Favard length, analytic capacity, and the Cauchy transform

Xavier Tolsa
(joint with Alan Chang)

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Removable singularities and analytic capacity

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Ahlfors (1947): E is removable for bounded analytic functions iff $\gamma(E) = 0$.

Rectifiability and Favard length

We say that $E \subset \mathbb{R}^2$ is rectifiable if it is $\mathcal{H}^1$-a.e. contained in a countable union of curves of finite length.
E is purely unrectifiable if any rectifiable subset $F \subset E$ satisfies $\mathcal{H}^1(F) = 0$.

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Favard length:

$$\mathrm{Fav}(E) = \int_0^\pi \mathcal{H}^1(P_\theta(E))\, d\theta,$$

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Besicovitch projection theorem:
A set E with $\mathcal{H}^1(E) < \infty$ is purely unrectifiable iff $\mathrm{Fav}(E) = 0$.

Analytic capacity and Favard length

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- He used that in this case $\mathrm{Fav}(E) = 0$ iff E is purely unrectifiable.
- He used the connection between analytic capacity, the Cauchy transform, and Melnikov's curvature of measures.
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Open problem: $\mathrm{Fav}(E) > 0 \Rightarrow \gamma(E) > 0$?

Analytic capacity, the Cauchy transform, and curvature I

The Cauchy transform of a measure μ is

$$\mathcal{C}(\mu)(z) = \int \frac{1}{\xi - z} \, d\mu(\xi).$$

For a function f, $\mathcal{C}_\mu(f) := \mathcal{C}(f\mu)$.

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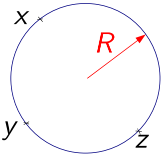
Curvature of μ (Melnikov, 1995):

$$c^2(\mu) := \iiint \frac{1}{R(x,y,z)^2}\, d\mu(x) d\mu(y) d\mu(z).$$



Analytic capacity, the Cauchy transform, and curvature II

Melnikov and Verdera (1995): If $\mu \in \Sigma_1(E)$, then

$$\|\mathcal{C}(\mu)\|^2_{L^2(\mu)} = \frac{1}{6}c^2(\mu) + O(\mu(E)).$$

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T. proved in 2003:

$$\begin{aligned}\gamma(E) &\approx \sup\{\mu(E) : \mu \in \Sigma_1(E),\ c^2(\mu) \le \mu(E)\} \\ &\approx \sup\{\mu(E) : \mu \in \Sigma_1(E),\ \|\mathcal{C}_\mu\|_{L^2(\mu)\to L^2(\mu)} \le 1\}.\end{aligned}$$

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- This implies the semiadditivity of analytic capacity.
- Extended to higher dimensions by Volberg.

Favard length and projection of measures

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For any set $I \subset [0,\pi]$ with $\mathcal{H}^1(I) > 0$, there is $c_I > 0$ such that

$$\mathrm{Fav}(E) \geq c_I \frac{\mu(E)^2}{\int_I \|P_\theta\mu\|_2^2\, d\theta}.$$

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Proof: If μ is supported on E, for all $\theta \in I$:

$$\mu(E) = \|P_\theta\mu\|_1 \leq \|P_\theta\mu\|_2\,\mathcal{H}^1(P_\Theta E)^{1/2}.$$

Thus,

$$\begin{aligned}
\mu(E)\,\mathcal{H}^1(I) &\leq \int_I \|P_\theta\mu\|_2\,\mathcal{H}^1(P_\Theta E)^{1/2}\,d\theta \\
&\leq \left(\int_I \|P_\theta\mu\|_2^2\,d\theta\right)^{1/2} \left(\int_I \mathcal{H}^1(P_\Theta E)\,d\theta\right)^{1/2} \\
&\leq \left(\int_I \|P_\theta\mu\|_2^2\,d\theta\right)^{1/2} \mathrm{Fav}(E)^{1/2}.
\end{aligned}$$

Our main result

Recall that for any set $I \subset [0, \pi]$ with $\mathcal{H}^1(I) > 0$,

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Theorem (Chang, T.)

For any interval $I \subset [0,\pi]$ with $\mathcal{H}^1(I) > 0$, there is $c_I > 0$ such that

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Remarks:

- We don't know if the result holds replacing the interval I by an arbitrary set $A \subset [0, \pi]$ with $\mathcal{H}^1(A) > 0$.

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Remarks:

- We don't know if the result holds replacing the interval I by an arbitrary set $A \subset [0, \pi]$ with $\mathcal{H}^1(A) > 0$.
- An analogous result is valid for the capacities $\Gamma_{d,n}$ associated to the kernels $x/|x|^{n+1}$ in $\mathbb{R}^d$.

