# *The Kato estimate for weighted parabolic operators*

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*The Kato estimate for weighted parabolic operators*

Work in progress, in collaboration with


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We consider parabolic operators in $\mathbb{R}^n \times \mathbb{R} = (\mathbb{R}^n)_x \times \mathbb{R}_t = \mathbb{R}^n_x \times \mathbb{R}_t$:

$$\mathcal{L}_w u = \partial_t u + \mathfrak{L}_w u = \partial_t u - \frac{1}{w(x)} \operatorname{div}_x w(x) \mathtt{A}(x,t) \nabla_x u.$$

Where $\mathtt{A}(x,t)$ is an $n \times n$ elliptic matrix with complex coefficients in $\mathbb{R}^{n+1}_+$

$$\kappa |\xi|^2 \leq \operatorname{Re}\left(\mathtt{A}(x,t)\xi, \overline{\xi}\right), \qquad |\mathtt{A}(x,t)\xi \cdot \zeta| \leq C|\xi||\zeta|, \qquad \text{for some } \kappa, C > 0,$$

and $w$ is a weight in the Muckemphout class $A_2(\mathbb{R}^n)$ :

$$[w]_{A_2} = \sup_B \left(\frac{1}{|B|}\int_B w\,dx\right)\left(\frac{1}{|B|}\int_B w^{-1}\,dx\right) < \infty,$$

where the sup is taken over all balls $B$ in $\mathbb{R}^n$.

Let $V = H^{1/2}\left(\mathbb{R}, L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}, H^1_w\left(\mathbb{R}^n\right)\right)$ and define the sesquilinear form in $V$ :

$$\mathfrak{a}_w\left(u,v\right) = \iint_{\mathbb{R}^{n+1}} \mathtt{A}\left(x,t\right) \nabla_x u\left(x,t\right) \cdot \overline{\nabla_x v\left(x,t\right)} + H_t D_t^{1/2} u \cdot \overline{D_t^{1/2} v}\, w\left(x\right) dx\, dt$$

The **parabolic operator** $\mathcal{L}_w = \partial_t + \mathfrak{L}_w$ can be defined as an operator $V \to V^*$ via this **accretive** sesquilinear form.

$$\mathcal{L}_w u = f \qquad \Longleftrightarrow \qquad \mathfrak{a}_w\left(u,v\right) = \left\langle f,v\right\rangle_{\mathbf{w}} = \iint_{\mathbb{R}^{n+1}} f\left(x,t\right) \overline{v\left(x,t\right)}\, w\left(x\right) dx\, dt.$$

- $D_t^{1/2}$ be the half-order derivative

$$D_t^{1/2} v\left(t\right) = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{v\left(t\right) - v\left(s\right)}{\left|t-s\right|^{3/2}} ds.$$

- $H_t$ the Hilbert transform with respect to the $t$ variable, so that $\partial_t = D_t^{1/2} H_t D_t^{1/2}$;

$$H_t D_t^{1/2} v\left(t\right) = -\frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\operatorname{sign}\left(t-s\right)\left(v\left(t\right) - v\left(s\right)\right)}{\left|t-s\right|^{3/2}} ds.$$

Let $V = H^{1/2}\left(\mathbb{R}, L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}, H^1_w\left(\mathbb{R}^n\right)\right)$ and define the sesquilinear form in $V$ :

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In the past few years there has been a *re-discovery* of a powerful technique to treat parabolic operators as $m$-accretive operators via a (non-local) accretive form.

- 1966 - **Kaplan**. Abstract boundary value problems for linear parabolic equations, Ann. Scuola Norm. Sup. Pisa (3).
- 1996 - **Hofmann** and **Lewis**. $L^2$ solvability and representation by caloric layer potentials in time-varying domains. Ann. of Math. (2).
- 2016 - **Castro**, **Nyström**, and **Sande**. Boundedness of single layer potentials associated to divergence form parabolic equations with complex coefficients. Calc. Var. Partial Differ. Equ.
- 2016 - **Nyström**. Square functions estimates and the Kato problem for second order parabolic operators in $\mathbb{R}^{n+1}$. Adv. Math.
- 2017 - **Auscher**, **Egert**, **Nyström**. $L^2$ Well-posedness of BVP for parabolic systems with measurable coefficients.

**Theorem (Parabolic Kato estimate)**

*The operator $\mathcal{L}_w = \partial_t + \mathfrak{L}_w = \partial_t - \frac{1}{w(x)} \operatorname{div}_x w(x) \mathtt{A}(x,t) \nabla_x$ arises from the* ***accretive form*** $\mathfrak{a}_w(u,v)$*, it is* ***maximal accretive*** *in $L^2(\mathbb{R}^{n+1}, dw\, dt)$, the* ***domain*** *of its square root is that of the accretive form, that is,*

$$\mathcal{D}\left(\sqrt{\mathcal{L}_w}\right) = H^{1/2}\left(\mathbb{R}, L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}, W^{1,2}_w(\mathbb{R}^n)\right).$$

*Denote* $\|f\|_{\mathbf{w}} = \langle f, f\rangle_{\mathbf{w}}^{1/2} = \left(\iint_{\mathbb{R}^{n+1}} |f(x,t)|^2\, w(x)\, dx\, dt\right)^{1/2}$*. The* ***estimates***

$$\left\|\sqrt{\mathcal{L}_w} u\right\|_{\mathbf{w}} \approx \|\nabla_x u\|_{\mathbf{w}} + \left\|D_t^{1/2} u\right\|_{\mathbf{w}}$$

