The duality between Poincaré type inequalities on Hamming cube and square function inequalities on dyadic lattice: How to use tree to climb on hypercube

based on works with Paata Ivanisvili

Harmonic Analysis and applications to PDE and GMT; on the occasion of 60th birthday of Steve Hofmann

May 29, 2018

Consider the Hamming cube $\{-1,1\}^n$ of an arbitrary dimension $n \geq 1$. For any $f : \{-1,1\}^n \to \mathbb{R}$ define the discrete gradient

$$|\nabla f|^2(x) = \sum_{y \sim x} \left(\frac{f(x) - f(y)}{2} \right)^2,$$

where the summation is over all neighbor vertices of x in $\{-1,1\}^n$. Set

$$\mathbb{E}f = \frac{1}{2^n} \sum_{x \in \{-1,1\}^n} f(x).$$

What follows is a joint work with Paata Ivanisvili.

Theorem

For $1 < p \le 2$, any $n \ge 1$ and any $f : \{-1, 1\}^n \to \mathbb{R}$ we obtain $s(p)^p(\mathbb{E}|f|^p - |\mathbb{E}f|^p) \le \|\nabla f\|_p^p$, where $s(p)$ is the smallest positive zero of the confluent hypergeometric function ${}_1F_1(p/2(1-p), 1/2, x^2/2)$.

Constant $s(2) = 1, s(1+) = 0$. The latter is not good.

Our approach is based on a certain duality between the classical square function estimates on Euclidean space and the gradient estimates on the Hamming cube.

As a corollary, we have the following estimate for the constant of Poincaré inequality. Let $c(p)$ be the largest constant such that any $n \geq 1$ and any $f : \{-1, 1\}^n \to \mathbb{R}$

$$c_{poincare}(p)^p \mathbb{E}|f - \mathbb{E}f|^p \leq \|\nabla f\|_p^p.$$

Let $\hat{s}(p)$ be the best (largest) constant in

$$\hat{s}(p)^p (\mathbb{E}|f|^p - |\mathbb{E}f|^p) \leq \|\nabla f\|_p^p.$$

Then we have immediately from the previous slide

$$s(p) \leq \hat{s}(p) \leq c_{poincare}(p).$$

There is some kind of converse inequality. Notice that if $1 \le p \le 2$ then

there exists $K(p) \le 2 : \; \forall x \in \mathbb{R},\; p(1-x) + |x|^p - 1 \le K(p)|1-x|^p$.

Put $x = \frac{f}{|\mathbb{E}f|}$, and apply $\mathbb{E}$. Then

$$\mathbb{E}|f|^p - |\mathbb{E}f|^p \le p\mathbb{E}(|\mathbb{E}f|^p - f|\mathbb{E}f|^{p-1}) + \mathbb{E}|f|^p - |\mathbb{E}f|^p \le K(p)\mathbb{E}|f - \mathbb{E}f|^p .$$

Then

$$s(p) \le \hat{s}(p) \le c(p) \le K(p)^{1/p}\hat{s}(p) .$$

5. Why $p(1-x)+|x|^p-1 \le C_p|1-x|^p$ implies $p \le 2$?

Consider $x = 1-\varepsilon$. Then $(1-\varepsilon)^p - 1 + p\varepsilon = a_p\varepsilon^2 + o(\varepsilon^2), a_p > 0$. This can be $\le C_p\varepsilon^p$ for $\varepsilon \to 0$ only if $p \le 2$.

The constant s_p is sharp when $p \to 2-$. On the other hand it degenerates to 0 when $p \to 1+$ which should not be the case for the best possible constant by a result of Talagrand.
It will be explained later that s_p in a "dual" sense coincides with the sharp constants found by B. Davis in L^q norm estimates between stopping times and Brownian motion

$$d_q \| T^{1/2} \|_q \leq \| B_T \|_q, \quad q \geq 2; \tag{1}$$

$$\| B_T \|_p \leq d_p \| T^{1/2} \|_p, \quad 0 < p \leq 2. \tag{2}$$

Here B_t is the Brownian motion starting at zero, and T is any stopping time. It was explained in B. Davis [4] that the same sharp estimates (3) and (4) hold with B_T replaced by an integrable function g on $[0, 1]$ with mean zero, and $T^{1/2}$ replaced by the dyadic square function of g.

Analogy with square function:

$$T^{1/2}=(|t_1-t_0|+\cdots+|t_n-t_{n-1}|)^{1/2}=(\mathbb{E}|B_{t_1}-B_{t_0}|^2+\cdots+\mathbb{E}|B_{t_n}-B_{t_{n-1}}|^2)^{1/2}$$

We notice the big difference between

$$d_q\|T^{1/2}\|_q \le \|B_T\|_q, \quad q\ge 2; \tag{3}$$

$$\|B_T\|_p \le d_p\|T^{1/2}\|_p, \quad 0<p\le 2. \tag{4}$$

and slide 2 inequality:

$$s(p)^p(\mathbb{E}|f|^p - |\mathbb{E}f|^p) \le \|\nabla f\|_p^p$$

that for the given power $p, 1<p\le 2$, we need "dual" constant $s_p = d_{\frac{p}{p-1}}$ in the theorem. Inequality of slide 2 cannot be extended to the full range of exponents p unlike (3-4).

7. Why slide 2 cannot be extended to the full range of p?

Notice that

$$(\mathbb{E}|f|^p - |\mathbb{E}f|^p) \le C(p)\|\nabla f\|_p^p$$

cannot be extended for the range of exponents $p > 2$ with some finite constant $C(p)$, $p > 2$. Indeed, assume the contrary. If this were true then Gaussian $\mathbb{E}$ would also work.
Take $f(x) = 1 + ax$. Using Jensen's inequality we obtain

$$(1+a^2)^{p/2} = \left(\int_{\mathbb{R}} |1+ax|^2 d\gamma\right)^{p/2} \le \int_{\mathbb{R}} |1+ax|^p d\gamma \le C(p)|a|^p + 1. \tag{5}$$

Therefore taking $a \to 0$ we obtain the contradiction because

$$pa^2/2 \le C(p)|a|^p$$

is false for for $p > 2$.

8. Gaussian isoperimetry of Sudakov–Tsirelson and Borell

Let $\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denote the density of $\gamma = \gamma_1$ and let Φ^{-1} denote the inverse of the standard Gaussian distribution function $\Phi(x) = \gamma_1((-\infty, x])$. The Gaussian isoperimetric inequality due to V.N. Sudakov, B.S. Tsirelson and C. Borell then asserts that, for any measurable set $A \subset \mathbb{R}^n$,

$$\gamma_n^+(A) \geq I(\gamma_n(A)),\ \ I(t) = \varphi(\Phi^{-1}(t)), t \in [0, 1]\,.$$

Here equality holds for an arbitrary halfspace A. Remarkable feature is that the function I is independent of the dimension n. $\gamma_n^+(A) = \liminf_{h\to 0} \frac{\gamma_n(A^h) - \gamma_n(A)}{h}$, where A^h is the h-neighborhood of A in Euclidean metric.

