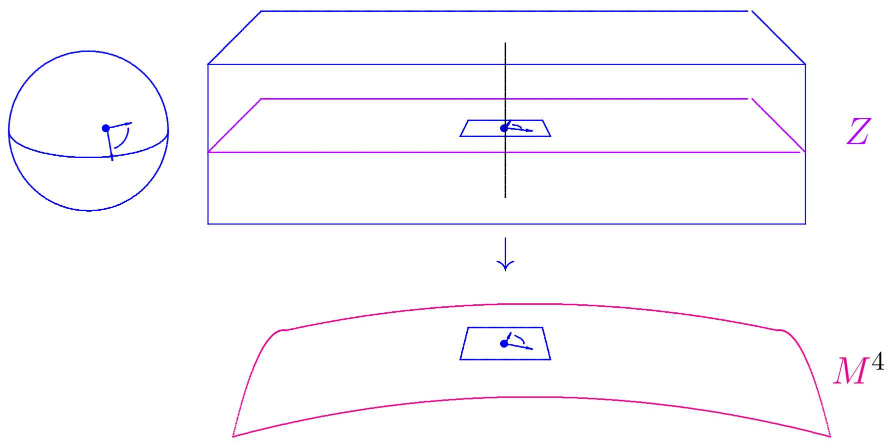


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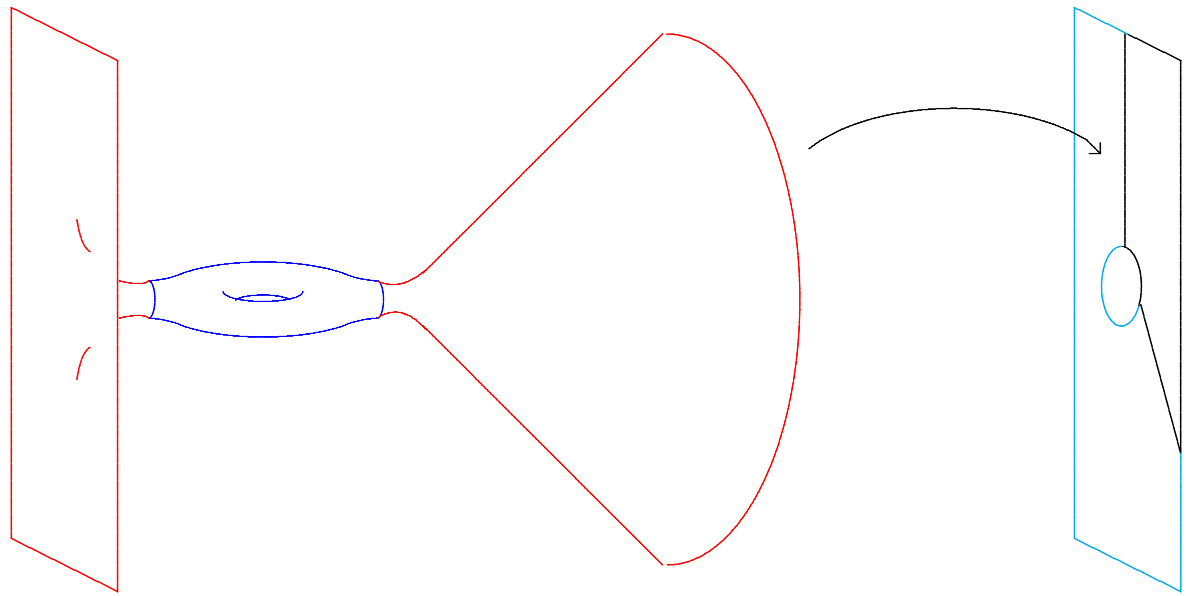
But full classification remains an open problem.

