Quadratic estimates for Hodge–Dirac operators with L^p-singular potentials

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Outline

1. The Kato square-root problem and quadratic estimates
   - Motivation and new results
2. Local quadratic estimates for inhomogeneous operators
   - Resolvent estimates
3. Quadratic estimates for homogeneous operators
   - Perturbed Hodge decompositions
   - Error terms in the principal part reduction

Theorem (Auscher–Hofmann–Lacey–McIntosh–Tchamitchian)

If $n \in \mathbb{N}$ and $A = A(x)$ is an $n \times n$ matrix of functions in $L^\infty(\mathbb{R}^n, \mathbb{C})$ satisfying the uniform ellipticity condition

$$Re\langle A\nabla u, \nabla u\rangle_{L^2} \gtrsim \|\nabla u\|_2^2 \qquad \forall u \in W^{1,2}(\mathbb{R}^n),$$

then $Dom(\sqrt{-\operatorname{div} A\nabla}) = W^{1,2}(\mathbb{R}^n)$ and $\|\sqrt{-\operatorname{div} A\nabla} u\|_2 \eqsim \|\nabla u\|_2$.

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Corollary

If fields $b_1, b_2 \in L^\infty(\mathbb{R}^n, \mathbb{C}^n)$ and a potential $V \in L^\infty(\mathbb{R}^n, \mathbb{C})$ satisfy

$$Re\langle A\nabla u + b_1 u, \nabla u\rangle + \langle b_2 \cdot \nabla u + Vu, u\rangle \gtrsim \|\nabla u\|_2^2 + \|u\|_2^2$$

then $Lu := -\operatorname{div} A\nabla u - \operatorname{div} b_1 u + b_2 \cdot \nabla u + Vu$ satisfies

$$Dom(\sqrt{L}) = W^{1,2}(\mathbb{R}^n) \quad \text{and} \quad \|\sqrt{L}u\|_2 \eqsim \|\nabla u\|_2 + \|u\|_2.$$

Theorem (Gesztesy–Hofmann–Nichols)

If $p \geq n \geq 3$ and $b_1, b_2 \in L^p + L^\infty$ and $V \in L^{p/2} + L^\infty$, then

$$Dom(\sqrt{L}) = W^{1,2}(\mathbb{R}^n) \quad \text{and} \quad \|\sqrt{L}u\|_2 \approx \|\nabla u\|_2 + \|u\|_2.$$

Moreover, there exists $\epsilon > 0$ such that

$$\max\{\|b_1\|_n, \|b_2\|_n, \|V\|_{n/2}\} < \epsilon \quad \Longrightarrow \quad \|\sqrt{L}u\|_2 \approx \|\nabla u\|_2.$$

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The critical index $p = n$ is determined by the Sobolev embedding

$$\|u\|_{2^*} \leq c_n \|\nabla u\|_2, \qquad 2^* := 2n/(n-2),$$

and Hölder's inequality:

$$\|b_i u\|_2 \leq \|b_i\|_n \, \|u\|_{2^*} \leq c_n \epsilon \|\nabla u\|_2$$
$$\|V^{1/2} u\|_2 \leq \|V\|_{n/2}^{1/2} \|u\|_{2^*} \leq c_n \epsilon \|\nabla u\|_2$$

The first-order system (on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, \mathbb{C}^n)$)

$$D_B := d + d^* B := \begin{bmatrix} 0 & 0 \\ \nabla & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\operatorname{div} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & A \end{bmatrix} = \begin{bmatrix} 0 & -\operatorname{div} A \\ \nabla & 0 \end{bmatrix}$$

is a bisectorial perturbation of $D := d + d^*$ and

$$\sqrt{D_B^2} = \begin{bmatrix} \sqrt{-\operatorname{div} A \nabla} & 0 \\ 0 & \sqrt{-\nabla A \operatorname{div}} \end{bmatrix}.$$

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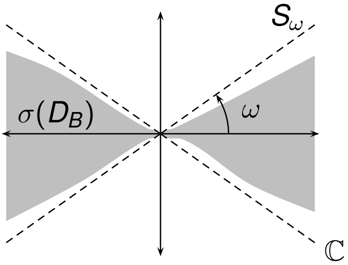
This means that

$$\|(zI - D_B)^{-1} u\|_2 \lesssim |z|^{-1} \|u\|_2$$

for all $z \in \mathbb{C} \setminus S_\omega$.