Sergei Bobkov proved the following generalization of the previous inequality.
Let $f : \to [0,1]$ then

$$I(\mathbb{E}f) \leq \mathbb{E}\sqrt{I^2(f) + |\nabla f|^2}\,,$$

where $|\nabla f|(x) := \sqrt{\sum_{i=1}^n \left(\frac{f(x)-f(s_i(x))}{2}\right)^2}$, and $s_i(x)$ are all n neighbors of x.
Obviously, for $f = \mathbf{1}_A, A \subset \{-1,1\}^n$, one gets $I(\mathbf{1}_A)(x) = 0 \forall x \in \{-1,1\}^n$, and we get

$$\varphi(\Phi^{-1}(|A|)) \leq \mathbb{E}|\nabla f| = \frac{1}{2}\mathbb{E}\sqrt{w_A(x)} =: \frac{1}{2}\,\text{surface measure of A}\,,$$

where $w_A(x) = |\text{neighbors of x from outside}|$.

In particular, two things can be derived:
1) If $|A| = 2^{n-1}$ then

$$\mathbb{E}(w_A(x))^{1/2} \geq \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}\,.$$

2) For any function $f : \{-1,1\}^n \to \mathbb{R}$ one has Talagrand-Poincaré inequality

$$\mathbb{E}|f - \mathbb{E}f| \leq C_1 \mathbb{E}|\nabla f|\,.$$

Also $1 \leq q < \infty$ Talagrand-Poincaré inequalities hold:

$$\mathbb{E}|f - \mathbb{E}f|^q \leq C_q^q \mathbb{E}|\nabla f|^q\,.$$

Seems like nobody knows sharp C_q on Hamming cube.

11. Combinatorics. What we know about $g(p) := \inf_n \inf_{A:\,|A|=2^{n-1}} \mathbb{E}(w_A)^{p/2}$?

1) $g(p) = \inf_n \inf_{A:\,|A|=2^{n-1}} \mathbb{E}(w_A)^{p/2} = 0$ if $p \in [0,1)$. Hamming balls are extremizers.
2) $g(1) \geq \sqrt{\frac{2}{\pi}}$, from Bobkov. Sharp?
3) $g(2) = 1$, discrete Poincaré. Sharp. Half-cube is extremal.
4) $g(p)$ is monotonically increasing.
5) $g(p) \geq s(p)^p$, from our theorem on slide 2. In fact apply theorem to $f(x) = 1, x \in A; f(x) = -1, x \in \{-1,1\}^n \setminus A$. Can be sharp only near $p = 2$.
6) $g(p) \geq \max\{s(p)^p, \left(\frac{2}{\pi}\right)^{\frac{p}{2}}\}$.

It is also interesting to remark that if one considers only nonnegative functions in Theorem 1 then one obtains **probably** better constant than s_p^p. For example, it was obtained in [5] that for any smooth $f \geq 0$ we have

$$(H'_{1/(p-1)}(R_{1/(p-1)}))^{p-1} \left(\int_{\mathbb{R}^n} f^p d\gamma - \left(\int_{\mathbb{R}^n} f \, d\gamma \right)^p \right) \leq \int_{\mathbb{R}^n} |\nabla f|^p d\gamma,$$

where H_q is the Hermite function, and R_q is the largest zero of H_q. Numerical computations show that the constant $(H'_{1/(p-1)}(R_{1/(p-1)}))^{p-1}$ is larger than s_p^p.
If $p = 3/2$, $(2x|x = 1)^{1/2} = \sqrt{2}$, and, therefore, $\left(\int_{\mathbb{R}^n} f^{3/2} d\gamma - \left(\int_{\mathbb{R}^n} f \, d\gamma \right)^{3/2} \right) \leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}^n} |\nabla f|^{3/2} d\gamma$ if $f \geq 0$.
We do not know whether the same constants work for positive functions on Hamming cube and $\mathbb{E}$ replacing Gaussian measure $\int \cdot d\gamma$. But for $p = 3/2$ we do know that.

13. An anonymous Bellman function

In this section we want to define a function $U : \mathbb{R}^2 \to \mathbb{R}$ that satisfies some special properties. Let $\alpha \geq 2$ and let $\beta = \frac{\alpha}{\alpha-1} \leq 2$ be the conjugate exponent of α. Let

$$N_\alpha(x) := {}_1F_1\left(-\frac{\alpha}{2}, \frac{1}{2}, \frac{x^2}{2}\right) = \sum_{m=0}^{\infty} \frac{(-2x^2)^m}{(2m)!} \frac{\alpha}{2}\left(\frac{\alpha}{2} - 1\right) \cdots \left(\frac{\alpha}{2} - m + 1\right)$$
$$= 1 - x^2 \frac{\alpha}{2} + \ldots$$

be the confluent hypergeometric function. $N_\alpha(x)$ satisfies the Hermite differential equation

$$N_\alpha''(x) - xN_\alpha'(x) + \alpha N_\alpha(x) = 0 \quad \text{for} \quad x \in \mathbb{R} \tag{6}$$

with initial conditions $N_\alpha(0) = 1$ and $N_\alpha'(0) = 0$. Let s_α be the smallest positive zero of $N_\alpha(z)$.

For $\alpha \geq 2$ set

$$u_\alpha(x) := \begin{cases} -\frac{\alpha s_\alpha^{\alpha-1}}{N'_\alpha(s_\alpha)} N_\alpha(x), & 0 \leq |x| \leq s_\alpha; \\ s_\alpha^\alpha - |x|^\alpha, & s_\alpha \leq |x|. \end{cases}$$

Clearly $u_\alpha(x)$ is $C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{s_\alpha\})$ smooth, even, concave function. Concavity follows from matching derivatives and

Lemma

For any $\alpha \geq 2$ we have $0 < s_\alpha \leq 1$. In addition s_α is decreasing in $\alpha > 0$, and $N'_\alpha(t), N''_\alpha(t) \leq 0$ on $[0, s_\alpha]$ for $\alpha > 0$.