*hold with constants depending only on n and the ellipticity constants of* $\mathtt{A}$*.*
*Here*

$$\|f\|_{\mathbf{w}} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}} |f(x,t)|^2 w(x)\, dx\, dt\right)^{\frac{1}{2}}.$$

# About the weighted elliptic operator

The elliptic operator $\mathfrak{L}_w$ in $\mathbb{R}^n$

$$\mathfrak{L}_w u = -\frac{1}{w(x)} \operatorname{div}_x w(x) \mathtt{A}(x) \nabla_x u$$

Is determined by the sesquilinear form

$$a_w(u, v) = \int_{\mathbb{R}^n} \mathtt{A}(x) \nabla u \cdot \overline{\nabla v}\, w(x)\, dx,$$

with domain $H^1_w(\mathbb{R}^n) = \left\{ f \in L^2_w(\mathbb{R}^n) : |\nabla f| \in L^2_w(\mathbb{R}^n) \right\} \subset L^2_w(\mathbb{R}^n)$.

- The operator $\mathfrak{L}_w$ is $\omega$-sectorial, $\omega = \omega(\lambda, \Lambda)$, and therefore it is m-accretive in $L^2_w(\mathbb{R}^n)$: we have $\left\| (1 + \mathfrak{L}_w)^{-1} \right\|_{\mathcal{B}(L^2_w)} \leq 1$, and

$$\sup_{z \in \Sigma_\theta} \left\| z (z + \mathfrak{L}_w)^{-1} \right\|_{\mathcal{B}(L^2_w)} \leq M_\theta, \qquad \theta \in \left( \frac{\pi}{2}, \pi - \omega \right).$$

The elliptic operator $\mathfrak{L}_w$ in $\mathbb{R}^n$

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Is determined by the sesquilinear form

$$a_w(u,v) = \int_{\mathbb{R}^n} \mathtt{A}(x) \nabla u \cdot \overline{\nabla v}\, w(x)\, dx,$$

with domain $H_w^1(\mathbb{R}^n) = \{f \in L_w^2(\mathbb{R}^n) : |\nabla f| \in L_w^2(\mathbb{R}^n)\} \subset L_w^2(\mathbb{R}^n)$.

- The operator $\mathfrak{L}_w$ is $\omega$-sectorial, $\omega = \omega(\lambda, \Lambda)$, and therefore it is m-accretive in $L_w^2(\mathbb{R}^n)$: we have $\left\| (1+\mathfrak{L}_w)^{-1} \right\|_{\mathcal{B}(L_w^2)} \leq 1$, and

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## Theorem (CU-R'15 )

*The operator* $\mathfrak{L}_w = \mathfrak{L}_w = -\frac{1}{w(x)} \operatorname{div}_x w(x) \mathtt{A}(x,t) \nabla_x$ *arises from the accretive form*

$$a_w(u,v) = \int_{\mathbb{R}^n} \mathtt{A}(x,t) \nabla_x u(x,t) \cdot \overline{\nabla_x v(x,t)}\, w(x)\, dx,$$

*it is maximal accretive in* $L^2(\mathbb{R}^n, dw)$, *the domain of its square root is that of the accretive form, that is,*

$$\mathcal{D}\left(\sqrt{\mathfrak{L}_w}\right) = H_w^1(\mathbb{R}^n) = \left\{ f \in L_w^2(\mathbb{R}^n) : |\nabla f| \in L_w^2(\mathbb{R}^n) \right\}.$$

*For* $\|f\|_w = \left(\int_{\mathbb{R}} |f(x)|^2\ w(x)\, dx\right)^{1/2}$. *The estimates*

$$\left\| \sqrt{\mathfrak{L}_w} u \right\|_w \approx \|\nabla_x u\|_w$$

*hold with constants depending only on n and the ellipticity constants of* $\mathtt{A}$.

## Theorem (Parabolic Kato estimate for time independent coefficients)

*Let* $\mathtt{A}(x)$ *be a complex valued* $n \times n$ *elliptic matrix in* $\mathbb{R}^n$*, i.e.*

$$\lambda |\xi|^2 \le \operatorname{Re} \langle \mathtt{A}(x) \xi, \xi \rangle, \qquad |\langle \mathtt{A}(x) \xi, \eta \rangle| \le \Lambda |\xi| |\eta| \qquad \text{for all } \xi, \eta \in \mathbb{C}^n.$$

*Let* $w$ *be an* $A_2$ *weight in* $\mathbb{R}^n$*, and let*

$$\mathcal{L}_w = \partial_t + \mathfrak{L}_w = \partial_t - \frac{1}{w} \operatorname{div} w \mathtt{A}(x) \nabla \qquad \text{in } \mathbb{R}^n \times \mathbb{R}$$

*then the square root operator* $\mathcal{L}_w^{1/2}$ *satisfies the estimates*

$$\left\| \mathcal{L}_w^{1/2} u \right\|_{\mathbf{w}} \approx \left\| D_t^{1/2} u \right\|_{\mathbf{w}} + \| \nabla_x u \|_{\mathbf{w}}$$