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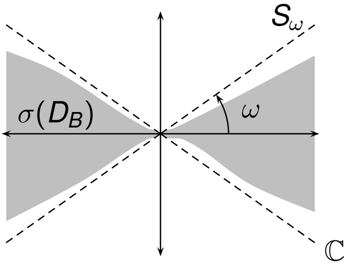

If D_B has an H^∞-functional calculus, then

$$\|\operatorname{sgn}(D_B) u\|_2 \lesssim \|u\|_2$$

$$\implies \|\sqrt{D_B^2} u\|_2 \lesssim \|D_B u\|_2$$

$$\implies \|\sqrt{-\operatorname{div} A\nabla} u_1\| \lesssim \|\nabla u_1\|_2.$$

Theorem (Axelsson–Keith–McIntosh)

If $D_B := d + d^*B$, where d is a constant-coefficient, homogeneous, first-order differential operator on $L^2(\mathbb{R}^n, \mathbb{C}^N)$ and B is a multiplication operator in $L^\infty(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ such that

- *(nilpotency)* $d^2 = (d^*)^2 = 0$ *and* $dBd = d^*Bd^* = 0$
- *(accretivity)* $Re\langle Bu, u\rangle \gtrsim \|u\|_2^2$ *for all* $u \in \overline{Ran}(D)$
- *(coercivity)* $\|\nabla \otimes u\|_2 \lesssim \|Du\|_2$ *for all* $u \in \overline{Ran}(D)$

then

$$\overline{Ran}(d) \oplus \overline{Ran}(d^*B) \oplus Nul(D_B) = L^2$$

and D_B has an H^∞-functional calculus, i.e.

$$\int_0^\infty \|tD_B(1 + t^2 D_B^2)^{-1}u\|_2^2 \frac{dt}{t} \eqsim \|u\|_2^2 \qquad \forall u \in \overline{Ran}(D_B).$$

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Example

If d is the exterior derivative on differential forms $L^2(\oplus_{k=0}^n \Lambda^k)$, then D_B is a metric perturbation of the Dirac operator $D = d + d^*$.

Let us incorporate additive perturbations $D + W$ in this framework to

1. Extend the results of Gesztesy–Hofmann–Nichols.
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Theorem (Gesztesy–Hofmann–M–Nichols)

If $p \geq n \geq 3$ and $D_{B,W} := (d + W_1) + (d^* + W_2)B$, where W_1, W_2 are multiplication operators in $L^p(\mathbb{R}^n, \mathcal{L}(\mathbb{C}^N))$ such that

$$L^2 \xrightarrow{W_1} Nul(d) \xrightarrow{W_1} 0, \qquad L^2 \xrightarrow{W_2} \overline{Ran}(d^*) \xrightarrow{W_2} 0,$$

and $W_1 B W_1 = W_1 B d = W_2 B d^* = W_2 B W_2 = 0$, then $\exists\, T > 0$ such that

$$\int_0^T \|tD_{B,W}(1 + t^2 D_{B,W}^2)^{-1} u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \qquad \forall u \in L^2.$$

Moreover, $\exists\, \epsilon > 0$ such that if $\max\{\|W_1\|_n, \|W_2\|_n\} < \epsilon$, then

$$\int_0^\infty \|tD_{B,W}(1 + t^2 D_{B,W}^2)^{-1} u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \qquad \forall u \in L^2.$$

