Harmonic measure and quantitative rectifiability

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ICMAT
Research Term on Real Harmonic Analysis
and its Applications to PDE and GMT

08/05/2018

Harmonic measure in $\Omega \subset \mathbb{R}^{n+1}$

Harmonic measure of a subset E of the boundary of a domain Ω in $\mathbb{R}^{n+1}$, is the probability that a Brownian motion started inside a domain first hits $\partial\Omega$ at E.

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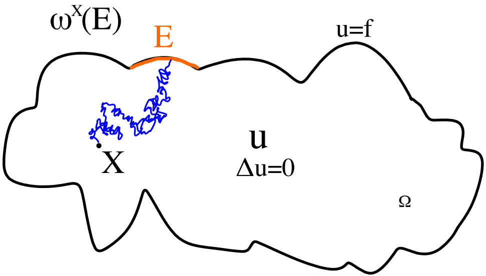

Geometry of the boundary $\partial\Omega$

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Definition

A Borel set $E \subset \mathbb{R}^d$ is ***n*-rectifiable** if there exist $E_i \subset \mathbb{R}^n$ and $f_i : E_i \to \mathbb{R}^d$ Lipschitz so that $\mathcal{H}^n(E \setminus \bigcup_{i=1}^{\infty} f_i(E_i)) = 0$.

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$E \subset \mathbb{R}^d$ is *n*-**AD-regular** if $\forall x \in E$ and $\forall r \in (0, \mathrm{diam}\,(E))$.

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The set $E \subset \mathbb{R}^d$ is **uniformly *n*-rectifiable** if

- E is n-AD-regular
- $\exists\, \theta, M > 0$ s.t. $\forall x \in E$ and $\forall r \in (0, \operatorname{diam}(E))$
  $\exists\, g_{x,r} : B_n(0,r) \subset \mathbb{R}^n \to \mathbb{R}^d$ an M-Lipschitz mapping s.t.

$$\mathcal{H}^n(E \cap B(x,r) \cap g_{x,r}(B_n(0,r))) \geq \theta r^n.$$

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Theorem (David-Semmes)

μ is uniformly n-rectifiable $\Leftrightarrow$
for all kernels K as above, $T_{K,\mu} : L^2(\mu) \to L^2(\mu)$ is bounded.

- **Riesz kernel**: $K(x) = \frac{x}{|x|^{n+1}}, \quad x \neq 0.$
- **Riesz transform**: $\mathcal{R}_\mu f(x) = \int K(x-y) f(y)\, d\mu(y).$

Uniform Rectifiability and Riesz Transform

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Question (David-Semmes Problem)

Let μ be an n-AD regular measure in $\mathbb{R}^d$.
$\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ *is bounded* $\overset{?}{\Rightarrow}$ μ *is uniformly n-rectifiable.*

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- $n = d - 1$, Nazarov, Tolsa and Volberg, '14.
- $2 \leq n \leq d - 2$ still OPEN!

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$$G(x,p) = \mathcal{E}(x-p) - \int \mathcal{E}(x-y)\, d\omega^p(y). \qquad (1)$$

Riesz transform and harmonic measure

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- Hence, differentiating both sides of (1),

$$\begin{aligned} \nabla G(x, p) &= \nabla \mathcal{E}(x - p) - \int K(x - y) d\omega^p(y) \\ &= \nabla \mathcal{E}(x - p) - \mathcal{R}\omega^p(x). \end{aligned}$$

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So the Riesz transform is naturally connected to harmonic measure and Green function.

“Qualitative” one-phase and two-phase free boundary problems (FBP) for harmonic measure

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Theorem (Azzam, Hofmann, Martell, Mayboroda, M., Tolsa, Volberg, '15)

Let $\Omega \subset \mathbb{R}^{n+1}$ *is a domain,* $E \subset \partial\Omega$ *with* $\mathcal{H}^n(E) < \infty$.
$\omega|_E \ll \mathcal{H}^n|_E \ll \omega|_E \implies E$ *is n-rectifiable.*

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Theorem (Azzam, M., Tolsa, ’17, & Azzam, M., Tolsa, Volberg, ’17)

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ are disjoint domains and let $E \subset \partial\Omega_1 \cap \partial\Omega_2$.
$\omega^1 \ll \omega^2 \ll \omega^1$ *on $E \Rightarrow \exists$ n-rectifiable $F \subset E$ s.t.*