Finally we define

$$U(p,q) := |q|^\alpha u_\alpha\left(\frac{p}{|q|}\right) \quad \text{with} \quad U(p,0) = -|p|^\alpha. \tag{7}$$

For the first time the function $U(p,q)$ appeared in Davis [4]. Later it was also used by Wang [4, 5] in the form $\widetilde{u}(p,t) = U(p,\sqrt{t})$, $t \geq 0$. It was explained in Davis [4] that $U(p,q)$ satisfies the following properties:

$$U(p,q) \geq |q|^\alpha s_\alpha^\alpha - |p|^\alpha \quad \text{for all} \qquad (p,q) \in \mathbb{R}^2, \tag{8}$$

$$\text{and when } q = 0 \text{ the equality holds;} \tag{9}$$

$$2U(p,q) \geq U(p+a, \sqrt{a^2+q^2}) + U(p-a, \sqrt{a^2+q^2}) \tag{10}$$

$$\text{for all} \quad (p,q,a) \in \mathbb{R}^3. \tag{11}$$

We should refer to (6-7) as the *obstacle condition*, and to (8-9) as the *main inequality*.

In Davis main inequality is not written explicitly but one will find its infinitesimal form

$$\frac{U_q}{q} + U_{pp} \leq 0 \text{ or } \widetilde{u}_t + \frac{\widetilde{u}_{pp}}{2} \leq 0 \quad \text{for} \quad \widetilde{u}(p,t) = U(p, \sqrt{t}), \quad (12)$$

which follows from the main inequality by expanding it into Taylor's series with respect to a near $a = 0$ and comparing the second order terms. Here $\widetilde{u}_{pp}$ is defined everywhere except the curve $|p/\sqrt{t}| = s_\alpha$, where $\widetilde{u}$ is only differentiable once.
In fact, the reverse implication also holds, i.e., one can derive

$$2U(p,q) \geq U(p+a, \sqrt{a^2+q^2}) + U(p-a, \sqrt{a^2+q^2})$$
$$\text{for all} \quad (p,q,a) \in \mathbb{R}^3.$$

from (12) for this special U.
This was done in the PhD thesis of Wang [5] but we will present a short proof of this.

The function $U(p,q)$ is essential in obtaining the result in the Davis paper, namely it is used in the proof of (3), slide 6, and the argument goes as follows. First one shows that

$$X_t = U(B_t, \sqrt{t}) \quad \text{for} \quad t \geq 0$$

is a **supermartingale** which is guaranteed by (12). Finally, by this property of being a supermartingale and by obstacle condition,

$$\mathbb{E}(T^{\frac{\alpha}{2}} s_\alpha^\alpha - |B_T|^\alpha) \overset{(9)}{\leq} \mathbb{E} U(B_T, \sqrt{T}) \leq U(0,0) = 0,$$

which yields (3) of slide 6. One may notice that $U(p,q)$ is the minimal function with properties obstacle (9) and main inequality (11) of slide 15.
Davis mentions that the proof presented in his paper was suggested by an anonymous referee, and this explains the title of slide 13.

18. Legendre transform of Bellman function $U(p, q)$

Set $\Psi(p, q, x, y) := px + qy + U(p, q)$ for $x \in \mathbb{R}$ and $y \geq 0$. We define

$$M(x, y) = \inf_{q \leq 0} \sup_{p \in \mathbb{R}} \Psi(p, q, x, y) \quad \text{for} \quad x \in \mathbb{R}, \ y \geq 0. \tag{13}$$

Lemma

For each $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, *we have*

$$\inf_{q \leq 0} \sup_{p \in \mathbb{R}} \Psi(p, q, x, y) = \min_{q \leq 0} \max_{p \in \mathbb{R}} \Psi(p, q, x, y) = \tag{14}$$

$$\max_{p \in \mathbb{R}} \min_{q \leq 0} \Psi(p, q, x, y) = \sup_{p \in \mathbb{R}} \inf_{q \leq 0} \Psi(p, q, x, y), \tag{15}$$

and the value is attained at a saddle point $(p^*, q^*) = (p^*(x, y), q^*(x, y))$ *such that*

$$\Psi(p, q^*, x, y) \leq \Psi(p^*, q^*, x, y) \leq \Psi(p^*, q, x, y) \textit{for all} (p, q) \in \mathbb{R} \times \mathbb{R}_-. \tag{16}$$

First let us show that for each fixed (x, y) the function $\Psi(p, q, x, y)$ is convex in q and concave in p. Concavity in p follows from Lemma on slide 14, and the fact that U is even and C^1 smooth in p.

To verify convexity in q it is enough to show that the map $q \to U(p, q)$ is convex for $|p| \le q s_\alpha$. Set $z = \frac{|p|}{q} \in [0, s_\alpha]$. Then we have

$$U_{qq} = q^{\alpha-2}\left[\alpha(\alpha-1)u_\alpha(z) - 2(\alpha-1)zu'_\alpha(z) + z^2 u''_\alpha(z)\right] \overset{(6)}{=}$$
$$q^{\alpha-2}\left[-(\alpha-1)zu'_\alpha(z) + (z^2 - \alpha + 1)u''_\alpha(z)\right].$$

Since $u_\alpha(z)$ coincides with $N_\alpha(z)$ up to a positive constant, the convexity follows from Lemma 2 and the fact that $\alpha \ge 2$. Notice that for each $(x, y) \in \mathbb{R} \times \mathbb{R}_+$ the map

$$(p, q) \to px + qy + |q|^\alpha u_\alpha\left(\frac{p}{|q|}\right) \tag{17}$$

is continuous and $(0, 0)$ *sup inf -compact*.

Let P, Q be not empty convex sets in $\mathbb{R}^d$.

Definition

A function $f : P \times Q \to \mathbb{R}$ is called (p_0, q_0)-*sup inf-compact* for a fixed $(p_0, q_0) \in X \times Y$ if the level sets $\{q \in Q : f(p_0, q) \le a\}$ and $\{p \in P : f(p, q_0) \ge a\}$ are compact for any $a \in \mathbb{R}$.