We recover $Lu := -\operatorname{div} A\nabla u - \operatorname{div} b_1 u + b_2 \cdot \nabla u + Vu$ by writting

$$V = V_1 V_2 := |V|^{1/2}(|V|^{1/2} e^{i \arg V}), \quad V_1, V_2 \in L^p,$$

and defining (on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n, \mathbb{C}^n) \oplus L^2(\mathbb{R}^n)$)

$$D_B := d + d^* B := \begin{bmatrix} 0 & 0 & 0 \\ \nabla & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\operatorname{div} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Next, choose $\beta_1, \beta_2, \gamma_1, \gamma_2 \in L^p$ with

$$W_1 + W_2 := \begin{bmatrix} 0 & 0 & 0 \\ \beta_1 & 0 & 0 \\ \gamma_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_2 \cdot & \gamma_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so that $(-\operatorname{div} A + \beta_2 \cdot A)(\nabla + \beta_1) + \gamma_2 \gamma_1 = L$, whence

$$(D_{B,W})^2 = \begin{bmatrix} 0 & -\operatorname{div} A + \beta_2 \cdot A & \gamma_2 \\ \nabla + \beta_1 & 0 & 0 \\ \gamma_1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} L & 0 & 0 \\ 0 & x & x \\ 0 & x & x \end{bmatrix}.$$

We need to verify the hypotheses of our theorem:

1. $L^2 \xrightarrow{W_1} \mathrm{Nul}(d) \xrightarrow{W_1} 0$:
   - $\mathrm{Nul}(d) = \mathrm{Nul}\begin{bmatrix} 0 & 0 & 0 \\ \nabla & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \{f : \nabla f = 0\} \oplus L^2(\mathbb{R}^n, \mathbb{C}^n) \oplus L^2(\mathbb{R}^n)$.
   - $W_1 u = W_1(f, F, g) = (0, \beta_1 f, \gamma_1 f) \in \mathrm{Nul}(d)$.
   - $u \in \mathrm{Nul}(d) \Longrightarrow \|\beta_1 f\|_2 + \|\gamma_1 f\|_2 \lesssim \|\nabla f\|_2 = 0 \Longrightarrow W_1 u = 0$.

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2. $L^2 \xrightarrow{W_2} \overline{\mathrm{Ran}}(d^*) \xrightarrow{W_2} 0$:
   - $\overline{\mathrm{Ran}}(d^*) = \overline{\mathrm{Ran}}\begin{bmatrix} 0 & -\mathrm{div} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = L^2(\mathbb{R}^n) \oplus \{0\} \oplus \{0\}$.
   - $W_2 u = W_2(f, F, g) = (\beta_2 \cdot F + \gamma_2 g, 0, 0) \in \overline{\mathrm{Ran}}(d^*)$.
   - $u \in \overline{\mathrm{Ran}}(d^*) \implies (F, g) = 0 \implies W_2 u = 0$.

Theorem (Auscher–Stahlhut, Frey–McIntosh–Portal, et. al.)

If $q \in [2_*, 2]$, then for all $t \in \mathbb{R} \setminus \{0\}$, it holds that

$$\|(1 + itD_B)^{-1}u\|_2 \lesssim \begin{cases} \|u\|_2 & \forall u \in L^2 \\ |t|^{-n(1/q-1/2)}\|u\|_q & \forall u \in L^q \cap \overline{Ran}(d^*). \end{cases}$$

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If $q \in [2_*, 2]$, then for all $t \in [-T, T] \setminus \{0\}$, it holds that

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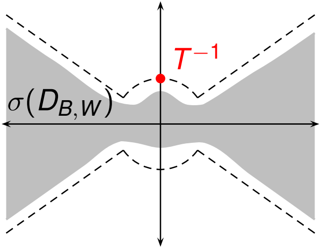
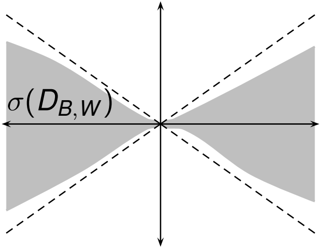


Inhomogeneous ($p > n$)



Homogeneous ($p = n$)

We need, for the Hodge projection $\mathbf{P} : L^2 \to \overline{\text{Ran}}(d^*)$:

Lemma

If $r \in [2, 2^*]$ and $q \in [2_*, 2]$, then

$$\|\mathbf{P}(1 + itD_B)^{-1}u\|_r \lesssim \begin{cases} |t|^{-n(1/2-1/r)}\|u\|_2 & \forall u \in L^2 \\ |t|^{-n(1/q-1/r)}\|u\|_q & \forall u \in L^q \cap \overline{Ran}(d^*). \end{cases}$$

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Proof.