*for all* $u \in V = H^{\frac{1}{2}}\left(\mathbb{R}; L_w^2(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; H_w^1(\mathbb{R}^n)\right)$*, where the constants only depend on* $n$*,* $\lambda$*,* $\Lambda$*, and* $[w]_{A_2}$*. Here*

$$\|f\|_{\mathbf{w}} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}} |f(x,t)|^2 w(x) \, dx \, dt \right)^{\frac{1}{2}}.$$

The operator $\mathcal{L}_w = \frac{\partial}{\partial t} + \mathfrak{L}_w$ is defined via the sesquilinear form,

$$\mathfrak{a}_w(u,v) := \int_{\mathbb{R}} \left\langle H_t D_t^{\frac{1}{2}} u, D_t^{\frac{1}{2}} v \right\rangle_w + \langle \mathtt{A} \nabla u, \nabla v \rangle_w \, dt.$$

The domain $\dot{\mathcal{D}}(\mathcal{L}_w) \subset \dot{V} \subset L^2_{\mathbf{w}} = L^2\left(\mathbb{R}, L^2_w(\mathbb{R}^n)\right)$ is dense in $L^2_{\mathbf{w}}$, where

$$\dot{V} = \dot{H}^{\frac{1}{2}}\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; \dot{H}^1_w(\mathbb{R}^n)\right).$$

If $u \in \dot{\mathcal{D}}(\mathcal{L}_w)$, we say that

$$f := \mathcal{L}_w u \in L^2_{\mathbf{w}} \qquad \Longleftrightarrow \qquad \mathfrak{a}_{\mathbf{w}}(u,v) = \langle f, v \rangle_{\mathbf{w}} \quad \text{for all } v \in \left(\dot{V}\right)^*.$$

Using that $D_t^{1/2}$ commutes with $\mathfrak{L}_w$ and an approximation argument, we obtain

$$\left\| D_t^{1/2} u \right\|_{\dot{E}} \lesssim \| \mathcal{L}_w u \|_{\mathbf{w}} \qquad \text{for all } u \in \mathcal{D}(\mathcal{L}_w).$$

In particular,

$$\| \partial_t u \|_{\mathbf{w}} \lesssim \| \partial_t u + \mathfrak{L}_w u \|_{\mathbf{w}} \qquad \text{for all } u \in \mathcal{D}(\mathcal{L}_w).$$

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$$\left\| D_t^{1/2} u \right\|_{\dot{E}} \lesssim \|\mathcal{L}_w u\|_{\mathbf{w}} \qquad \text{for all } u \in \mathcal{D}(\mathcal{L}_w).$$

In particular,

$$\|\partial_t u\|_{\mathbf{w}} \lesssim \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}} \qquad \text{for all } u \in \mathcal{D}(\mathcal{L}_w).$$

Let $\varphi$ be a radial mollifier, write $f_\varepsilon = \varphi_\varepsilon * f$. Given $u \in \dot{\mathcal{D}}(\mathcal{L}_w)$ and $v \in \dot{V}$, let $f = \mathcal{L}_w u$, we have

$$\mathfrak{a}_w(u_\varepsilon, v) = \mathfrak{a}_w(u, v_\varepsilon) = \langle f, v_\varepsilon \rangle = \langle f_\varepsilon, v \rangle,$$

so $\mathcal{L}_w u_\varepsilon = f_\varepsilon$. In particular $u_\varepsilon \in \dot{\mathcal{D}}(\mathcal{L}_w) \bigcap C^\infty$.

Let now $v \in \dot{V} \bigcap \dot{H}^1(\mathbb{R}; L^2_w(\mathbb{R}^n))$, then $D_t^{1/2} v \in \dot{V}$, and so

$$\mathfrak{a}_w\left(D_t^{1/2} u_\varepsilon, v\right) = \mathfrak{a}_w\left(u_\varepsilon, D_t^{1/2} v\right) = \left\langle f_\varepsilon, D_t^{1/2} v \right\rangle_{\mathbf{w}} = \left\langle D_t^{1/2} f_\varepsilon, v \right\rangle_{\mathbf{w}},$$

so $\mathcal{L}_w D_t^{1/2} u_\varepsilon = D_t^{1/2} f_\varepsilon$, and $D^{1/2} f_\varepsilon \in (\dot{V})^*$ with $\left\| D^{1/2} f_\varepsilon \right\|_{V^*} \lesssim \|f\|_{\mathbf{w}}$. In particular, taking $v = D_t^{1/2} u_\varepsilon$

$$\left\| D_t^{1/2} u_\varepsilon \right\|_{\dot{V}}^2 = \mathfrak{a}_w\left(D_t^{1/2} u_\varepsilon, D_t^{1/2} u_\varepsilon\right) = \left\langle D_t^{1/2} f_\varepsilon, D_t^{1/2} u_\varepsilon \right\rangle_{\mathbf{w}} \lesssim \|f_\varepsilon\|_{\mathbf{w}} \left\| D_t^{1/2} u_\varepsilon \right\|_{\dot{V}}.$$

Letting $\varepsilon \to 0$ we obtain

$$\|\partial_t u\|_{\mathbf{w}} \le \left\| D_t^{1/2} u \right\|_V \lesssim \|\mathcal{L} u\|_{\mathbf{w}} = \|\partial_t u + \mathfrak{L}_w u\|.$$