- $\omega^1(E \setminus F) = \omega^2(E \setminus F) = 0$
- $\omega^1 \ll \omega^2 \ll \mathcal{H}^n \ll \omega^1$ *on F.*

“Quantitative” one-phase (FBP) for ω

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Theorem (Hofmann, Martell, ’14 and Hofmann, Martell, Uriarte-Tuero, ’14 (proved in ’12))

If $\Omega \subset \mathbb{R}^{n+1}$ is 1-NTA domain with AD-regular boundary, then $\omega^{x_B} \in w - A_\infty(B) \Leftrightarrow \partial\Omega \in UR$, for some corkscrew point $x_B \in cB \cap \Omega$.

Theorem (Azzam, Hofmann, Martell, Nyström, Toro ’17 (proved in ’14))

If $\Omega \subset \mathbb{R}^{n+1}$ is 1-NTA domain with UR boundary, then it is NTA $\Rightarrow \omega \in A_\infty$.

Theorem (Hofmann, Martell, ’15–Hofmann, Le, Martell, Nystrom, ’17)

Let $\Omega \subset \mathbb{R}^{n+1}$ open with the interior corkscrew condition and ADR boundary. If $\omega \ll \sigma$ and for every B, $\exists\, x_B$ Corkscrew point in B, $\int_{2B\cap\Omega} (k^{x_B})^q \, d\sigma \lesssim \sigma(B)^{1-q}$, then $\partial\Omega$ is uniformly rectifiable.

Theorem (M., Tolsa, ’15)

Assume that $\Omega \subset \mathbb{R}^{n+1}$ *is open and* μ *is a measure s.t.*
$\mu(B(x,r) \lesssim r^n$ *and* $\operatorname{supp}(\mu) \subset \partial\Omega$.
If $\exists\, \varepsilon, \varepsilon' \in (0,1)$ *s.t.*
$\forall \mu$*-doubling ball* B *centered at* $\operatorname{supp}\mu$ *with*
$\operatorname{diam}(B) \leq \operatorname{diam}(\operatorname{supp}\mu)$,
$\exists\, x_B \in \kappa B \cap \Omega$ *s.t.:for any* $E \subset B$,
if $\quad \mu(E) \leq \varepsilon\, \mu(B), \quad$ *then* $\quad \omega^{x_B}(E) \leq \varepsilon'\, \omega^{x_B}(B).$
Then the Riesz transform $\mathcal{R}_\mu : L^2(\mu) \to L^2(\mu)$ *is bounded.*

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Ω is **semiuniform domain** if for every $x \in \Omega$ and $y \in \partial\Omega$, there exists a rectifiable curve $\gamma \subset \overline{\Omega}$ connecting x and y such that

- $\lambda \min(\ell(x,z), \ell(z,y)) \leq \mathrm{dist}(z, \partial\Omega)$
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- *Ω has very big pieces of interior chord-arc subdomains,*

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- *$\omega \in A_\infty$.*

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Definition

Given $x \in \Omega$, $y \in \partial\Omega$, and $\lambda > 0$, a λ**-carrot curve** (or just *carrot curve*) from x to y is a curve $\gamma \subset \Omega \cup \{y\}$ with end-points x and y such that $\delta_\Omega(z) := \mathrm{dist}(z, \partial\Omega) \geq \kappa\, \mathcal{H}^1(\gamma(y, z))$ for all $z \in \gamma$, where $\gamma(y, z)$ is the arc in γ between y and z.

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Definition

Ω satisfies the **weak local John condition** (with parameters λ, θ, Λ) if there are constants $\lambda, \theta \in (0, 1)$ and $\Lambda \geq 2$ such that for every $x \in \Omega$ there is a Borel subset $F \subset B(x, \Lambda\delta_\Omega(x)) \cap \partial\Omega$) with $\mathcal{H}^n(F) \geq \theta\, \mathcal{H}^n(B(x, \Lambda\delta_\Omega(x)) \cap \partial\Omega)$ such that every $y \in F$ can be joined to x by a λ-carrot curve.

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- *Ω has big pieces of interior chord-arc subdomains.*

“Quantitative” two-phase (FBP) for ω

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Theorem (Azzam, M., Tolsa, 2017)

Let $\Omega_1, \Omega_2 \subsetneq \mathbb{R}^{n+1}$ disjoint CDC domains and $\omega_i^{p_i}$ be harmonic measures of Ω_i with poles $p_i \in \Omega_i$, i=1,2. Set $F := \partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$.