Let $R_t^B := (1 + itD_B)^{-1}$ and $P_t^B := R_t^B R_{-t}^B = (1 + t^2 D_B^2)^{-1}$.

Since $[d^* B, P_t^B] = 0$:

$$\mathbf{P} R_t^B \mathbf{P} = \mathbf{P}(1 - itD_B) R_{-t}^B R_t^B \mathbf{P} = \mathbf{P} P_t^B \mathbf{P} - itd^* B P_t^B \mathbf{P} = \mathbf{P} P_t^B \mathbf{P}.$$

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Interpolate the end-point cases:

1. $\|\mathbf{P}R_t^B u\|_{2^*} \lesssim \|\nabla \otimes \mathbf{P}R_t^B u\|_2 \lesssim \|D_B R_t^B u\|_2 \lesssim |t|^{-1}\|u\|_2$.
2. $\|\mathbf{P}R_t^B\mathbf{P}u\|_{2^*} = \|\mathbf{P}P_t^B\mathbf{P}u\|_{2^*} \lesssim |t|^{-1}\|R_{-t}^B\mathbf{P}u\|_2 \lesssim |t|^{-2}\|u\|_{2_*}$. □

Proof of $\|(1+itD_{B,W})^{-1}u\|_2 \lesssim |t|^{-n(1/q-1/2)}\|u\|_q$.

Write $\mathrm{Nul}(d) \xrightarrow{W_1} 0$ and $L^2 \xrightarrow{W_2} \overline{\mathrm{Ran}}(d^*)$ as

$$W_1 = W_1\mathbf{P} \qquad \text{and} \qquad W_2 = \mathbf{P}W_2$$

to obtain

$$\begin{aligned}\|tW_1R_t^Bu\|_2 &= \|tW_1\mathbf{P}R_t^Bu\|_2 \\ &\leq |t|\|W_1\|_p\|\mathbf{P}R_t^Bu\|_{2p/(p-2)} \\ &\lesssim \begin{cases}|t|^{1-n/p}\|W_1\|_p\|u\|_2 & \forall u \in L^2 \\ |t|^{-n/p}\|W_1\|_p\|u\|_{2_*} & \forall u \in L^{2_*} \cap \overline{Ran}(d^*).\end{cases}\end{aligned}$$

and

$$\begin{aligned}\|tR_t^BW_2u\|_2 &= |t|\|R_t^B\mathbf{P}W_2u\|_2 \\ &\lesssim |t|^{1-n/p}\|W_2u\|_{2p/(p+2)} \\ &\leq |t|^{1-n/p}\|W_2\|_p\|u\|_2.\end{aligned}$$

Proof of $\|(1+itD_{B,W})^{-1}u\|_2 \lesssim |t|^{-n(1/q-1/2)}\|u\|_q$.

Step 1: $D_B \mapsto D_B + W_1$.

$$\|(1+it(D_B+W_1))^{-1}u\|_2 = \left\|\sum_{k=0}^{\infty}(1+itD_B)^{-1}[itW_1(1+itD_B)^{-1}]^k u\right\|$$

$$\leq (k=0) + \sum_{k=1}^{\infty}\|(tW_1R_t^B)^{k-1}(tW_1R_t^B)u\|_2$$

$$\lesssim 1 + \sum_{k=1}^{\infty}(|t|^{1-n/p}\|W_1\|_p)^k \begin{cases} \|u\|_2 & \forall u \in L^2 \\ |t|^{-1}\|u\|_{2_*} & \forall u \in L^{2_*} \cap \overline{Ran}(d^*). \end{cases}$$

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Step 2: $(D_B + W_1) \mapsto (D_B + W_1) + W_2B$.

$$(1+itD_{B,W})^{-1}u = \sum_{k=0}^{\infty}[R_t^{B,W_1} itW_2B]^k R_t^{B,W_1}u.$$

□

We now prove the local (inhomogeneous) quadratic estimate.