Let $\varphi$ be a radial mollifier, write $f_\varepsilon = \varphi_\varepsilon * f$. Given $u \in \dot{\mathcal{D}}(\mathcal{L}_w)$ and $v \in \dot{V}$, let $f = \mathcal{L}_w u$, we have

$$\mathfrak{a}_w(u_\varepsilon, v) = \mathfrak{a}_w(u, v_\varepsilon) = \langle f, v_\varepsilon \rangle = \langle f_\varepsilon, v \rangle,$$

so $\mathcal{L}_w u_\varepsilon = f_\varepsilon$. In particular $u_\varepsilon \in \dot{\mathcal{D}}(\mathcal{L}_w) \bigcap C^\infty$.
Let now $v \in \dot{V} \bigcap \dot{H}^1(\mathbb{R}; L^2_w(\mathbb{R}^n))$, then $D_t^{1/2} v \in \dot{V}$, and so

$$\mathfrak{a}_w\left(D_t^{1/2} u_\varepsilon, v\right) = \mathfrak{a}_w\left(u_\varepsilon, D_t^{1/2} v\right) = \left\langle f_\varepsilon, D_t^{1/2} v \right\rangle_{\mathbf{w}} = \left\langle D_t^{1/2} f_\varepsilon, v \right\rangle_{\mathbf{w}},$$

so $\mathcal{L}_w D_t^{1/2} u_\varepsilon = D_t^{1/2} f_\varepsilon$, and $D^{1/2} f_\varepsilon \in (\dot{V})^*$ with $\left\| D^{1/2} f_\varepsilon \right\|_{V^*} \lesssim \|f\|_{\mathbf{w}}$. In particular, taking $v = D_t^{1/2} u_\varepsilon$

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Letting $\varepsilon \to 0$ we obtain

$$\|\partial_t u\|_{\mathbf{w}} \leq \left\| D_t^{1/2} u \right\|_V \lesssim \|\mathcal{L} u\|_{\mathbf{w}} = \|\partial_t u + \mathfrak{L}_w u\|.$$

Let $\varphi$ be a radial mollifier, write $f_\varepsilon = \varphi_\varepsilon * f$. Given $u \in \dot{\mathcal{D}}(\mathcal{L}_w)$ and $v \in \dot{V}$, let $f = \mathcal{L}_w u$, we have

$$\mathfrak{a}_w(u_\varepsilon, v) = \mathfrak{a}_w(u, v_\varepsilon) = \langle f, v_\varepsilon \rangle = \langle f_\varepsilon, v \rangle ,$$

so $\mathcal{L}_w u_\varepsilon = f_\varepsilon$. In particular $u_\varepsilon \in \dot{\mathcal{D}}(\mathcal{L}_w) \bigcap C^\infty$.
Let now $v \in \dot{V} \bigcap \dot{H}^1(\mathbb{R}; L^2_w(\mathbb{R}^n))$, then $D_t^{1/2} v \in \dot{V}$, and so

$$\mathfrak{a}_w\left(D_t^{1/2} u_\varepsilon, v\right) = \mathfrak{a}_w\left(u_\varepsilon, D_t^{1/2} v\right) = \left\langle f_\varepsilon, D_t^{1/2} v \right\rangle_{\mathbf{w}} = \left\langle D_t^{1/2} f_\varepsilon, v \right\rangle_{\mathbf{w}} ,$$

so $\mathcal{L}_w D_t^{1/2} u_\varepsilon = D_t^{1/2} f_\varepsilon$, and $D^{1/2} f_\varepsilon \in (\dot{V})^*$ with $\left\| D^{1/2} f_\varepsilon \right\|_{V^*} \lesssim \|f\|_{\mathbf{w}}$. In particular, taking $v = D_t^{1/2} u_\varepsilon$

$$\left\| D_t^{1/2} u_\varepsilon \right\|_{\dot{V}}^2 = \mathfrak{a}_w\left(D_t^{1/2} u_\varepsilon, D_t^{1/2} u_\varepsilon\right) = \left\langle D_t^{1/2} f_\varepsilon, D_t^{1/2} u_\varepsilon \right\rangle_{\mathbf{w}} \lesssim \|f_\varepsilon\|_{\mathbf{w}} \left\| D_t^{1/2} u_\varepsilon \right\|_{\dot{V}} .$$

Letting $\varepsilon \to 0$ we obtain

$$\|\partial_t u\|_{\mathbf{w}} \leq \left\| D_t^{1/2} u \right\|_V \lesssim \|\mathcal{L} u\|_{\mathbf{w}} = \|\partial_t u + \mathfrak{L}_w u\| .$$

Then also $\|\mathfrak{L}_w u\|_{\mathbf{w}} \leq \|\partial_t u\|_{\mathbf{w}} + \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}} \leq \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}}$, and we obtain the equivalences

$$\|\partial_t u\|_{\mathbf{w}} + \|\mathfrak{L}_w u\|_{\mathbf{w}} \lesssim \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}} \leq \|\partial_t u\|_{\mathbf{w}} + \|\mathfrak{L}_w u\|_{\mathbf{w}} .$$

This yields

$$\dot{\mathcal{D}}\left(\partial_t + \mathfrak{L}_w\right) = \dot{H}^1\left(\mathbb{R}; L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}\left(\mathfrak{L}_w\right)\right).$$