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$$\text{if} \quad \omega_1^{x_1}(E) \leq \varepsilon\, \omega_1^{x_1}(B), \qquad \text{then} \quad \omega_2^{x_2}(E) \leq \varepsilon'\, \omega_2^{x_2}(B). \tag{2}$$

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If ε' is small enough (depending only on n and the CDC constant), then $\exists\, \theta_i \in (0,1)$ and a uniformly rectifiable set $\Sigma_B \subset \mathbb{R}^{n+1}$ such that

$$\omega_i^{x_i}(\Sigma_B \cap F \cap B) \geq \theta_i, \quad i = 1,2.$$

Theorem (Azzam, M., Tolsa, 2017)

Let $\Omega_1, \Omega_2 \subsetneq \mathbb{R}^{n+1}$ *disjoint CDC domains and* $\omega_i^{p_i}$ *be harmonic measures of* Ω_i *with poles* $p_i \in \Omega_i$, *i=1,2. Set* $F := \partial\Omega_1 \cap \partial\Omega_2 \neq \varnothing$. *Suppose* $\exists\, \varepsilon, \varepsilon' \in (0,1)$ *s.t. if B ball centered at F with* $\operatorname{diam}(B) \leq \min(\operatorname{diam}(\partial\Omega_1), \operatorname{diam}(\partial\Omega_2))$, $\exists\, x_i \in \frac{1}{4}B \cap \Omega_i$ *for* $i = 1, 2$, *s.t.:* $\forall E \subset B$,

$$\text{if} \quad \omega_1^{x_1}(E) \leq \varepsilon\, \omega_1^{x_1}(B), \qquad \text{then} \quad \omega_2^{x_2}(E) \leq \varepsilon'\, \omega_2^{x_2}(B). \tag{2}$$

If ε' *is small enough (depending only on n and the CDC constant), then* $\exists\, \theta_i \in (0,1)$ *and a uniformly rectifiable set* $\Sigma_B \subset \mathbb{R}^{n+1}$ *such that*

$$\omega_i^{x_i}(\Sigma_B \cap F \cap B) \geq \theta_i, \quad i = 1, 2.$$

Moreover, there is $c > 0$ *so that if* x_1 *is also corkscrew points for* $\frac{1}{4}B$, *then*

$$\mathcal{H}^d(\Sigma_B \cap F \cap B) \geq c r(B)^n. \tag{3}$$

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- $P_{\gamma,\mu}(B) = \sum_{j\geq 0} 2^{-j\gamma}\,\Theta^n_\mu(2^j B)$,
- **Jones' β function**:
  For a measure μ, $\xi \in \operatorname{supp}\mu$ and $r > 0$, we define

$$\beta_{\mu,1}(\xi, r) := \inf_L \beta^L_{\mu,1}(B(\xi, r)) := \inf_L \frac{1}{r^n} \int_{B(\xi,r)} \frac{\operatorname{dist}(x, L)}{r} d\mu(x)$$

where the infimum is over all n-dimensional planes L.

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- If B is arbitrary ball, $G \subset B$ so that $\omega_2(B \backslash G) < 2\varepsilon' \omega_2(B)$, and

$$A^{-1} \frac{\omega_1(B)}{\omega_2(B)} \leq \frac{\omega_1(B(x,r))}{\omega_2(B(x,r))} \leq A \frac{\omega_1(B)}{\omega_2(B)} \quad x \in G, 0 < r < r(B).$$

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  (stopping type argument)
- There is a ball $B_2 \subset B_1$ centered at F, with $r(B_1) \approx r(B_2)$ and $r(B_2) \lesssim \tau\, |p_i - c_{B_2}|$, for $i = 1, 2$, s.t. $\omega_i(B_2) \gtrsim \omega_i(B_1)$, $\beta^n_{1,\omega_2}(B_2) < \eta\, \Theta^n_{\omega_2}(B_2)$. (compactness argument-limits of harmonic measure-use [AMTV16])

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- $G^{sd} = G \setminus G^{bd}$.