**Definition.** *Complete, non-compact $n$-manifold $(M^n, g)$ is* asymptotically locally Euclidean *(ALE) if $\exists$ compact set $K \subset M$ such that $M - K \approx \coprod_i (\mathbb{R}^n - D^n)/\Gamma_i$, where $\Gamma_i \subset \mathbf{O(n)}$, such that*



$$g_{jk} = \delta_{jk} + O(|x|^{1-\frac{n}{2}-\varepsilon})$$

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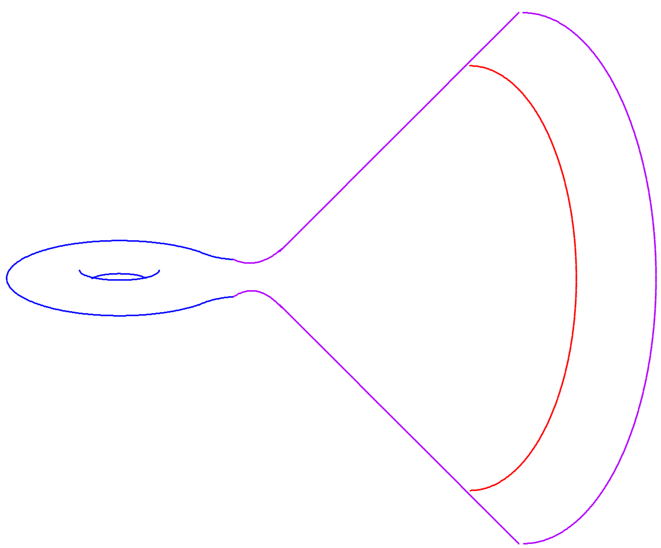

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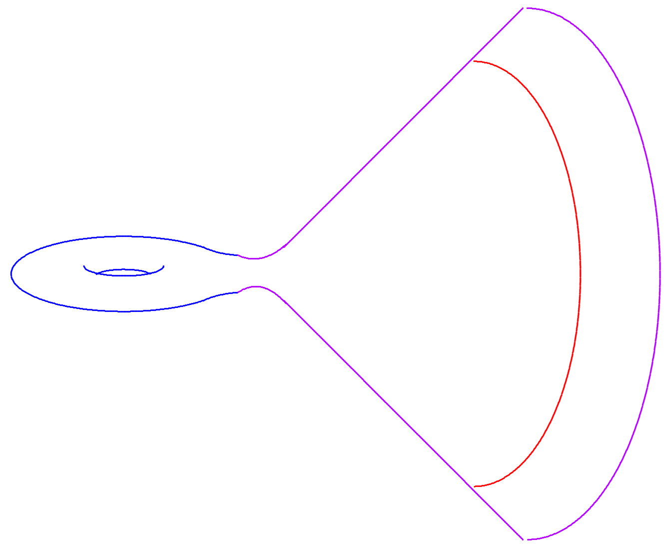

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**Chruściel-type fall-off:**

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We'll see a new proof of this in the Kähler case.

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**Theorem C.** *Any ALE Kähler manifold* $(M, g, J)$ *of complex dimension* $m$ *has mass given by*

$$\mathfrak{m}(M, g) = -\frac{\langle \clubsuit(c_1), [\omega]^{m-1}\rangle}{(2m-1)\pi^{m-1}} + \frac{(m-1)!}{4(2m-1)\pi^m}\int_M s_g d\mu_g$$

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- $\clubsuit : H^2(M) \xrightarrow{\cong} H^2_c(M)$ *inverse of natural map.*

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Scalar-flat Kähler case:

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Today: What does this mean in practice?

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**Bartnik:** Ricci-flat $\Longrightarrow$ faster fall-off of metric!

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Exploit Poincaré duality...