By **homogeneous interpolation**, we have

$$\begin{aligned} \dot{\mathcal{D}}\left(\left(\partial_t + \mathfrak{L}_w\right)^{\frac{1}{2}}\right) &= \left[\dot{H}^1\left(\mathbb{R}; L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}\left(\mathfrak{L}_w\right)\right)\right]_{\frac{1}{2}} \\ &\subset \left[\dot{H}^1\left(\mathbb{R}; L^2_w\left(\mathbb{R}^n\right)\right)\right]_{\frac{1}{2}} \bigcap \left[L^2\left(\mathbb{R}; \dot{\mathcal{D}}\left(\mathfrak{L}_w\right)\right)\right]_{\frac{1}{2}} \\ &= \dot{H}^{1/2}\left(\mathbb{R}; L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}\left(\mathfrak{L}_w^{1/2}\right)\right). \end{aligned}$$

From the Kato estimates for the weighted elliptic operators $\mathfrak{L}_w$[CU-R'15], we have that $\dot{\mathcal{D}}\left(\mathfrak{L}_w^{1/2}\right) = \dot{H}^1_w\left(\mathbb{R}^n\right)$, hence

$$\dot{\mathcal{D}}\left(\left(\partial_t + \mathfrak{L}_w\right)^{\frac{1}{2}}\right) \subset \dot{H}^{1/2}\left(\mathbb{R}; L^2_w\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}; \dot{H}^1_w\left(\mathbb{R}^n\right)\right).$$

Then also $\|\mathfrak{L}_w u\|_{\mathbf{w}} \le \|\partial_t u\|_{\mathbf{w}} + \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}} \le \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}}$, and we obtain the equivalences

$$\|\partial_t u\|_{\mathbf{w}} + \|\mathfrak{L}_w u\|_{\mathbf{w}} \lesssim \|\partial_t u + \mathfrak{L}_w u\|_{\mathbf{w}} \le \|\partial_t u\|_{\mathbf{w}} + \|\mathfrak{L}_w u\|_{\mathbf{w}}.$$

This yields

$$\dot{\mathcal{D}}(\partial_t + \mathfrak{L}_w) = \dot{H}^1\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}(\mathfrak{L}_w)\right).$$

By **homogeneous interpolation**, we have

$$\begin{aligned} \dot{\mathcal{D}}\left((\partial_t + \mathfrak{L}_w)^{\frac{1}{2}}\right) &= \left[\dot{H}^1\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}(\mathfrak{L}_w)\right)\right]_{\frac{1}{2}} \\ &\subset \left[\dot{H}^1\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right)\right]_{\frac{1}{2}} \bigcap \left[L^2\left(\mathbb{R}; \dot{\mathcal{D}}(\mathfrak{L}_w)\right)\right]_{\frac{1}{2}} \\ &= \dot{H}^{1/2}\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; \dot{\mathcal{D}}\left(\mathfrak{L}_w^{1/2}\right)\right). \end{aligned}$$

From the Kato estimates for the weighted elliptic operators $\mathfrak{L}_w$[CU-R'15], we have that $\dot{\mathcal{D}}\left(\mathfrak{L}_w^{1/2}\right) = \dot{H}^1_w(\mathbb{R}^n)$, hence

$$\dot{\mathcal{D}}\left((\partial_t + \mathfrak{L}_w)^{\frac{1}{2}}\right) \subset \dot{H}^{1/2}\left(\mathbb{R}; L^2_w(\mathbb{R}^n)\right) \bigcap L^2\left(\mathbb{R}; \dot{H}^1_w(\mathbb{R}^n)\right).$$

By a similar estimate for the adjoint operator, we obtain the equality

$$\dot{\mathcal{D}}\left(\left(\partial_t+\mathfrak{L}_w\right)^{\frac{1}{2}}\right)=\dot{H}^{1/2}\left(\mathbb{R};L^2_w\left(\mathbb{R}^n\right)\right)\bigcap L^2\left(\mathbb{R};\dot{H}^1_w\left(\mathbb{R}^n\right)\right).$$

Since the equality above is *meant as interpolation spaces*, we have the equivalence of norms

$$\left\|\left(\partial_t+\mathfrak{L}_w\right)^{\frac{1}{2}}u\right\|_{\mathbf{w}}\approx\left\|D_t^{1/2}u\right\|_{\mathbf{w}}+\left\|\nabla_x u\right\|_{\mathbf{w}}.$$

The constants in the elliptic Kato estimates $\left\|\mathfrak{L}_w^{1/2}u\right\|_w\approx\left\|\nabla_x u\right\|_w$ depend only on $n$, $\lambda$, $\Lambda$, and $[w]_{A_2}$ [CU-R'15], hence the equivalence above has the same dependence. □

# The parabolic Kato estimate and its first order system formulation

- Alan McIntosh first proposed that solving boundary value problems involving accretive sesquilinear forms could be addressed via a first order system of Cauchy-Riemann type (**Auscher**, **Axelsson**, and **Hofmann**. J. Funct. Anal. 255 (2008)).
- The sesquilinear form formulation of the parabolic problem, together with McIntosh's first order technique was recently applied to obtain several results on well-posedness of BVP for parabolic systems, and also the Kato estimate. (2017) - [AEN] **Auscher**, **Egert**, **Nyström**. "$L^2$ Well-posedness of BVP for parabolic systems with measurable coefficients."
- We generalized Auscher, Egert, and Nyström's results to weighted parabolic operators.