Rectifiability criterion of Nazarov, Tolsa and Volberg

Theorem

There is $C > 0$ *depending only on* n *such that for all* $p, q > 1$ *the following holds. Let* ν *be a measure supported in a ball* B *such that* $\nu(B(x,r)) \leq r^n$ *for all* $x \in \mathbb{R}^{n+1}$ *and* $r > 0$*, and define*

$$F_p = \{x \in \operatorname{supp}(\nu) : \nu(B(x,r)) \geq r^n/p\},$$

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If $\mathcal{R}_\nu : L^2(\nu) \to L^2(\nu)$ is bounded, then there is a $\frac{C}{pq}$-AD regular measure σ so that $\sigma|_{F_{p,q}} = \nu|_{F_{p,q}}$ and $\mathcal{R}_\sigma : L^2(\sigma) \to L^2(\sigma)$ is bounded. That is, σ is uniformly rectifiable.
If $\nu(F_p) \geq \delta\|\nu\|$ then there exists q large depending on δ s.t. $\nu(F_{p,q}) \geq \frac{1}{2}\nu(F_p)$.

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- The result now follows.

Rectifiability criterion of Girela-Sarrión and Tolsa

Let μ be Radon and $B \subset \mathbb{R}^{n+1}$ a ball with $\mu(B) > 0$ s.t.

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(c) For some constant $C_1 > 0$, there is $G_B \subset B$ such that

$$\sup_{0<r\leq 2r(B)} \frac{\mu(B(x,r))}{r^n} + \mathcal{R}_*(\chi_{2B}\,\mu)(x) \leq C_1\,\Theta_\mu(B), \text{ for all } x \in G_B$$

and $\mu(B \setminus G_B) \leq \delta\,\mu(B)$.

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(d) For some constant $0 < \tau \ll 1$,

$$\int_{G_B} |\mathcal{R}\mu(x) - m_{\mu,G_B}(\mathcal{R}\mu)|^2\, d\mu(x) \leq \tau\, \Theta_\mu(B)^2 \mu(B).$$

If δ, τ small enough, there is a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that $\mu(G_B \cap \Gamma) \geq \theta\, \mu(B)$.

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- Have shown that: $P_{\gamma,\omega_2}(B) \lesssim \Theta_{\omega_2}(B)$, for any $\gamma \in [0,1]$, $\beta_{\omega_2,1}(B) < \eta\,\Theta_{\omega_2}(B)$, and $r(B) \lesssim \tau\,|x_2 - c_B|$.

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- It is enough to prove (c) and (d) in Girela-Sarrión and Tolsa for $\mu = \omega_2$ and $G_B = G^{sd}$.

Proof of (c)–bounded density of ω_i

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Theorem

Let $B(x,R) \subset \mathbb{R}^{n+1}$, and let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative subharmonic functions. Suppose that $u_1(x) = u_2(x) = 0$ and that $u_1 \cdot u_2 \equiv 0$. Set

$$\gamma(x,r) = \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}}dy\right) \cdot \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}}dy\right). \tag{4}$$

Then $\gamma(x,r)$ is a non-decreasing function of $r \in (0,R)$ and $\gamma(x,r) < \infty$ for all $r \in (0,R)$. That is,

$$\gamma(x,r_1) \leq \gamma(x,r_2) < \infty \quad \text{for} \quad 0 < r_1 \leq r_2 < R. \tag{5}$$

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$$\frac{\omega_i(B(\xi, r))}{r^n} \lesssim \left(\frac{1}{r^2} \int_{B(\xi,2r)} \frac{|\nabla u_i(y)|^2}{|y - \xi|^{n-1}} dy \right)^{\frac{1}{2}} \lesssim \left(\frac{1}{r^{n+3}} \int_{B(\xi,4r)} |u_i|^2 \right)^{\frac{1}{2}}$$

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$$\left(\frac{1}{r^{n+3}}\int_{B(\xi,\rho)\cap\Omega_i} |u_i|^2\right)^{\frac{1}{2}} \lesssim \frac{\omega_i(B(\xi,C_1\rho))}{\rho^n}.$$

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Thus, by mutual absolute continuity, for $r < R/(8C_1)$

$$\frac{\omega_i(B(\xi,r))}{r^n} \lesssim \gamma(\xi,2r)^{1/2} \leq \gamma(\xi,R/2C_1)^{1/2} \lesssim \frac{\omega_i(B(\xi,R))}{R^n}$$

Thank you!!!

$\varepsilon\upsilon\chi\alpha\rho\iota\sigma\tau\omega \quad \pi o\lambda\upsilon$!!!

Muchas gracias!!!