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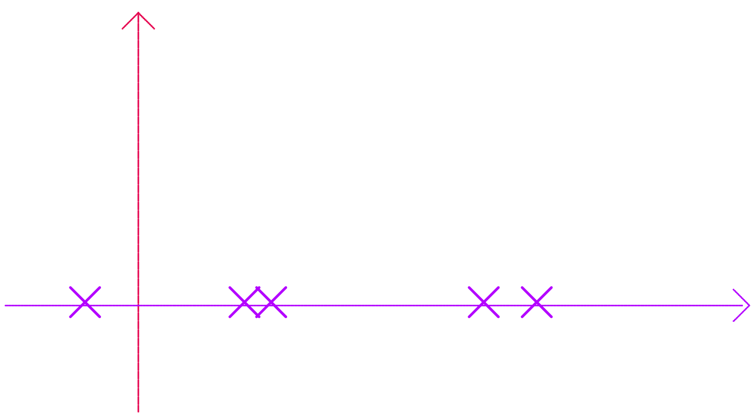

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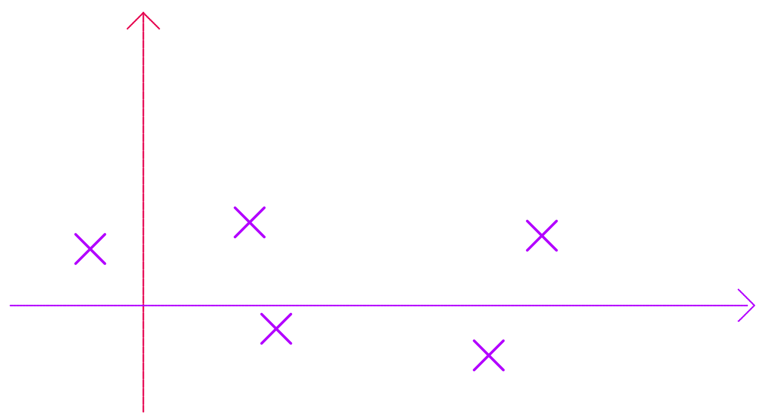

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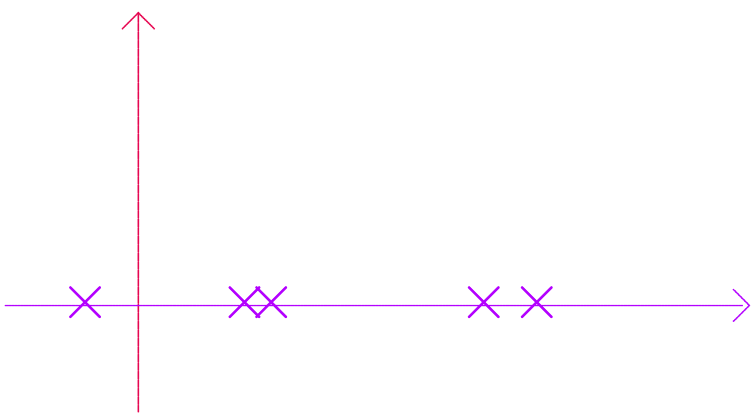

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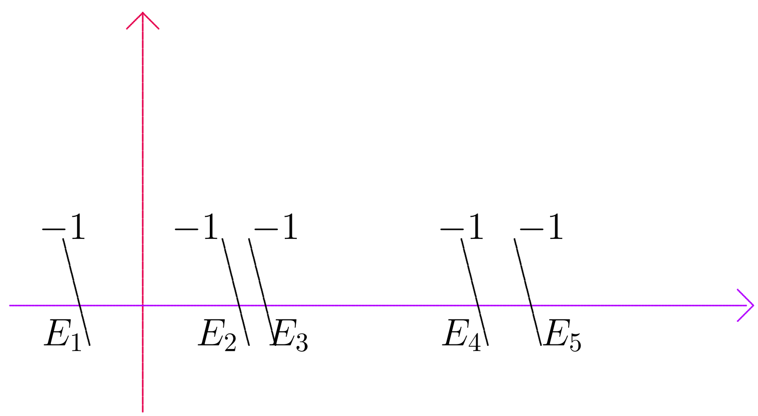

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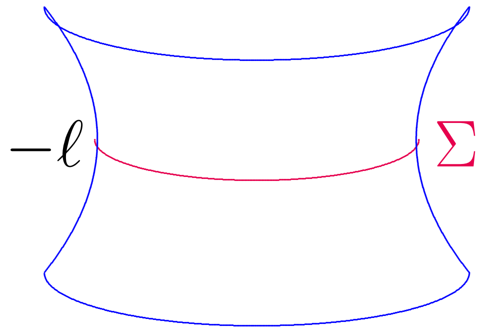