Proof of $\int_0^T \|tD_{B,W}(1+t^2D_{B,W}^2)^{-1}u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2$.

Set $Q^{B,W} := tD_{B,W}(1+t^2(D_{B,W}^2))^{-1}$ and write

$$\int_0^T \|Q^{B,W}u\|_2^2 \frac{dt}{t} \leq \int_0^T \|Q^{B,W}u - Q^B u\|_2^2 \frac{dt}{t} + \|u\|_2^2.$$

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It suffices, since $Q_t = \frac{1}{2i}(R_{-t} - R_t)$, to note that

$$\begin{aligned}
\|(R_t^{B,W} - R_t^B)u\|_2 &\leq \|(R_t^{B,W} - R_t^{B,W_1})u\|_2 + \|(R_t^{B,W_1} - R_t^B)u\|_2 \\
&= \left\|\sum_{k=1}^{\infty}(R_t^{B,W_1}tW_2B)^k R_t^{B,W_1}u\right\|_2 + \left\|\sum_{k=1}^{\infty} R_t^B(tW_1R_t^B)^k u\right\|_2 \\
&\lesssim |t|^{1-n/p}\|u\|_2,
\end{aligned}$$

since $1 - n/p > 0$ when $p > n$. □

To obtain the full (homogeneous) quadratic estimate we need to modify the Hodge decomposition $\overline{\mathrm{Ran}}(d) \oplus \overline{\mathrm{Ran}}(d^*B) \oplus \mathrm{Nul}(D_B) = L^2$.

Lemma

If $\max\{\|W_1\|_n, \|W_2\|_n\} < \epsilon$, *then* $\begin{cases} Nul((d^* + W_2)B) \oplus \overline{Ran}(d) = L^2 \\ Nul(d^*) \oplus \overline{Ran}(d + W_1) = L^2 \end{cases}$.

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Proof.

If $v \in \mathrm{Nul}((d^* + W_2)B)$, then

$$\begin{aligned} \|du\|_2^2 &\lesssim \langle du, B^* du\rangle = |\langle v + du, B^*(d + W_2^*)u\rangle - \langle du, B^* W_2^* u\rangle| \\ &\lesssim \|v + du\|_2(\|du\|_2 + \|W_2^* \mathbf{P} u\|_2) + \|du\|_2 \|W_2^* \mathbf{P} u\|_2 \\ &\lesssim \|v + du\|_2 \|du\|_2 + \underbrace{\epsilon \|du\|_2^2}_{\text{Hide}}, \end{aligned}$$

so $\|v\|_2 + \|du\|_2 \lesssim \|v + du\|_2$ and $\mathrm{Nul}((d^* + W_2)B) \oplus \overline{\mathrm{Ran}}(d) \subseteq L^2$. □

Set $\Theta_t^{B,W} := t(d^* + W_2)B(1 + t^2(D_{B,W}^2))^{-1}$.

Lemma

If $\displaystyle\int_0^\infty \|\Theta_t^{B,W} u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \quad \forall u \in \overline{Ran}(d),$

then $\displaystyle\int_0^\infty \|Q^{B,W} u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \quad \forall u \in Nul(d).$

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Proof.

