Harmonic measure via blow up methods and monotonicity formulas

Xavier Tolsa

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Plan of the course

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- Geometric characterization of the weak-A_∞ condition.
  Proof of the weak local John condition via the ACF formula.

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- Geometric characterization of the weak-A_∞ condition.
  Proof of the weak local John condition via the ACF formula.
- Tsirelson's theorem. Proof by blowup methods.

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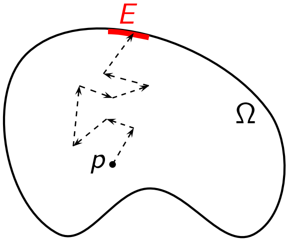
Probabilistic interpretation [Kakutani]:
When Ω is bounded, $\omega^p(E)$ is the probability that a particle with a Brownian movement leaving from $p \in \Omega$ escapes from Ω through E.



Rectifiability

We say that $E \subset \mathbb{R}^d$ is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.

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E is n-AD-regular if

$$\mathcal{H}^n(B(x,r) \cap E) \approx r^n \quad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

E is uniformly n-rectifiable if it is n-AD-regular and there are $M, \theta > 0$ such that for all $x \in E$, $0 < r \leq \operatorname{diam}(E)$, there exists a Lipschitz map

$$g : \mathbb{R}^n \supset B_n(0,r) \to \mathbb{R}^d, \qquad \|\nabla g\|_\infty \leq M,$$

such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta\, r^n.$$

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Uniform n-rectifiability is a quantitative version of n-rectifiability introduced by David and Semmes.

Metric properties of harmonic measure

- In the plane if Ω is simply connected and $\mathcal{H}^1(\partial\Omega) < \infty$, then $\mathcal{H}^1 \approx \omega^p$. (F.& M. Riesz)
- Many results in $\mathbb{C}$ using complex analysis (Carleson, Makarov, Jones, Bishop, Wolff,...).
- The analogue of Riesz theorem fails in higher dimensions (counterexamples by Wu and Ziemer).
- In higher dimensions, need real analysis techniques.
- A basic result of Dahlberg: If Ω is a Lipschitz domain, then $\omega \in A_\infty(\mathcal{H}^n|_{\partial\Omega})$.

Uniform, semiuniform, and NTA domains

Let $\Omega \subset \mathbb{R}^{n+1}$ be open.

- For $x, y \in \overline{\Omega}$, a curve $\gamma \subset \overline{\Omega}$ from x to y is a C-cigar curve with bounded turning if
  - $\min(\mathcal{H}^1(\gamma(x,z)), \mathcal{H}^1(\gamma(y,z))) \leq C \operatorname{dist}(z, \Omega^c)$ for all $z \in \gamma$, and
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$$\text{NTA} \quad \subsetneq \quad \text{uniform} \quad \subsetneq \quad \text{semiuniform}.$$

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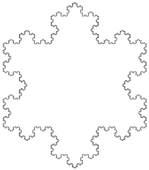
A non trivial NTA domain:



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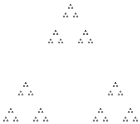
Example: The complement of this Cantor set is uniform but not NTA:



Harmonic measure in different types of domains

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Theorem (David, Jerison / Semmes)

If Ω is NTA and $\partial\Omega$ is uniformly n-rectifiable, then $\omega \in A_\infty$.

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- A previous partial result by Aikawa and Hirata.

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Theorem (Hofmann, Le)

Let $\Omega \subset \mathbb{R}^{n+1}$, with $\partial\Omega$ n-AD-regular, satisfying the interior corkscrew condition. TFAE:

(a) *For some $p > 1$, the Dirichlet problem is L^p-solvable, i.e.*

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(b) $\omega \in \mathrm{weak}{-}A_\infty$.

Remarks

- Ω satisfies the interior corkscrew condition if for every ball B centered at $\partial\Omega$ with $r(B) \leq \text{diam}(\Omega)$ there is another ball $B' \subset B \cap \Omega$ with $r(B') \approx r(B)$.

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- We say that $\omega \in \mathrm{weak}{-}A_\infty$ if for every $\varepsilon \in (0,1)$ there exists $\delta \in (0,1)$ such that for every ball B centered at $\partial\Omega$, all $p \in \Omega \setminus 4B$, and all $E \subset B \cap \partial\Omega$, the following holds:

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- Problem: Find a geometric characterization of the weak$-A_\infty$ condition.