# The parabolic Kato estimate and its first order system formulation

As in [AEN], we consider parabolic equations in $\mathbb{R}^{n+1}_+ \times \mathbb{R} = \left(\mathbb{R}^{n+1}_+\right)_X \times \mathbb{R}_t = (\mathbb{R}^+)_\lambda \times \mathbb{R}^n_x \times \mathbb{R}_t$, $(X,t) = ((\lambda, x), t)$:

$$\mathcal{L}_w u = \partial_t u - \frac{1}{w(x)} \operatorname{div}_X w(x) \mathtt{A}(X,t) \nabla_X u = 0.$$

Where $\mathtt{A}(X,t) = \mathtt{A}((\lambda,x),t)$ is an $(n+1)\times(n+1)$ elliptic matrix with complex coefficients in $\mathbb{R}^{n+1}_+$ and $w \in A_2(\mathbb{R}^n)$.

We also say that $u$ is a **reinforced weak solution** if $u \in V := H^{1/2}\left(\mathbb{R}; L^2_{\text{loc}}\left(\mathbb{R}^{n+1}_+, d\lambda dw\right)\right) \bigcap L^2_{\text{loc}}\left(\mathbb{R}; H^1_{\text{loc}}\left(\mathbb{R}^{n+1}_+, d\lambda dw\right)\right)$ and

$$\int_0^\infty \iint_{\mathbb{R}^{n+1}} \mathtt{A}\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x}\phi}\, dw\, dt\, d\lambda - \int_0^\infty \iint_{\mathbb{R}^{n+1}} H_t D_t^{1/2} u\, \overline{D_t^{1/2}\phi}\, dw\, dt\, d\lambda = 0$$

for all $\phi \in C_0^\infty\left(\mathbb{R}^{n+2}_+\right)$.

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# A characterization of weighted Sobolev spaces

**Nicholas Miller**, "Weighted Sobolev spaces and pseudodifferential operators with smooth symbols", Trans. Amer. Math. Soc. 269 (1982).

## Theorem (Miller)

*Given a positive integer k and $1 < p < \infty$, let $\Lambda^k$ denote the Bessel potential of order $-k$, i.e. $\Lambda^k$ is a p.d.o. with symbol $\left(1+|\xi|^2\right)^{-\frac{k}{2}}$, then for every $w \in A_p\left(\mathbb{R}^n\right)$*

$$\Lambda^k\left(L_p\left(\mathbb{R}^n, dw\right)\right) = W^{k,p}\left(\mathbb{R}^n, dw\right),$$

*where $W^{k,p}\left(\mathbb{R}^n, dw\right)$ is the weighted Sobolev space of functions with k distributional derivatives satisfying*

$$\|f\|_{k,p,w} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p(w)} < \infty.$$

# Parabolic weighted Sobolev spaces

Considering the parabolic metric

$$|\xi|_{\cup}^{4} = \xi_1^4 + \xi_2^4 + \cdots + \xi_n^4 + \xi_{n+1}^2,$$

we can define p.d.o. adapted to parabolic problems. For example, defining $\Lambda_{\cup}^{s}$ as the p.d.o. with symbol $\left(1 + |\xi|_{\cup}^{4}\right)^{-\frac{s}{4}}$, and identifying the $n+1$-variable with $t$, we have, for integer $k \geq 0$

$$H_{\cup}^{2k}\left(\mathbb{R}^{n+1}\right) := \Lambda_{\cup}^{2k}\left(L^2\left(\mathbb{R}^{n+1}\right)\right) = H^k\left(\mathbb{R}, L^2\left(\mathbb{R}^n\right)\right) \bigcap L^2\left(\mathbb{R}, H^{2k}\left(\mathbb{R}^n\right)\right).$$

Similarly, given an $A_{p,\cup}$-weight (satisfies the $A_p$-condition on parabolic balls), we can define weighted parabolic Sobolev spaces. In our applications, we take $\mathbf{w}(x,t) = w(x) \in A_2(\mathbb{R}^n) \implies \mathbf{w} \in A_{2,\cup}\left(\mathbb{R}^{n+1}\right)$. Defining

$$H_{\cup}^{s}\left(\mathbb{R}^{n+1}, d\mathbf{w}\right) = \Lambda_{\cup}^{s}\left(L^2\left(\mathbb{R}^{n+1}, d\mathbf{w}\right)\right),$$

we have that

$$H_{\cup}^{2k}\left(\mathbb{R}^{n+1}, d\mathbf{w}\right) = H^k\left(\mathbb{R}, L^2\left(\mathbb{R}^n, dw\right)\right) \bigcap L^2\left(\mathbb{R}, H^{2k}\left(\mathbb{R}^n, dw\right)\right).$$

Interpolation techniques provide inclusions (equalities, considering dual

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Interpolation techniques provide inclusions (equalities, considering dual

# The parabolic Kato estimate and its first order system formulation

**Remark** A (classical) **weak solution** $u$ of $\mathcal{L}_w u = 0$ satisfies $u \in L^2_{\text{loc}}\left(\mathbb{R}; H^1\left(\mathbb{R}^{n+1}_+, d\lambda dw\right)\right)$ and

$$\int_{\mathbb{R}} \iint_{\mathbb{R}^{n+1}_+} \mathtt{A}\nabla_{\lambda,x} u \cdot \overline{\nabla_{\lambda,x}\phi}\, w(x)\, dx\, d\lambda\, dt - \int_{\mathbb{R}} \iint_{\mathbb{R}^{n+1}_+} u\, \overline{\partial_t \phi}\, w(x)\, dx\, d\lambda\, dt = 0$$

for all $\phi \in C_0^\infty\left(\mathbb{R}^{n+2}_+\right)$.