Theorem

If $f : P \times Q \to \mathbb{R}$ is upper semi-continuous and concave in p, lower semi-continuous and convex in q, and (p_0, q_0)-sup inf-compact *for a fixed $(p_0, q_0) \in P \times Q$ then we have*

$$\max_{p\in P}\min_{q\in Q} f(p,q) = \sup_{p\in P}\inf_{q\in Q} f(p,q) = \inf_{q\in Q}\sup_{p\in P} f(p,q) = \min_{q\in Q}\max_{p\in P} f(p,q). \tag{18}$$

21. From U to M

Lemma

For $\beta = \frac{\alpha}{\alpha-1}$, any $x, a, b \in \mathbb{R}$, and any $y \geq 0$ we have

$$M(x,y) \geq \left(\frac{\alpha-1}{\alpha^\beta}\right)\left(|x|^\beta - \frac{y^\beta}{s_\alpha^\beta}\right) \quad \text{and when } y = 0 \text{ the equality holds;} \tag{19}$$

$$2M(x,y) \geq M(x+a, \sqrt{a^2+(y+b)^2}) + M(x-a, \sqrt{a^2+(y-b)^2}) . \tag{20}$$

Notices that the Legendre transform (13) produces from $U(p,q)$ function $M(x,y)$ with inequality (20) that **seems to be very close to** but which is **diifferent from**

$$2U(p,q) \geq U(p+a, \sqrt{a^2+q^2}) + U(p-a, \sqrt{a^2+q^2})$$
$$\text{for all} \quad (p,q,a) \in \mathbb{R}^3.$$

Set

$$(x_\pm, y_\pm) := (x \pm a, \sqrt{a^2 + (y \pm b)^2}).$$

Lemma of slide 18 gives saddle points (p^*, q^*) and $(p^\pm, q^\pm)$ corresponding to (x, y) and $(x_\pm, y_\pm)$. It follows from (16) that to prove (20) it would be enough to find numbers $p \in \mathbb{R}$ and $q_1, q_2 \leq 0$ such that

$$2\Psi(p, q^*, x, y) \geq \Psi(p^+, q_1, x_+, y_+) + \Psi(p^-, q_2, x_-, y_-).$$

The right choice will be

$$p = \frac{p^+ + p^-}{2} \quad \text{and} \quad q_1 = q_2 = -\sqrt{\left(\frac{p^+ - p^-}{2}\right)^2 + (q^*)^2}, \quad (21)$$

but let us explain it in details.

Indeed, we have

$$q_1\sqrt{a^2+(y+b)^2}+q_2\sqrt{a^2+(y-b)^2}-2q^*y=$$

$$-\sqrt{(q_1^2-(q^*)^2)+(q^*)^2}\sqrt{a^2+(y+b)^2}-$$

$$\sqrt{(q_2^2-(q^*)^2)+(q^*)^2}\sqrt{a^2+(y-b)^2}-2q^*y\le$$

$$-|a|\sqrt{q_1^2-(q^*)^2}-|q^*(y+b)|-|a|\sqrt{q_2^2-(q^*)^2}-|q^*(y-b)|-2q^*y$$

$$-|a|\left(\sqrt{q_1^2-(q^*)^2}+\sqrt{q_2^2-(q^*)^2}\right).$$

Denote $r_j^2=q_j^2-(q^*)^2$ for $j=1,2$. From above it is enough to find $p\in\mathbb{R}$ and $r_1,r_2\ge 0$ such that

$$2(px+U(p,q^*))\ge -|a|(r_1+r_2)+p^+x_++U\left(p^+,\sqrt{r_1^2+(q^*)^2}\right)+$$

$$p^-x_-+U\left(p^-,\sqrt{r_2^2+(q^*)^2}\right).$$

24. Verifying main inequality for $M(x, y)$

Choose $p = \frac{p^+ + p^-}{2}$, and substituting the values for $x_\pm = x \pm a$ we see that it would suffice to find $r_1, r_2 \geq 0$ such that

$$2U\left(\frac{p^+ + p^-}{2}, q^*\right) \geq -|a|(r_1 + r_2) + a(p^+ - p^-) + U\left(p^+, \sqrt{r_1^2 + (q^*)^2}\right)$$
$$+ U\left(p^-, \sqrt{r_2^2 + (q^*)^2}\right).$$

We choose $r_1 = r_2 = \frac{|p^+ - p^-|}{2}$. It follows from $-|a|(r_1 + r_2)| + a(p^+ - p^-) \leq 0$ that we only need to have the inequality–which is (8-9) on slide 15 (main inequality for U):

$$2U\left(\frac{p^+ + p^-}{2}, q^*\right) \geq U\left(p^+, \sqrt{\left(\frac{p^+ - p^-}{2}\right)^2 + (q^*)^2}\right)$$
$$+ U\left(p^-, \sqrt{\left(\frac{p^+ - p^-}{2}\right)^2 + (q^*)^2}\right).$$

25. Verifying obstacle condition for $M(x, y)$

To verify the obstacle condition (19) of slide 21 notice that (9) for $U(p, q)$ gives

$$M(x, y) \geq \inf_{q \leq 0} \sup_{p} \left(px + qy + |q|^{\alpha} s_{\alpha}^{\alpha} - |p|^{\alpha}\right) = \left(\frac{\alpha - 1}{\alpha^{\beta}}\right) \left(|x|^{\beta} - \frac{y^{\beta}}{s_{\alpha}^{\beta}}\right). \tag{22}$$

Finally if $y = 0$, then we obtain

$$M(x, 0) = \sup_{p} \inf_{q \leq 0} (px + U(p, q)) \stackrel{(*)}{=} \sup_{p} (px + U(p, 0)) = \sup_{p} (px - |p|^{\alpha})$$
$$= \left(\frac{\alpha - 1}{\alpha^{\beta}}\right) |x|^{\beta}.$$

Equality (*) follows from the fact that

$$q \to px + U(p, q)$$

is an even convex map.

Corollary

For any $a, x \in \mathbb{R}$, *all* $y, b \in \mathbb{R}^N$, *and any* $N \geq 1$, *we have*

$$M(x+a, \sqrt{a^2 + \|y+b\|^2}) + M(x-a, \sqrt{a^2 + \|y-b\|^2}) \leq 2M(x, \|y\|). \tag{23}$$

Proof.

It follows from the definition of M that the map $y \to M(x,y)$ is decreasing in y for $y \geq 0$. Therefore by the lemma and the triangle inequality we obtain

$$\frac{1}{2}\left(M(x+a, \sqrt{a^2 + \|y+b\|^2}) + M(x-a, \sqrt{a^2 + \|y-b\|^2})\right) \leq$$
$$M\left(x, \frac{\|y+b\| + \|y-b\|}{2}\right) \leq M(x, \|y\|).$$

$\square$

follows from the previous corollary.
The inequality (23) of the previous slide gives rise to the estimate

$$\mathbb{E}M(f,|\nabla f|) \le M(\mathbb{E}f,0) \quad \text{for all} \quad f:\{-1,1\}^n \to \mathbb{R}. \tag{24}$$

In fact, inequality (23) is the same as the following pointwise inequality on $\{-1,1\}^{n-1}$:

$$\mathbb{E}_{x_j}M(f,|\nabla f|) \le M(\mathbb{E}_{x_j}f,|\nabla \mathbb{E}_{x_j}f|) \text{ for any } f:\{-1,1\}^n \to \mathbb{R}, \tag{25}$$

where $\mathbb{E}_{x_j}$ takes the average in the coordinate x_j, i.e.,

$$\mathbb{E}_{x_j}f = \frac{1}{2}\left(f\underbrace{(x_1,\dots,1,\dots,x_n)}_{\text{set 1 on the } j\text{-th place}} + f\underbrace{(x_1,\dots,-1,\dots,x_n)}_{\text{set } -1 \text{ on the } j\text{-th place}} \right).$$

The rest follows by iterating (25), the fact that $\mathbb{E} = \mathbb{E}_{x_1}\dots\mathbb{E}_{x_n}$ and $|\nabla \mathbb{E}f| = 0$.