Geometric characterization of the weak$-A_\infty$ condition I

- $\omega \in \mathrm{weak}{-}A_\infty$ + interior corkscrew condition $\Longrightarrow$ $\partial\Omega$ is uniformly n-rectifiable [Hofmann, Martell], [Mourgoglou-T.].

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- But $\partial\Omega$ uniformly n-rectifiable $\not\Longrightarrow \omega \in \mathrm{weak}{-}A_\infty$ (Bishop, Jones).

Geometric characterization of the weak$-A_\infty$ condition II

- Given $x \in \Omega$, $y \in \partial\Omega$, a c-carrot curve from x to y is a curve $\gamma \subset \Omega \cup \{y\}$ with end-points x and y such that $\text{dist}(z, \partial\Omega) \geq c\,\mathcal{H}^1(\gamma(y,z))$ for all $z \in \gamma$, where $\gamma(y,z)$ is the arc in γ between y and z.

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- We say that Ω satisfies the weak local John condition if there are $\lambda, \theta \in (0,1)$ such that for every $x \in \Omega$ there is a Borel set $F \subset B(x, 2\delta_\Omega(x)) \cap \partial\Omega$ with $\mathcal{H}^n(F) \geq \theta\,\mathcal{H}^n(B(x, 2\delta_\Omega(x)) \cap \partial\Omega)$ such that every $y \in F$ can be joined to x by a λ-carrot curve.

The main results I

Theorem (Hofmann, Martell)

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with uniformly n-rectifiable boundary satisfying the weak local John condition. Then $\omega \in \mathrm{weak}{-}A_\infty$.

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Theorem (Azzam, Mourgoglou, T.)

Let $\Omega \subset \mathbb{R}^{n+1}$ be open with n-AD-regular boundary. If $\omega \in \mathrm{weak}{-}A_\infty$, then Ω satisfies the weak local John condition.

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Putting all together:

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Since BPCAS $\Rightarrow$ $\omega \in \mathrm{weak}-A_\infty$ (Bennewitz, Lewis), we have

$$(a) \iff (b) \iff \text{BPCAS}.$$

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(c) *Ω has BPCAS.*

Some ideas for the proof of the weak local John condition

- For $p \in \Omega$, we have to build carrot curves that connect a big proportion of the points from $B(p, 2\delta_\Omega(p)) \cap \partial\Omega$ to p.

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  A fundamental property:
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  A fundamental property:
  For all $\lambda > 0$, $\{x \in \Omega : g(p, x) > \lambda\}$ is connected and contains p.
- Important difficulties:
  ω^p may be non doubling.
  ω^{p_1} and ω^{p_2} may be mutually singular.
  Otherwise we could argue with different poles $p_1, p_2, \ldots$

The ACF formula

Theorem (Alt-Caffarelli-Friedman)

Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative subharmonic functions. Suppose that that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$. Set

$$J_i(x,r) = \frac{1}{r^2} \int_{B(x,r)} \frac{|\nabla u_i(y)|^2}{|y-x|^{n-1}} dy,$$

and $J(x,r) = J_1(x,r)\, J_2(x,r)$.

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Then $J(x,\cdot)$ is non-decreasing in $r\in(0,R]$.
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This formula is a basic tool in free boundary problems.

The case of equality in the ACF formula

Theorem

Let $B(x,R)$ and u_1, u_2 be as in the previous theorem. Suppose that $J(x,r_a) = J(x,r_b)$ for some $0 < r_a < r_b < R$. Then either one or the other of the following holds:

(a) *$u_1 = 0$ in $B(x,r_b)$ or $u_2 = 0$ in $B(x,r_b)$;*

(b) *there exists a unit vector e and constants $k_1, k_2 > 0$ such that*

$$u_1(y) = k_1\,((y-x)\cdot e)^+, \qquad u_2(y) = k_2\,((y-x)\cdot e)^-, \qquad \text{in } B(x,r_b).$$

A quantification of the previous result

Theorem

Let $B(x,R)$ and u_1, u_2 be as in the previous theorem, such that each u_i is harmonic in $\{y \in B(x,R) : u_i(y) > 0\}$. Assume also that

$$\|u_i\|_{\infty,B(x,R)} \leq C_1\,R \quad \text{and} \quad \|u_i\|_{\mathrm{Lip}^\alpha,B(x,R)} \leq C_1\,R^{1-\alpha} \quad \text{for } i = 1,2.$$