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Given a reinforced weak solution $u$, we define the $(n+2)$-vector

$$D_{\mathtt{A}} u(\lambda, x, t) := \begin{bmatrix} F_\perp \\ F_\parallel \\ F_\theta \end{bmatrix} = \begin{bmatrix} \partial_{\nu_{\mathtt{A}}} u(\lambda, x, t) \\ \nabla_x u(\lambda, x, t) \\ H_t D_t^{1/2} u(\lambda, x, t) \end{bmatrix}$$

(parabolic conormal differential),

where $\partial_{\nu_{\mathtt{A}}} u = \left(\mathtt{A} \nabla_{\lambda,x} u(\lambda, x, t)\right)_\perp$.
Then

$$D_{\mathtt{A}} u \in L^2_{\text{loc}}\left(\mathbb{R}; L^2_{\text{loc}}\left(\mathbb{R}^{n+1}_+, \mathbb{C}^{n+1}\right)\right) \times L^2\left(\mathbb{R}; L^2_{\text{loc}}\left(\mathbb{R}^{n+1}_+, \mathbb{C}\right)\right)$$

and, moreover,

$$\operatorname{curl}_x F_\parallel = 0, \qquad \nabla_x F_\theta = H_t D_t^{1/2} F_\parallel$$

in the sense of distributions on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$.
Split the coefficient matrix $A$ as

$$\mathtt{A}(\lambda, x, t) = \begin{bmatrix} \mathtt{A}_{\perp\perp}(\lambda, x, t) & \mathtt{A}_{\perp\parallel}(\lambda, x, t) \\ \mathtt{A}_{\parallel\perp}(\lambda, x, t) & \mathtt{A}_{\parallel\parallel}(\lambda, x, t) \end{bmatrix} = \begin{bmatrix} 1\times 1 & 1\times n \\ n\times 1 & n\times n \end{bmatrix}.$$

# The first order formulation

Given a reinforced weak solution $u$, we define the $(n+2)$-vector

$$D_{\mathtt{A}}u\,(\lambda,x,t) := \begin{bmatrix} F_{\perp} \\ F_{\parallel} \\ F_{\theta} \end{bmatrix} = \begin{bmatrix} \partial_{\nu_{\mathtt{A}}} u\,(\lambda,x,t) \\ \nabla_x u\,(\lambda,x,t) \\ H_t D_t^{1/2} u\,(\lambda,x,t) \end{bmatrix}$$

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# The first order formulation

Let

$$\widehat{\mathtt{A}} := \begin{bmatrix} 1 & 0 \\ \mathtt{A}_{\parallel\perp} & \mathtt{A}_{\parallel\parallel} \end{bmatrix} \begin{bmatrix} \mathtt{A}_{\perp\perp} & \mathtt{A}_{\perp\parallel} \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathtt{A}_{\perp\perp}^{-1} & -\mathtt{A}_{\perp\perp}^{-1}\mathtt{A}_{\perp\parallel} \\ \mathtt{A}_{\parallel\perp}\mathtt{A}_{\perp\perp}^{-1} & \mathtt{A}_{\parallel\parallel} - \mathtt{A}_{\parallel\perp}\mathtt{A}_{\perp\perp}^{-1}\mathtt{A}_{\perp\parallel} \end{bmatrix}$$

Introduce the operators

$$P_w = \begin{bmatrix} 0 & \frac{1}{w}\operatorname{div}_x w & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} \widehat{\mathtt{A}}_{\perp\perp} & \widehat{\mathtt{A}}_{\perp\parallel} & 0 \\ \widehat{\mathtt{A}}_{\parallel\perp} & \widehat{\mathtt{A}}_{\parallel\parallel} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $$P_w^* = \begin{bmatrix} 0 & \frac{1}{w}\operatorname{div}_x w & H_t D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -D_t^{1/2} & 0 & 0 \end{bmatrix},$$ and

- $P_w$ does not depend on $\lambda$.
- $M$ is a bounded accretive operator.
- $D_t^{1/2}$ is non-local and $H_t D_t^{1/2}$ is not self-adjoint.