We have

$$\left(\frac{\alpha-1}{\alpha^\beta}\right)\mathbb{E}\left(|f|^\beta-\frac{|\nabla f|^\beta}{s_\alpha^\beta}\right)\overset{(19)}{\leq}\mathbb{E}M(f,|\nabla f|)\overset{(24)}{\leq}$$
$$M(\mathbb{E}f,0)\overset{(19)}{=}\left(\frac{\alpha-1}{\alpha^\beta}\right)|\mathbb{E}f|^\beta,$$

and this gives inequality of slide 2:

$$s(p)^p(\mathbb{E}|f|^p-|\mathbb{E}f|^p)\leq\|\nabla f\|_p^p.$$

29. Going from U to M: from Square function to the Hamming cube

Let g be an integrable function on $[0,1]$. Let $D([0,1])$ denote all dyadic intervals in $[0,1]$. Consider the dyadic martingale g_n defined as follows

$$g_n(x) = \sum_{|I|=2^{-n},\, I\in D([0,1])} \langle g\rangle_I \mathbf{1}_I(x), \tag{26}$$

where $\langle g\rangle_I = \frac{1}{|I|}\int_I g$. The square function $S(g)$ is defined as follows

$$S(g)(x) = \left(\sum_{n=0}^{\infty}(g_{n+1}(x)-g_n(x))^2\right)^{1/2}.$$

For convenience we always assume that the number of nonzero terms in (26) is finite so that $S(g)(x)$ makes sense. Let $O(p,q)$ be a continuous real valued function, and suppose one wants to estimate the quantity from above $\int_0^1 O(g, S(g))$ in terms of $\int_0^1 g$.

30. Correct Bellman function is equivalent to correct square function estimate

Clearly, if one finds a function

$$U(p,q) \geq O(p,q), U(p,0) \leq 0 \tag{27}$$

$$2U(p,q) \geq U(p+a, \sqrt{a^2+q^2}) + U(p-a, \sqrt{a^2+q^2}), \tag{28}$$

then one obtains the bound (for g, $\int_0^1 g = 0$)
$\int_0^1 O(g, S(g)) \leq \int_0^1 U(g, S(g)) \leq U\left(\int_0^1 g, 0\right) \leq 0$.
Typically, $O(x,y) = c|y|^p - |x|^p$, so we get
$\int_0^1 O(g, S(g)) = c\int S(g)^p - \int |g|^p \leq 0$. Conversely, suppose that the inequality $\int_0^1 O(g, S(g)) \leq F\left(\int_0^1 g\right)$ holds for all integrable functions g on $[0,1]$, and some F $(F(0) \leq 0)$. Then there exists $U(p,q)$ such that the conditions (27), (28) are satisfied and $U(p,0) \leq F(p)$, so $U(0,0) \leq 0$.

31. Correct Bellman function is equivalent to correct square function estimate

Indeed, consider the extremal problem

$$U(p,q) = \sup_g \left\{ \int_0^1 O(g, \sqrt{S(g)^2 + q^2}), \quad \int_0^1 g = p \right\}. \qquad (29)$$

This U satisfies (27) (take $g = p$ constant), and, in fact, it satisfies (28). The latter fact can be proved by using the standard Bellman principle (see Chapter 8, [5], and survey [4]). Besides

$$U(p,0) = \sup_g \left\{ \int_0^1 O(g, S(g)), \quad \int_0^1 g = p \right\} \le F(p)$$

follows from (29). Therefore there is one-to-one correspondence between the extremal problems for the square function estimates of the form (29) and the functions $U(p,q)$ with the properties (27) and (28).

32. Bellman function for problems on Hamming cube?

The extremal problems for the gradient estimates on the Hamming cube are more subtle. Take any real valued $\widetilde{O}(x, y)$ and suppose we want to estimate from above $\mathbb{E}\widetilde{O}(f, |\nabla f|)$ in terms of $\mathbb{E}f$ for any $f : \{-1, 1\}^n \to \mathbb{R}$ and for all $n \geq 1$. Clearly, if one finds $M(x, y)$ such that

$$2M(x,y)\geq M(x+a,\sqrt{a^2+(y+b)^2})+M(x-a,\sqrt{a^2+(y-b)^2}), \quad (30)$$

then one gets (31) by induction on cube's dimension (slide 27)

$$\mathbb{E}M(f, |\nabla f|) \leq M(\mathbb{E}f, 0)\,. \qquad (31)$$

if in addition $M(x, y) \geq \widetilde{O}(x, y)$ then $\mathbb{E}\widetilde{O}(f, |\nabla f|) \leq M(\mathbb{E}f, 0)$

32a. Bellman function for problems on Hamming cube. Example.

Beckner's inequality in L^p on Hamming cube with $p = 3/2$ is that for any positive f on $\{-1,1\}^n$ $\mathbb{E}f^{3/2} - \frac{3}{8}\mathbb{E}\frac{|\nabla f|^2}{f^{1/2}} \le (\mathbb{E}f)^{3/2}$.
Consider

$$M(x,y) = \frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{x+\sqrt{x^2+y^2}},\ x \ge 0,$$

satisfies pointwise inequality

$$x^{3/2} - \frac{3}{8}\frac{y^2}{x^{1/2}} \le \frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{x+\sqrt{x^2+y^2}},\ x \ge 0\,. \tag{32}$$

The following improves Beckner's inequality because of pointwise estimate (32).

$$\mathbb{E}\frac{1}{\sqrt{2}}\left((2f - \sqrt{f^2+|\nabla f|^2})\sqrt{f+\sqrt{f^2+|\nabla f|^2}}\right) \le (\mathbb{E}f)^{3/2},$$

32b. Bellman function for problems on Hamming cube. Example.