For any $\varepsilon > 0$, there exists some $\delta > 0$ such that if

$$J(x, \tfrac{1}{2}R) \leq (1+\delta)\,J(x, \tfrac{1}{4}R),$$

then either one or the other of the following holds:

(a) $\|u_1\|_{\infty,B(x,\frac{1}{2}R)} \leq \varepsilon\,R$ *or* $\|u_2\|_{\infty,B(x,\frac{1}{2}R)} \leq \varepsilon\,R$*;*

(b) *there exists a unit vector e and constants $k_1, k_2 > 0$ such that*

$$\|u_i - k_i\,((\cdot - x)\cdot e)^+\|_{\infty,B(x,\frac{1}{2}R)} \leq \varepsilon\,R \quad \text{for } i = 1,2.$$

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(a) *The family of roots of $\mathcal{T} \in I$ fulfils the packing condition*

$$\sum_{\mathcal{T} \in I : \mathrm{Root}(\mathcal{T}) \subset S} \mu(\mathrm{Root}(\mathcal{T})) \leq C\,\mu(S) \quad \textit{for all } S \in \mathcal{D}_\mu.$$

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(b) *In each $\mathcal{T} \in I$, E is "very well approximated" by an n-dimensional Lipschitz graph $\Gamma_{\mathcal{T}}$.*
That is, for all $Q \in \mathcal{T}$, $\mathrm{dist}(Q, \Gamma_{\mathcal{T}}) \leq \ell(Q)$.

How to build another corona decomposition

Fix $0 < \varepsilon \ll 1$.
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Given and $\mathrm{Top}_k \subset \mathcal{D}_\mu(R_0)$, with $\bigcup_{P\in \mathrm{Top}_k} P = R_0$, Top_{k+1} is the maximal subfamily of cubes Q strictly contained in cubes from Top_k such that $b\beta(100Q) > \varepsilon$, where

$$b\beta(100Q) = \inf_{L \text{ n-plane}} \frac{\mathrm{dist}_H(L\cap 100B_Q,\, \partial\Omega \cap 100B_Q)}{r(100B_Q)}.$$

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We set $\text{Top} = \bigcup_{k \geq 0} \text{Top}_k$.
For $R \in \text{Top}_k$,
$\text{Tree}(R) = \{Q \subset R : Q \text{ not contained in any cube from } \text{Top}_{k+1}\}$.
R is called *root* of $\text{Tree}(R)$.

Long and short trees

We have:

$$\mathcal{D}_\mu(R_0) = \bigcup_{R\in\mathsf{Top}} \mathsf{Tree}(R)$$

and

$$\sum_{R\in\mathsf{Top}} \mu(R) \leq C(\varepsilon)\,\mu(R_0).$$

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For $R \in \mathsf{Top}$, we set

$$R \in \mathsf{Top}_s \quad \text{if} \quad \mathsf{Tree}(R) = \{R\} \qquad \text{(short tree)},$$

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If $R \in \mathrm{Top}_s$, we may have $b\beta(50Q) \gg \varepsilon$.

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Let $\eta, \lambda > 0$ *and* $R \in \mathrm{Top}_\ell$*. Then there exists* $\mathrm{Ex}(R) \subset \mathrm{Stop}(R) \cap G$ *such that*

$$\sum_{P \in \mathrm{Ex}(R)} \mu(P) \leq \eta\, \mu(R)$$

and such that every $Q \in \mathrm{Stop}(R) \cap G \setminus \mathrm{Ex}(R)$ *can be joined to a good* (λ', τ_0)*-good corkscrew* x_R *by a* C*-nice curve, with* $\lambda' = \lambda'(\eta, \lambda)$*,* $C = C(\eta, \lambda)$*.*

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A key fact: we do not ask ε to depend on λ or η.