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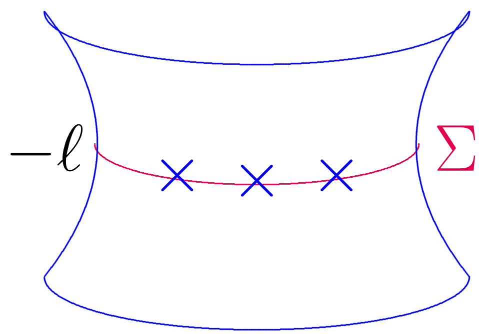

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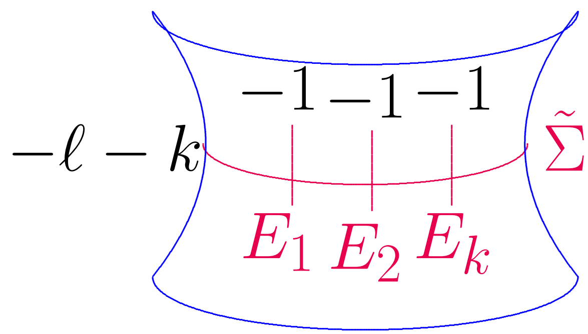

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Calderbank-Singer metrics generalize for $k \neq \pm 1$.

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Brought to our attention by C. Spotti.

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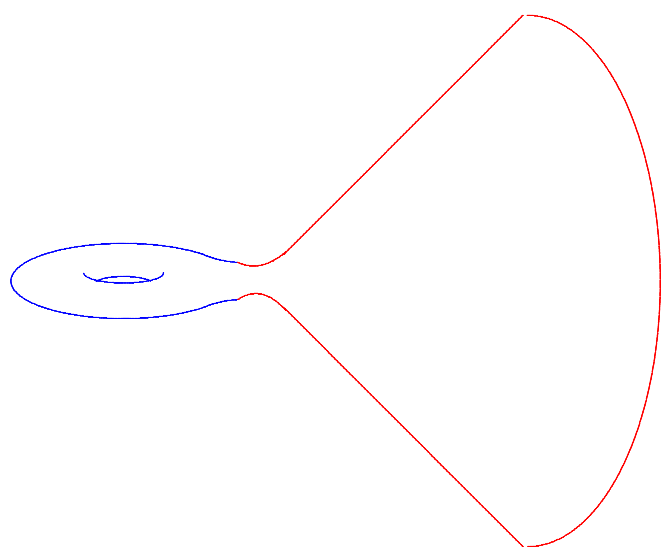

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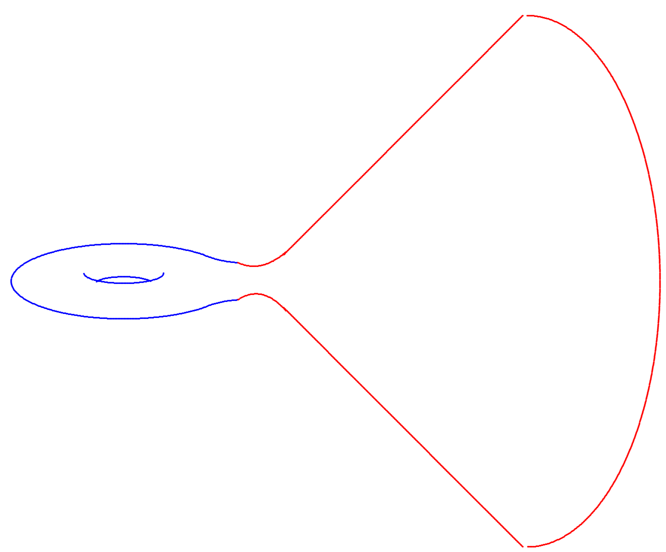

**End, Part II**