$$M(x,y)=\frac{1}{\sqrt{2}}(2x-\sqrt{x^2+y^2})\sqrt{x+\sqrt{x^2+y^2}},\ M(x,0)=x^{3/2}, x\geq 0,$$

$$M(x,y)\geq x^{3/2}-\frac{1}{\sqrt{2}}y^{3/2}=:\tilde{O}(x,y)\,.$$

And looking at the previous slide we get two inequalities: improved Beckner inequality:

$$1)\ \mathbb{E}\left((2f-\sqrt{f^2+|\nabla f|^2})\sqrt{f+\sqrt{f^2+|\nabla f|^2}}\right)\leq\sqrt{2}(\mathbb{E}f)^{3/2},$$

and new sharp Poincaré inequality for functions $f:\{-1,1\}^n\to\mathbb{R}_+$:

$$2)\ \mathbb{E}f^{3/2}-(\mathbb{E}f)^{3/2}\leq\frac{1}{\sqrt{2}}\mathbb{E}|\nabla f|^{3/2}\Rightarrow\mathbb{E}(w_A(x))^{3/4}\geq(2-\sqrt{2})$$

33. Bellman function for problems on Hamming cube?

Thus finding such M is sufficient to obtain the estimate but it is unclear whether condition (30) is **necessary** to obtain the bound $\mathbb{E}\widetilde{O}(f, |\nabla f|) \le M(\mathbb{E}f, 0)$.
In other words we do not know what is the corresponding extremal problem for M, i.e., what is the right Bellman function M. The reason lies in the fact that there is an essential difference between the Hamming cube as a graph and the dyadic tree, i.e., test functions do not concatenate in a good way on $\{-1,1\}^n$ as it happens for dyadic martingales.

34. Abstract way to pass from dyadic tree to Hamming cube

Theorem

Let $I, J \subseteq \mathbb{R}$ be convex sets. Take an arbitrary $O(p,q) \in C(I \times \mathbb{R}_+)$, and let $U(p,q) : I \times \mathbb{R}_+ \to \mathbb{R}$ satisfy properties (27) and (28). Assume for each $(x,y) \in J \times \mathbb{R}_+$ the map $(p,q) \to px + qy + U(p,|q|)$ has a saddle point $(p^*(x,y), q^*(x,y))$ such that

$$\inf_{q\leq 0}\sup_{p\in I}(px+qy+U(p,|q|)) = \sup_{p\in I}\inf_{q\leq 0}(px+qy+U(p,|q|))$$

$$= p^*x + q^*y + U(p^*,|q^*|).$$

$M(x,y) = \inf_{q\leq 0}\sup_{p\in I}\ (px+qy+U(p,|q|))\,;\,,\ \widetilde{O}(x,y) = \inf_{q\leq 0}\sup_{p\in I}\ (px+qy+O(p,|q|))\,;$ *satisfy (30), and thereby (31) for any $f : \{-1,1\}^n \to J$ and any $n \geq 1$.*

35. The main inequality for U and inverse heat

One may think that finding $U(p,q)$ with the property (28) is a difficult problem. Let us make a quick remark here that if it happens that $t \to U(p, \sqrt{t})$ is convex for each fixed $p \in I$ then (28) is automatically implied by its infinitesimal form, i.e., $U_{pp} + U_q/q \le 0$, or $\frac{1}{2}u_{pp} + u_t \le 0, u(p,t) = U(p, \sqrt{t})$, which is the inverse heat equation.

Another interesting observation is that the equality

$$M(x,y) = \inf_{q\le 0} \sup_{p\in I} \left(px + qy + U(p,|q|)\right)$$

was lurking in a solution of a certain Monge–Ampère equation. For example, taking $a, b \to 0$ in (30) of slide 32, and using the Taylor's series expansion (assuming that M is smooth enough) one obtains

$$\begin{pmatrix} M_{xx} + \frac{M_y}{y} & M_{xy} \\ M_{xy} & M_{yy} \end{pmatrix} \le 0\,. \tag{33}$$

37. Suppose we guessed M but cannot prove the main inequality for it?

When looking for the least function M with $M \geq \widetilde{O}$ and (33), it is reasonable to assume that condition (33) should degenerate except, possibly, on the set where M coincides with its obstacle $\widetilde{O}$. The degeneracy of (33) means that the determinant of the matrix in (33) is zero. This is a general Monge–Ampère type equation and, after a successful application of the exterior differential systems of Bryant–Griffiths (see [4]), we obtain that the solutions can be locally characterized as follows

$$\begin{aligned} x &= -U_p(p,q), \\ y &= -U_q(p,q), \\ M(x,y) &= px + qy + U(p,q), \end{aligned} \tag{34}$$

where U satisfies the equation $U_{pp} + \frac{U_q}{q} = 0$. We will not formulate a formal statement but we do make a remark that such a reasoning allows us to guess the *dual* of M, i.e., find U given M, and how this guess works will be illustrated on next slide.

38. Here is M for which it was difficult to prove the main inequality. Improved Beckner for $p = 3/2$

Beckner–Poincaré inequality $3/2$: a simple proof via duality

It was proved in [3] that for any $f : \{-1, 1\}^n \to \mathbb{R}_+$ we have

$$\mathbb{E}\,\Re\,(f + i|\nabla f|)^{3/2} \le (\mathbb{E}f)^{3/2}, \tag{35}$$

where $z^{3/2}$ is taken in the sense of principal brunch. Inequality (35) improves Beckner's bound [3]. Consider

$$M(x, y) = \Re(x + iy)^{3/2} = \frac{1}{\sqrt{2}}(2x - \sqrt{x^2 + y^2})\sqrt{\sqrt{x^2 + y^2} + x}.$$

It was explained in [3] that to prove (35) it is enough to check that $M(x, y)$ satisfies (30), and the latter fact involved careful investigation of the roots of several very high degree polynomials with integer coefficients. Let us give a simple proof of (30) using our duality technique.

39 Improved Beckner for $p = 3/2$

Proposition

Function $M(x,y) = \Re(x+iy)^{3/2}$ satisfies (30) for all $x, y, a, b \in \mathbb{R}$.

Proof.