Use $\mathrm{Nul}((d^* + W_2)B) \oplus \overline{\mathrm{Ran}}(d) = L^2$ to define

$$\mathbf{P}^W : L^2 \to \overline{\mathrm{Ran}}(d), \qquad I - \mathbf{P}^W : L^2 \to \mathrm{Nul}((d^* + W_2)B).$$

If $u \in \mathrm{Nul}(d)$, then

$$\begin{aligned} Q^{B,W} u &:= (1 + t^2(D_{B,W}^2))^{-1} t(d + W_1 + (d^* + W_2)B)u \\ &= (1 + t^2(D_{B,W}^2))^{-1} t(d^* + W_2)B(\mathbf{P}^W u + (I - \mathbf{P}^W)u) \\ &= \Theta_t^{B,W} \mathbf{P}^W u. \end{aligned}$$

Setting $P_t := (1 + t^2 D^2)^{-1}$, write (for $u \in \overline{\mathrm{Ran}}(d)$):

$$\int_0^\infty \|\Theta_t^{B,W} u\|_2^2 \frac{dt}{t} \leq \underbrace{\int_0^\infty \|\Theta_t^{B,W}(I - P_t)u\|_2^2 \frac{dt}{t}}_{\text{High frequency}} + \underbrace{\int_0^\infty \|\Theta_t^{B,W} P_t u\|_2^2 \frac{dt}{t}}_{\text{Low frequency}}.$$

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Low frequency ingredients (minor modifications):

1. Off-diagonal estimates

$$\|\mathbf{1}_E R_t^{B,W} \mathbf{1}_F\|_{2\to 2} \lesssim \left(\frac{|t|}{\text{dist}(E,F)}\right)^M.$$

   - Commutators $[D_B + W, \eta] = (D_B + W)\eta - \eta(D_B + W) = [D_B, \eta]$.

2. Carleson measure estimates

$$\sup_Q \int_0^{\ell(Q)} ⨍_Q |\Theta_t^{B,W} \mathbf{1}|^2 \frac{dxdt}{t} \lesssim 1.$$

   - Use test functions adapted D_B.
   - Need $\|R_t^B\|_{2+\delta\to 2+\delta}$ for $2 + \delta \in (2, p^H)$ to offset W.

Lemma (High frequency estimate)

$$\int_0^\infty \|\Theta_t^{B,W}(I - P_t)u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \qquad \forall u \in \overline{Ran}(d).$$

Lemma (High frequency estimate)

$$\int_0^\infty \|\Theta_t^{B,W}(I - P_t)u\|_2^2 \frac{dt}{t} \lesssim \|u\|_2^2 \qquad \forall u \in \overline{Ran}(d).$$

Proof.

Using $[P_t, d]u = [(1 + t^2 D^2)^{-1}, d]u = 0$:

$$\begin{aligned}
\Theta_t^{B,W}(I - P_t)u &= Q_t^{B,W}\mathbf{P}_{\overline{\mathrm{Ran}(d)}}(I - P_t)u \\
&= Q_t^{B,W}\mathbf{P}_{\overline{\mathrm{Ran}(d)}}t(d + d^*)Q_t u \\
&= Q_t^{B,W}\, td\, Q_t u \\
&= Q_t^{B,W} t(D_B + W_1 + W_2 B)\mathbf{P}_{\overline{\mathrm{Ran}(d^* B)}} Q_t u - Q_t^{B,W} tW_1 Q_t u \\
&= \underbrace{(I - P_t^B)\mathbf{P}_{\overline{\mathrm{Ran}(d^* B)}}}_{\text{Uniform } L^2 \text{ estimate}} \underbrace{Q_t u}_{\text{Quadratic estimate}} \underbrace{-Q_t^{B,W} tW_1 Q_t u}_{\text{Schur estimate}}
\end{aligned}$$

for all $u \in \overline{\mathrm{Ran}}(d)$. □

The Schur estimate uses $\int_0^\infty Q_s^2 u \frac{ds}{s} \approx u$ and quasi-orthogonality:

Lemma

There exists $\alpha > 0$ *such that*

$$\|Q_t^{B,W} tW_1 Q_t(Q_s v)\|_2 \lesssim \left(\min\left\{\frac{s}{t}, \frac{t}{s}\right\}\right)^\alpha \|v\|_2 \qquad \forall v \in \overline{Ran}(D).$$

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Proof.

