Elliptic measure and rectifiability

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Workshop on Real Harmonic Analysis and its Applications to Partial Differential Equations and Geometric Measure Theory: on the occasion of the 60th birthday of Steve Hofmann

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Motivation

What is the relationship between the geometry of a domain and the boundary regularity of the solutions to a differential operator on this domain? (*regularity=degree of smoothness*.)

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Can the regularity at the boundary of a "general harmonic function" distinguish between a rectifiable and a purely unrectifiable boundary?

Some history

- F&M Riesz (1916): Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain bounded by a Jordan curve. If $\mathcal{H}^1(\partial\Omega) < \infty$ then the harmonic measure ω and the surface measure $\sigma = \mathcal{H} \llcorner \partial\Omega$ are mutually absolutely continuous, i.e.

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- Lavrentiev (1936): Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected chord arc domain. Then $\omega \in A_\infty(\sigma)$.
- What happens in higher dimensions?

Domains

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  - Ω is uniform
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- A domain $\Omega \subset \mathbb{R}^n$ has Ahlfors regular boundary if there exists $c_0 > 1$ such that for $q \in \partial\Omega$ and $r \in (0, \operatorname{diam}\Omega)$

$$c_0^{-1} r^{n-1} \leq \sigma(B(q,r)) \leq c_0 r^{n-1}.$$

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Divergence form elliptic operators

Let $\Omega \subset \mathbb{R}^n$ be a bounded Wiener regular domain and $Lu = -\mathrm{div}\,(A(x)\nabla u)$ with $A(x) = (a_{ij}(x))$ an **uniformly elliptic** symmetric matrix with bounded measurable coefficients, i.e.

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle, \; \langle A(x)\xi, \zeta\rangle\Lambda \leq |\xi||\zeta| \text{ for } x \in \Omega \text{ and } \xi, \zeta \in \mathbb{R}^n.$$

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Let ω_L be the corresponding elliptic measure. Recall that if $f \in C(\partial\Omega)$ there exists $u \in C(\overline{\Omega})$ such that

$$\begin{cases} Lu &= 0 \text{ in } \Omega \\ u &= f \text{ on } \partial\Omega \end{cases} \tag{1}$$

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Moreover

$$u(x) = \int_{\partial\Omega} f(q)\, d\omega_L^x(q)$$

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Questions: *Characterize the operators L for which $\omega_L \in A_\infty(\sigma)$. To what extent does this characterization depend on the domain?*

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- *Behavior of A and the corresponding elliptic measure on interior Lipschitz domains whose boundaries coincide with $\partial\Omega$ in big pieces.*

Approach based on the oscillation of the matrix A

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$$\sup\{\delta(z)|\nabla A(z)|^2 : z \in B(x, \delta(x)/2)\}$$

  is a Carleson measure, then $\omega_L \in A_\infty(\sigma)$. Here $\delta(x) = \mathrm{dist}(x, \partial\Omega)$ and $\sigma = \mathcal{H}^{n-1} \llcorner \partial\Omega$.

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- Similar results hold on chord arc domains. Key: good approximation by interior Lipschitz domains + maximum principle.

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Results:

- Hofmann-Martell-Toro
- Azzam-Garnett-Mourgoglou-Tolsa
- Akman-Badger-Hofmann-Martell

Theorems: Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $L\cdot = -\mathrm{div}\,(A(x)\nabla\cdot)$ be uniformly elliptic.

1. Zhao-Toro: If $A \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and $\sigma \ll \omega_L$ then $\partial\Omega$ is $(n-1)$-rectifiable.

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2. Azzam-Mourgoglou: If A satisfies the [KP] condition and $\sigma \ll \omega_L$ then $\partial\Omega$ is $(n-1)$-rectifiable.
3. Zhao-Toro: If $A \in C(\overline{\Omega})$ and $\omega_L \in A_\infty(\sigma)$, there exists $r_\Omega > 0$ s.t. Ω satisfies the exterior corkscrew condition for balls of radius less than r_Ω. In particular $\partial\Omega$ is *locally uniformly rectifiable.*

Key idea: Understand the structure of the tangent objects

Let $\Omega \subset \mathbb{R}^n$ be a bounded uniform domain with Ahlfors regular boundary. Let $x_0 \in \Omega$, $u(y) = G_L(x_0, y)$, $\omega_L^{x_0}$, and $q \in \partial\Omega$. Let $q_j \in \partial\Omega$, $q_j \to q$, $r_j \to 0^+$ and consider

$$\Omega_j = \frac{1}{r_j}(\Omega - q_j), \quad \partial\Omega_j = \frac{1}{r_j}(\partial\Omega - q_j), \quad \sigma_j(E) = \frac{\sigma(r_j E + q_j)}{r_j^{n-1}},$$

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$$u_j(z) = r_j^{n-2}\frac{u(r_j z + q_j)}{\omega(B(q_j, r_j))}, \quad \text{and} \quad \omega_j(E) = \frac{\omega(r_j E + q_j)}{\omega(B(q_j, r_j))}.$$

Properties of the pseudo-tangents in the case $A \in C(\overline{\Omega})$

Modulo passing to a subsequence we have:

- There exists $u_\infty \in C(\mathbb{R}^n)$ such that $u_j \to u_\infty$ uniformly on compact sets and $\nabla u_j \rightharpoonup \nabla u_\infty$ in $L^2_{\text{loc}}(\mathbb{R}^n)$.