Function $M(x,y)$ is a solution of the homogeneous Monge–Ampère equation (33), and therefore it has a representation of the form (34) (see Section 3.1.4 in [4]). This leads us to the following guess

$$\frac{1}{\sqrt{2}}(2x - \sqrt{x^2+y^2})\sqrt{\sqrt{x^2+y^2}+x}$$
$$= \inf_{q\le 0}\sup_{p\ge 0}\left(xp + qy - \frac{4}{27}(p^3 - 3pq^2)\right),$$

which can be directly checked. Notice that in this case $U(p,q) = -\frac{4}{27}(p^3 - 3pq^2)$. Following Theorem of slide 34 it is enough to check that $U(p,q)$ satisfies (28). Notice that (28) is identity for $U(p,q) = -\frac{4}{27}(p^3 - 3pq^2)$. We are done. □

40. Dual to Log-Sobolev is Chang–Wilson–Wolff: superexponential bounds

The function $M(x, y) = x \ln x - \frac{y^2}{2x}$ satisfies (33) and therefore it gives the log-Sobolev inequality [4]. Its dual in the sense of (34) is $U(p, q) = e^{p-q^2/2}$ (see Section 3.1.1 in [4] where $t = q^2/2$). Notice that for this U inequality (28) simplifies to

$$2e^{a^2/2} \geq e^a + e^{-a}$$

which is true since $(2k)! \geq 2^k k!$ for $k \geq 0$. Therefore we obtain

Corollary

For any integrable g on $[0, 1]$ we have

$$\int_0^1 \exp\left(g - \frac{S^2(g)}{2}\right) \leq \exp\left(\int_0^1 g\right).$$

The corollary immediately recovers the result of Chang-Wilson-Wolf [3] well-known for probabilists, namely for any g with $\int_0^1 g = 0$ and $\|S(g)\|_\infty < \infty$ we have

$$\int_0^1 e^g \leq e^{\|S(g)\|_\infty^2/2} \tag{36}$$

Next, repeating a standard argument, namely, considering tg, applying Chebyshev inequality (see Theorem 3.1 in [3]) one obtains the superexponential bound

Corollary

Suppose $\|Sg\|_\infty < \infty$. Then for any $\lambda \geq 0$ we have

$$|\{x \in [0,1]] \,:\, g(x) - \int_0^1 g \geq \lambda\}| \leq e^{-\frac{1}{2}\lambda^2/\|Sg\|_\infty^2}.$$

We should remind that log-Sobolev inequality via the Herbst argument [1] gives Gaussian concentration inequalities, namely

$$\gamma\left(x \in \mathbb{R}^n \; : f(x) - \int_{\mathbb{R}^n} fd\gamma \geq \lambda\right) \leq e^{-\frac{1}{2}\lambda^2/\|\nabla f\|_\infty^2} \qquad (37)$$

for any $\lambda \geq 0$, and any smooth $f : \mathbb{R}^n \to \mathbb{R}$ with $\|\nabla f\|_\infty < \infty$.
Here $d\gamma$ is the standard Gaussian measure on $\mathbb{R}^n$.
In other words we just illustrated that estimates (37) and (36) are dual to each other in the sense of duality between functions $M = x \ln x - \frac{y^2}{2x}$ and $U = e^{p-q^2/2}$.

43. Gaussian measure on $\mathbb{R}^d$

Application of the Central Limit Theorem to our main inequality gives dimension independent Sobolev inequality

Corollary

For any smooth bounded $f : \mathbb{R}^n \to \mathbb{R}$ and any $n \geq 1$ we have

$$s_{p'}^p \left(\int_{\mathbb{R}^n} |f|^p d\gamma - \left| \int_{\mathbb{R}^n} f d\gamma \right|^p \right) \leq \int_{\mathbb{R}^n} |\nabla f|^p d\gamma. \qquad (38)$$

Behavior of $s_{p'}^p$ is sharp when $p \to 2-$. However, the best possible constant in (38) unlike $s_{p'}^p$ should not degenerate when $p \to 1+$. Indeed, Cheeger's isoperimetric inequality (see [2], pp. 115) claims

$$\sqrt{\frac{2}{\pi}} \int_{\mathbb{R}^n} \left| f - \int_{\mathbb{R}^n} f d\gamma \right| d\gamma \leq \int_{\mathbb{R}^n} |\nabla f| d\gamma, \qquad (39)$$

where the constant $\sqrt{\frac{2}{\pi}}$ is the best possible in the left hand side of (39). We should also mention that estimate (39) can be also easily obtained by co-area formula and Bobkov's estimate (gaussian

44. From Brownian motion stopping times to Square function estimates

Lemma (Barthe–Mauery [1])

Let J be a convex subset of $\mathbb{R}$, and let $V(p,q) : J \times \mathbb{R}_+ \to \mathbb{R}$ be such that

$$V_{pp} + \frac{V_q}{q} \le 0 \quad \text{for all} \quad (p,q) \in J \times \mathbb{R}_+; \tag{40}$$

$$t \mapsto V(p, \sqrt{t}) \quad \text{is convex for each fixed} \quad p \in J. \tag{41}$$

Then for all (p,q,a) with $p \pm a \in J$ and $q \ge 0$, we have

$$2V(p,q) \ge V(p+a, \sqrt{a^2+q^2}) + V(p-a, \sqrt{a^2+q^2}). \tag{42}$$

The lemma says that the global discrete inequality (42) is in fact implied by its infinitesimal form (40) under the extra condition (41).

45. From Brownian motion stopping times to Square function estimates

The argument is borrowed from [1]. The similar argument was used by Davis [4] in obtaining sharp square function estimates from the ones for the Brownian motion.
Without loss of generality assume $a \geq 0$. Consider the process

$$X_t = V(p + B_t, \sqrt{q^2 + t}), \quad t \geq 0.$$

Here B_t is the standard Brownian motion starting at zero. It follows from Ito's formula together with (40) that X_t is a supermartingale. Let τ be the stopping time

$$\tau = \inf\{t \geq 0 : B_t \notin (-a, a)\}.$$

$$V(p,q) = X_0 \geq \mathbb{E}X_\tau = \mathbb{E}V(p+B_\tau, \sqrt{q^2+\tau}) =$$
$$P(B_\tau = -a)\mathbb{E}(V(p-a, \sqrt{q^2+\tau})|B_\tau = -a) +$$
$$P(B_\tau = a)\mathbb{E}(V(p+a, \sqrt{q^2+\tau})|B_\tau = a) =$$
$$\frac{1}{2}\left(\mathbb{E}(V(p-a, \sqrt{q^2+\tau})|B_\tau = -a) + \mathbb{E}(V(p+a, \sqrt{q^2+\tau})|B_\tau = a)\right) \geq$$
$$\frac{1}{2}\left(V\left(p-a, \sqrt{q^2+\mathbb{E}(\tau|B_\tau = -a)}\right) + V\left(p+a, \sqrt{q^2+\mathbb{E}(\tau|B_\tau = a)}\right)\right.$$
$$= \frac{1}{2}\left(V\left(p-a, \sqrt{q^2+a^2}\right) + V\left(p+a, \sqrt{q^2+a^2}\right)\right) .$$

Notice that we have used $P(B_\tau = a) = P(B_\tau = -a) = 1/2$, $\mathbb{E}(\tau|B_\tau = a) = \mathbb{E}(\tau|B_\tau = -a) = a^2$, and the fact that the map $t \mapsto V(p, \sqrt{t})$ is convex together with Jensen's inequality.