Write

$$Q_t^{B,W} := P_t^{B,W} tD_{B,W} = P_t^{B,W}\mathbf{P} td^* B + P_t^{B,W}\mathbf{P} tW_2 B$$

and estimate (with $2_* < q^* < 2$)

$$\begin{aligned}\|P_t^{B,W}\mathbf{P} tW_2 BtW_1 Q_t Q_s v\|_2 &\lesssim t^{-n(1/q-1/2)}\|tW_2 BtW_1 Q_t Q_s v\|_q \\ &\lesssim t^{-n(1/q-1/2)}\|W_2\|_n\|W_1\|_n\|t^3 D^2 P_t(sP_s v)\|_{q^{**}} \\ &\lesssim t^{-n(1/q-1/2)+1}\|sP_s v\|_{q^{**}} \\ &\lesssim (s/t)^{n(1/q-1/2)-1}\|v\|_2.\end{aligned}$$

We need to justify using $2_* < q^* < 2$, since $q = 2_* - \delta$!

Lemma

If $\|W\|_n < \epsilon$, then there exists $\delta > 0$ such that

$$\|P_t^{B,W} u\|_2 \lesssim |t|^{-n(1/(2_*-\delta)-1/2)} \|u\|_{2_*-\delta} \qquad \forall u \in L^{2_*-\delta} \cap \overline{Ran}(d^*).$$

Proof (when $W = 0$).

Choose $q := 2_* - \delta$ so that $q^* \in (p_H, 2)$.

Then

$$\begin{aligned} \|P_t^B u\|_2 &= \|R_t^B \mathbf{P} R_{-t}^B \mathbf{P} u\|_2 \\ &\lesssim |t|^{-n(1/q^*-1/2)} \|R_{-t}^B \mathbf{P} u\|_{q^*} \\ &\lesssim |t|^{-n(1/(2_*-\delta)-1/2)} \|u\|_{2_*-\delta} \end{aligned}$$

for all $u \in \overline{\text{Ran}}(d^*)$. □

Lemma

If $\|W\|_n < \epsilon$, *then there exists* $\delta > 0$ *such that*

$$\|P_t^{B,W} u\|_2 \lesssim |t|^{-n(1/(2_* - \delta) - 1/2)} \|u\|_{2_* - \delta} \qquad \forall u \in L^{2_* - \delta} \cap \overline{Ran}(d^*).$$

Proof (when $\|W\|_n < \epsilon$).

Using $W_1 = W_1 \mathbf{P}$ and $W_2 = \mathbf{P} W_2$ we obtain

$$(D_{B,W}^2 - D_B^2) P_t^B u = \mathbf{P}(D_B W_1 + W_2 B D_B + W_2 B W_1) \mathbf{P} P_t^B u$$

for all $u \in \overline{\mathrm{Ran}}(d^*)$, so

$$P_t^{B,W} u = \sum_{k=0}^{\infty} P_t^B [t^2 (D_{B,W}^2 - D_B^2) P_t^B]^k u$$

$$= \sum_{k=0}^{\infty} \underbrace{R_t^B}_{L^{q^*} \to L^2} \underbrace{[R_{-t}^B t^2 \mathbf{P}(D_B W_1 + W_2 B D_B + W_2 B W_1) \mathbf{P} R_t^B]^k}_{L^{q^*} \to L^{q^*}} \underbrace{R_{-t}^B u}_{L^{2_* - \delta} \to L^{q^*}} .$$

□

Corollary (Conclusion)

There exists $R > 0$ *such that* $D_{B,W}$ *has a bounded* $H^\infty(S_\theta \cup D_R)$ *functional calculus,*

$$\|f(D_{B,W})u\|_2 \lesssim \|u\|_2 \qquad \forall u \in H^\infty(S_\theta \cup D_R),$$

e.g. $f(z) := \dfrac{z}{\sqrt{z^2 + R^2}}$.

If $\max\{\|W_1\|_n, \|W_2\|_n\} < \epsilon$, *then* $D_{B,W}$ *has a bounded* $H^\infty(S_\theta)$ *functional calculus, e.g.* $f(z) := \dfrac{z}{\sqrt{z^2}}$.

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Thank-you.