Strategy for the proof of the Theorem

Given $N > 1$, set

$$VG_0 = \{x \in G_0 : \sum_{R \in \mathrm{Top}} \chi_R \leq N\}.$$

Since $\sum_{R \in \mathrm{Top}} \mu(R) \leq C(\varepsilon)\, \mu(R_0)$ choosing $N = N(\varepsilon)$ big enough, by Chebyshev

$$\mu(VG_0) \geq \frac{1}{2}\, \mu(G) \gtrsim \mu(R_0).$$

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By a suitable algorithm which combines a repeated application of the "short paths" Lemma and the Key Lemma, we will be able to connect a big piece of VG_0 to p by carrot curves, modulo an small exceptional set.

Idea of proof of the Key Lemma (1)

Let $R \in \mathsf{Top}_\ell$, Γ_R approximating chord surface, and Ω_R^1, Ω_R^2 approximating chord-arc domains.

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Consider a C^∞ bump function such that $\chi_{B_R} \leq \varphi \leq \chi_{2B_R}$.
Using the identity

$$\begin{aligned} g(p, x_Q^1) &= g(p, x_Q^1)\, \varphi(x_Q^1) \\ &= \int_{\Omega_R^1} \nabla(g(p,\cdot)\, \varphi)(y) \cdot \nabla g_{\Omega_R^1}(q,y)\, dy + \int g(p,y)\, \varphi(y)\, d\omega_{\Omega_R}^{x_Q^1}(y), \end{aligned}$$

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we prove

$$\sum_{Q \in \mathrm{Stop}(R)} g(p, x^1_Q)\, \ell(Q)^{n-1} \leq C\, \frac{g(p, x^1_R)}{\ell(R)}\, \mu(R) + \mathrm{Err},$$

with

$$\mathrm{Err} \leq C\, \varepsilon^a\, \frac{\mu(R)}{\ell(R_0)^n}, \qquad a > 0.$$

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Since $g(p,x_Q^1) \geq \lambda\,\ell(R)/\ell(R_0)^n$ for each Q, we get

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So for $\varepsilon > 0$ small enough,

$$g(p,x_R^1) \gtrsim \lambda\,\frac{\ell(R)}{\ell(R_0)^n}.$$

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To solve this, we need to split $\mathrm{Tree}(R)$.

Given $0 < \kappa \ll \varepsilon \ll 1$ and $Q \in \mathrm{Tree}(R)$, we write $Q \in \mathrm{WSBC}(\kappa)$ if there does not exists any curve Γ joining the "big corkscrews" x_Q^1 and x_Q^2 such that

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Let $\mathrm{Stop}_{\mathrm{WSBC}}(R)$ be the layer of maximal cubes $Q \in \mathrm{Tree}(R)$ such that $Q \notin \mathrm{WSBC}(\kappa)$, and let $\mathrm{Tree}_{\mathrm{WSBC}}(R)$ be the cubes from $\mathrm{Tree}(R)$ above the layer $\mathrm{Stop}_{\mathrm{WSBC}}(R)$.

Idea of proof of the Key Lemma (4)

Suppose that for each $Q \in \mathrm{Stop}_{\mathrm{WSBC}}(R)$, there exists a (λ, τ_0)-good corkscrew $x_Q^1 \in \Omega_R^1$.

Idea of proof of the Key Lemma (4)

Suppose that for each $Q \in \mathrm{Stop}_{\mathrm{WSBC}}(R)$, there exists a (λ, τ_0)-good corkscrew $x_Q^1 \in \Omega_R^1$.

Arguing as above, we get

$$\sum_{Q \in \mathrm{Stop}_{\mathrm{WSBC}}(R)} g(p, x_Q^1)\, \ell(Q)^{n-1} \leq C\, \frac{g(p, x_R^1)}{\ell(R)}\, \mu(R) + \mathrm{Err},$$

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We deduce, for κ small enough,

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Further, any $P \in \mathrm{Stop}(R)$ contained in some cube $Q \in \mathrm{Stop}_{\mathrm{WSBC}}(R)$ can be connected to both corkscrews x_Q^1, x_Q^2 by a "nice" curve, because x_Q^1 and x_Q^2 are joined by a nice curve Γ.

Idea of proof of the Key Lemma (5)

An important difficulty:

We need a delicate geometric argument to approximate Ω by a domains Ω_R^1, Ω_R^2 at the level of the cubes $Q \in \mathrm{Stop}_{\mathrm{WSBC}}(R)$, so that $g(p,\cdot)$ is very small near $\partial\Omega_R^i$.

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We also need the ACF formula in this construction.