47. Here is $P(x; b, y)$.

$$P(x) =$$

$$-128b^3y^3(b^2y^2 + y^2 + 2 + 4by + 3b^2 + 2b^3y +$$
$$b^4)(b^2y^2 + y^2 + 2 - 4by + 3b^2 - 2b^3y + b^4)x^3 +$$
$$(-64y^8b^8 + 1088b^6y^6 - 3392b^8y^4 + 8128b^{10}y^2 +$$
$$384b^{10}y^6 - 704b^{12}y^4 + 960b^8y^6 - 3136b^{10}y^4$$
$$+ 3392b^{12}y^2 + 512b^{14}y^2 - 64y^8b^6 + 64y^8b^4 +$$
$$64y^8b^2 - 960b^4y^6 + 960b^6y^4 + 64b^2y^6$$
$$- 2816b^4y^2 + 1280b^4y^4 + 1088b^6y^2 -$$
$$640b^2y^4 + 7872b^8y^2 - 1280b^2y^2 - 10880b^8$$
$$- 8960b^{10} - 3072b^4 - 128b^{16} - 7808b^6 - 512b^2 -$$
$$4352b^{12} - 1152b^{14})x^2$$
$$(-1792b^5y^3 + 256b^7y^7 - 5504b^7y^3 - 1408b^5y^7 +$$

$$+3456b^7y^5 - 384y^7b^3 + 640b^9y^5 +$$
$$2752b^5y^5 + 1536b^3y^3 - 5760b^9y^3 - 3840b^{11}y^3 -$$
$$768b^3y^5 + 512by + 3072b^3y +$$
$$1024by^3 + 1984b^{13}y + 384b^{15}y + 32b^{17}y +$$
$$32by^9 + 10272b^9y + 768by^5 +$$
$$5760b^{11}y + 256by^7 + 32b^9y^9 - 128b^{11}y^7 -$$
$$1408b^{13}y^3 - 64b^5y^9$$
$$-640b^9y^7 + 1664b^{11}y^5 + 192b^{13}y^5 - 128b^{15}y^3 +$$
$$7936b^5y + 11520b^7y)x +$$
$$-256 - 144b^{18} - 16y^{10} + 688y^8b^8 + 1504b^6y^6 -$$
$$1920b^8y^4 - 3440b^{10}y^2$$
$$-2304b^{10}y^6 + 2592b^{12}y^4 - 192b^8y^6 + 3264b^{10}y^4 -$$

$$
\begin{aligned}
&-4448b^{12}y^2-352b^{14}\\
&y^2-288y^8b^6-224y^8b^4+48y^8b^2-736b^4y^6-\\
&1376b^6y^4-320b^2y^6-2816b^4y^2\\
&-480b^4y^4+2496b^6y^2-1792b^2y^4+3056b^8y^2-\\
&3072b^2y^2-768y^2-512y^6-896y^4\\
&-144y^8-3344b^8+1584b^{10}-4992b^4-336b^{16}-\\
&6656b^6-1792b^2+2528b^{12}+\\
&608b^{14}-64b^{16}y^4+96b^14y^6+16y^{10}b^2+32y^{10}b^4+\\
&624b^{16}y^2-864b^{14}y^4\\
&+416b^{12}y^6-64b^{12}y^8-16b^{10}y^8-16b^8y^{10}+\\
&16b^{10}y^{10}-32y^{10}b^6+16b^{18}y^2
\end{aligned}
$$

F. Barthe, B. Maurey, *Some remarks on isoperimetry of Gaussian type*, Annales de l'Institut Henri Poincare (B) Probability and Statistics, Vol. 36, Iss. 4, pp. 419–434.

S. G. Bobkov, F. Götze, *Discrete isoperimetric and Poincaré-type inequalities*, Probab. Theory Relat. Fields 114, 245–277 (1999).

A. Chang, J. M. Wilson, Th. Wolff, *Some weighted norm inequalities concerning the Schrödinger operators* , Comment. Math. Helvetici, Vol. 60, 1985, 217–246.

B. Davis, *On the L^p norms of stochastic integrals and other martingales*, Duke Math. J. Vol. 43, pp. 697–704 (1976).

I. Ekelan, R. Temam, *Convex analysis and variational problems*, North-Holland Publishing Company, Amsterdam, 1976.

- L. H. Harper, *Optimal numberings and isoperimetric problems on graphs*, J. Combin. Theory, !, pp. 385–393, 1996.
- J. Hartung, *An extension of Sion's minimax theorem with an application to a method for constrained games*, Pacific J. Math. Vol. 103, No. 2 (1982), 401–408.
- P. Ivanisvili, A. Volberg, *Poincaré inequality 3/2 on the Hamming cube*, preprint arXiv:1608.4021.
- P. Ivanisvili, A. Volberg, *Isoperimetric functional inequalities via the maximum principle: the exterior differential systems approach*, preprint arXiv: 1511.06895.
- P. Ivanisvili, A. Volberg, *Improving Beckner's bound via Hermite functions*, Analysis & PDE 10-4 (2017), 929–942.

M. Ledoux, *Concentration of measure and logarithmic Sobolev inequalities*, In Séminare de Probabilités XXXIII, 120–216. Lecture Notes in Math. Vol. 1709. Springer, Berlin, 1999.

M. Ledoux, *Isoperimetry and Gaussian Analysis*, lecture notes, Univ. de Toulouse.

A. Naor G. Schechtman, *Remarks on Non linear Type and Pisier's Inequality*, Jooournal für die reine und angevandte Mathematik (Crelle's Journal) 552 (2002) 213–236.

F. Nazarov, S. Treil, A. Volberg, *Bellman function in stochastic control and harmonic analysis*, Systems, approximation, singular integral operators, and related topics (Bordeaux, 2000), 393–423, Oper. Theory Adv. Appl., 129, Birkhäuser, Basel, 2001.

A. Osekowski, *Sharp Martingale and Semimartingale Inequalities*, Monografie Matematyczne Vo. 72, Springer, Basel.

G. Pisier, *Probabilistic methods in the geometry of Banach spaces*, in "Probability and Analysis, Varenna (Italy) 1985", Lecture Notes in Math. 1206, 167–241, Springer (1986).

M. Sion, *On general minimax theorems*, Pac. J. Math. 8 (1958) 171–176.

M. Talagrand, *Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis' graph connectivity*, Geom. Funct. Anal. 3, No. 3, 295–314 (1993).

G. Wang, *Sharp square function inequalities for conditionally symmetric martingales*, TAMS, Vol. 328, No. 1 (1991)

G. Wang, *Some sharp inequalities for conditionally symmetric martingales*, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1989.

S.-T.Yau, *Isoperimetric constants and the first eigenvalue of a compact Riemannian manifold*, Ann. scient. Éc. Norm. Sup. 8, 487–507 (1975).