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- There exists an Ahlfors regular measure μ_∞ such that $\sigma_j \rightharpoonup \mu_\infty$ and $\operatorname{spt}\mu_\infty = \partial\Omega_\infty$. Moreover $\mu_\infty \sim \sigma_\infty = \mathcal{H}^{n-1} \llcorner \partial\Omega_\infty$, and $\partial\Omega_\infty$ is Ahlfors regular.

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- $L_\infty u_\infty = -\mathrm{div}(A(q)\nabla u_\infty) = 0$ in Ω_∞, $u_\infty > 0$ in Ω_∞ and $u_\infty = 0$ on $\partial\Omega_\infty$.

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- This combined with a contradiction argument yields that there exists a $r_\Omega > 0$ s.t. Ω satisfies the exterior corkscrew condition for balls of radius less than r_Ω.

Take away

There is an underlying compactness argument which guarantees that objects in a given class (in this case the dilations $(\Omega_j, \partial\Omega_j, \sigma_j, u_j, \omega_j, A_j)$*) converge to an object in the class* $(\Omega_\infty, \partial\Omega_\infty, \sigma_\infty, u_\infty, \omega_{L_\infty}, A_\infty)$*. Under the correct assumptions on* A *(for example) the limiting object is more regular. This allows us to draw information about the sequence of dilations and the original object.*

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Corollary: Let $\Omega \subset \mathbb{R}^n$ be a bounded chord arc domain and let A satisfy the [KP] condition then $\omega_L \in A_\infty(\sigma)$.

Main result

Theorem: Let $\Omega \subset \mathbb{R}^n$ be a uniform domain with Ahlfors regular boundary. Let A be a symmetric uniformly elliptic bounded matrix in Ω satisfying the [KP] condition. Then the following are equivalent:

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3) $\omega_L \in A_\infty(\sigma)$.

Remarks:

1) $\Longleftrightarrow$ 2) [DJ], [S], [HMU], [AHMMT]

2) $\Longrightarrow$ 3) [KP], [DJ], [HMU]

3) $\Longrightarrow$ 1) Hofmann-Martell-Mayboroda-Toro-Zhao [HMMTZ]

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Theorem [HMMTZ] : If $\Omega \subset \mathbb{R}^n$ is a uniform domain with Ahlfors regular boundary, $L = -div(A\nabla\,)$ is a symmetric elliptic bounded operator with constants $1 \leq \lambda \leq \Lambda < \infty$, $\omega_L \in A_\infty(\sigma)$ and A satisfies [KP] with small constant then Ω satisfies the exterior corkscrew condition.

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Definition: We say that $\omega_L \in A_\infty(\sigma)$ with constants κ and θ if for $E \subset \Delta$ where $\Delta = B(q, r) \cap \partial\Omega$, $q \in \partial\Omega$ and $r > 0$

$$\frac{\omega(E)}{\omega(\Delta)} \leq \kappa \left(\frac{\sigma(E)}{\sigma(\Delta)} \right)^{\theta}$$

Small Carleson constant case II

Theorem: [HMMTZ] Given $n \geq 3$, $M > 1$, $c_o > 1$, $1 \leq \lambda \leq \Lambda < \infty$, $\kappa > 1$ and $\theta \in (0,1)$ there exist $N > 1$ and $\varepsilon > 0$ such that if $\Omega \subset \mathbb{R}^n$ is a bounded M-uniform domain whose boundary is Ahlfors regular with constant c_o, $L = -\mathrm{div}\,(A\nabla\,)$ is a symmetric elliptic bounded operator with constants λ and Λ, $\omega_L \in A_\infty(\sigma)$ with constants κ and θ and

$$\sup_{0<r<\mathrm{diam}\,\Omega} \sup_{q\in\partial\Omega} \frac{1}{r^{n-1}} \int_{B(q,r)\cap\Omega} \delta(x)|\nabla A|^2\, dx < \varepsilon,$$

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then Ω satisfies the exterior corkscrew condition with constant N. Here N only depends on n, M, c_0, λ, Λ, κ, and θ.

Proof by contradiction I

Assume there is a set of allowable constants M, c_o, λ, Λ, κ, θ and sequences Ω_j of M uniform domains with c_o Ahlfors regular boundary, $L_j = -\mathrm{div}\,(A_j \nabla\,)$ symmetric elliptic bounded operators with constants λ and Λ, $\omega_j = \omega_{L_j} \in A_\infty(\sigma_j)$ with constants κ and θ and $\varepsilon_j \to 0$, such that

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and contrary to the conclusion there are $q_j \in \partial\Omega_j$ and $r_j \in (0, \mathrm{diam}\,\Omega_j)$ such that Ω_j has no exterior corkscrew ball with constant N at the point q_j and radius r_j.

Proof by contradiction II

Define

$$\widetilde{\Omega_j} = \frac{1}{r_j}\left(\Omega_j - q_j\right), \quad \partial\widetilde{\Omega_j} = \frac{1}{r_j}\left(\partial\Omega_j - q_j\right), \quad \widetilde{A}_j(x) = A_j(r_j x + q_j),$$

and $\widetilde{\sigma}_j$, $\widetilde{\omega}_j$ and $\widetilde{u}_j$ accordingly.

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Thus Ω_∞ admits exterior corkscrew condition. This leads to a contradiction.

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- Extrapolation $+$[HMM] $+$ [GMT] ensure that Ω is chord arc.

HAPPY BIRTHDAY STEVE!

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Thank you to the organizers,
and participants for a wonderful conference.