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### Theorem (Functional Calculus for the Parabolic Dirac Operator)

*The operator $MP_w$ is bisectorial on $\mathbb{L}^2_w$ with range $\mathrm{R}(P_w M) = \mathrm{R}(P_w)$. It satisfies the quadratic estimate*

$$\int_0^\infty \left\| \lambda P_w M \left(1 + (\lambda P_w M)^2\right)^{-1} h \right\|_{\mathbf{w}}^2 \frac{d\lambda}{\lambda} \sim \|h\|_{\mathbf{w}}^2 \qquad \left(h \in \overline{\mathrm{R}(P_w M)}\right).$$

*The angle $\omega$ of bisectoriality and constants in the quadratic estimate depend on $n$, the ellipticity of $\mathtt{A}$, and the $A_2$-norm $[w]_{A_2}$ of $w$. In particular $P_w M$ has a bounded holomorphic functional calculus on $\overline{\mathrm{R}(P_w M)} = \overline{\mathrm{R}(P_w)}$ on open double sectors $S_\mu$for all $\mu \in (\omega, \pi/2)$. The same holds true for $MP_w$ on $\overline{\mathrm{R}(MP_w)} = M\overline{\mathrm{R}(P_w)}$.*

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# From the first order estimate to Kato

In order to obtain the Kato estimate from the square function estimate for $P_w M$, we take

$$\mathtt{A}(\lambda, x, t) = \mathtt{A}(x, t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A(x,t) & 0 \\ 0 & 0 & 1 \end{bmatrix} = M.$$

Then

$$P_w M = \begin{bmatrix} 0 & \frac{1}{w} \operatorname{div}_x w\, A & -D_t^{1/2} \\ -\nabla_x & 0 & 0 \\ -H_t D_t^{1/2} & 0 & 0 \end{bmatrix},$$

and

$$(P_w M)^2 = \begin{bmatrix} \mathcal{L}_w & 0 & 0 \\ 0 & -\nabla_x \frac{1}{w} \operatorname{div}_x w\, A & \nabla_x D_t^{1/2} \\ 0 & -H_t D_t^{1/2} \frac{1}{w} \operatorname{div}_x w\, A & \partial_t \end{bmatrix}.$$

# From the first order estimate to Kato

1. Since $P_w M$ has a bounded functional calculus on $\overline{R(P_w)}$, the operator sign $(P_w M)$ is a bounded involution in this space.
2. Therefore, $[P_w M] := \text{sign}(P_w M) P_w M$ and $P_w M$ have the same domain and
$$\|[P_w M]\| \approx \|P_w M\|.$$
3. Since, moreover, $[P_w M] = \sqrt{(P_w M)^2}$, it follows that
$$[P_w M] = \begin{bmatrix} \sqrt{\mathcal{L}_w} & 0 \\ 0 & \sqrt{T_w} \end{bmatrix}, \quad T_w = \begin{bmatrix} -\nabla_x \frac{1}{w} \operatorname{div}_x w\, A & \nabla_x D_t^{1/2} \\ -H_t D_t^{1/2} \frac{1}{w} \operatorname{div}_x w\, A & \partial_t \end{bmatrix}.$$
4. Specializing to $\mathbf{h} = \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \in \overline{R(P_w)}$, it follows that
$$\begin{aligned} \left\|\sqrt{\mathcal{L}_w h}\right\|_{\mathbf{w}} &= \|[P_w M]\,\mathbf{h}\| \approx \|(P_w M)\,\mathbf{h}\| \\ &= \|\nabla_x h\|_{\mathbf{w}} + \left\|H_t D_t^{1/2} h\right\|_{\mathbf{w}} \approx \|\nabla_x h\|_{\mathbf{w}} + \left\|D_t^{1/2} h\right\|_{\mathbf{w}}. \ \square \end{aligned}$$

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# Observations about the proof of the square function estimate

- Our proof follows closely the original proof in [AEN] for unweighted operators, based on the algorithm for the elliptic problem from **Auscher**, **Axelsson**, and **McIntosh**, Contemp. Math. (2010).
- The classic Sobolev spaces $W^{1,p}$ are replaced by the weighted Sobolev spaces $W^{1,p}(w)$.
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- Unweighted estimates based on the parabolic metric (or modified metric like in the off-diagonal decay of the resolvent) naturally extend to the weighted setting (C-Z. theory, spaces of homogeneous type).
- The final decomposition for the proof of the Carleson measure estimate in the $\mathtt{T}(b)$ theorem required a corona decomposition of the weight, based on a construction due to **Garnett**, "Bounded analytic functions" (1981), as applied in **Auscher**, **Rosén**, and **Rule**, "Boundary value problems for degenerate elliptic equations and systems", Ann. Sci. Éc. Norm. Supér. (2015), for elliptic operators.
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- The classic Sobolev spaces $W^{1,p}$ are replaced by the weighted Sobolev spaces $W^{1,p}(w)$.
- Some square function estimates required weighted norm inequalities techniques, and interpolation with change of measure.
- Unweighted estimates based on the parabolic metric (or modified metric like in the off-diagonal decay of the resolvent) naturally extend to the weighted setting (C-Z. theory, spaces of homogeneous type).
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# Open problems

A general framework as proposed in [AEN] is to consider the operators

$$\mathcal{L}_{w_1,w_2}u = w_1(x,t)\,\partial_t u - \operatorname{div} w_2(x,t)\,\mathtt{A}(x,t)\,\nabla u.$$

- We considered the case $w_1 = w_2 = w(x) \in A_2(\mathbb{R}^n)$.
- If $w_1 = w_2 = w(x,t)$, a weak formulation of $\mathcal{L}_{w_1,w_2}u = 0$ is

$$\iint \left(\mathtt{A}(x,t)\,\nabla_x u \cdot \nabla_x v + \partial_t u\, v\right)\, dw(x,t),$$

this was studied by **Chiarenza** and **Serapioni** (1984), (1987). They considered $w(x,t) \in A_2(t) \bigcap A_{1+\frac{2}{n}}(x)$.
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