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Remarks:

- We don't know if the result holds replacing the interval I by an arbitrary set $A \subset [0,\pi]$ with $\mathcal{H}^1(A) > 0$.
- The connection between γ and Fav is one of the motivations that has led to study Fav in Cantor self-similar sets (for example, by Nazarov, Peres, and Volberg).

About the proof

The proof has two main ingredients:

- A Fourier estimate.
- A geometric argument involving a corona type decomposition.

The Fourier estimate

Theorem

Let μ be a finite Radon measure in $\mathbb{C}$ and I an interval. Then

$$2\iint_{x-y\in K_I\setminus\{0\}} \frac{1}{|x-y|}\,d\mu(x)\,d\mu(y) \leq \int_I \|P_\theta\mu\|_2^2\,d\theta,$$

where $\quad K_I = \{re^{i\theta} : r\in\mathbb{R},\, \theta\in I\}.$

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  So in this case the Theorem is equivalent to the easy estimate

$$\gamma(E)\geq c\,C_1(E),$$

  where $C_1(E)$ is the capacity associated with the Riesz kernel $\frac{1}{|x|}$.

More remarks

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- The use of Fourier transform is a usual tool in the study of projections, although usually integrals are over the whole $[0, \pi]$.
- The preceding result is partially inspired by a result of Martikainen and Orponen, where they obtained a related estimate in the case when μ is AD-regular.
  Using this estimate, they characterized when an AD-regular set has big pieces of Lipschitz graphs (which is stronger than being uniformly rectifiable).

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- The use of Fourier transform is a usual tool in the study of projections, although usually integrals are over the whole $[0, \pi]$.
- Examples:

  - If μ is supported on Lipschitz graph and I is chosen suitably,

$$\iint_{x-y \in K_I} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y) = 0.$$

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  - If $C_1(E) > 0$, there exists μ supported on E such that

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  - One can construct sets with $C_1(E) = 0$, purely unrectifiable, such that

$$\iint_{x-y\in K_I} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y) < \infty$$

for a suitable I.

The geometric argument

We prove the following:

Theorem

For $\mu \in \Sigma(E)$, we have

$$c^2(\mu) \lesssim \mu(E) + \iint_{x-y\in K_I} \frac{1}{|x-y|}\,d\mu(x)\,d\mu(y).$$

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- The main theorem follows by combining the result above and the Fourier estimate for a suitable big piece of μ.

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(a) *The roots of $\mathcal{T} \in J$ are doubling cubes and fulfil the packing condition*

$$\sum_{\mathcal{T}\in J} \Theta_\mu(\mathrm{Root}(\mathcal{T}))\, \mu(\mathrm{Root}(\mathcal{T})) \lesssim \mu(E) + \iint_{x-y\in K_I} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y).$$

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(b) *For all $\mathcal{T} \in J$ and $Q \in \mathcal{T}$,* $\quad \Theta_\mu(2Q) \leq C\, \Theta_\mu(\mathrm{Root}(\mathcal{T}))$.

(c) *In each $\mathcal{T} \in J$, E is "well approximated" by a Lipschitz graph $\Gamma_{\mathcal{T}}$.*

The corona decomposition II

(c) means that for any $Q \in \mathcal{T}$ there exists a ball $\widetilde{B}_Q$ concentric with Q such that $\widetilde{B}_Q \cap \Gamma_{\mathcal{T}} \neq \varnothing$ and

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The lemma is proven by a stopping time argument, using $\iint_{x-y\in K_I} \dfrac{1}{|x-y|}\, d\mu(x)\, d\mu(y)$ as a square function that controls the measure μ of the points far from a Lipschitz graph.

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Note that if $x - y \notin K_I$ for all $x, y \in \operatorname{supp}\mu$, then μ is supported on a Lipschitz graph.

Estimating the curvature by the corona decomposition

By an approximating argument, taking into account that the curvature and the Cauchy transform are bounded in Lipschitz graphs, we derive

$$
\begin{aligned}
c^2(\mu) &\lesssim \sum_{\mathcal{T}\in J} \Theta_\mu(\mathrm{Root}(\mathcal{T}))^2\, \mu(\mathrm{Root}(\mathcal{T})) \\
&\leq \sup_Q \Theta_\mu(Q) \sum_{\mathcal{T}\in J} \Theta_\mu(\mathrm{Root}(\mathcal{T}))\, \mu(\mathrm{Root}(\mathcal{T})) \\
&\lesssim \mu(E) + \iint_{x-y\in K_I} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y).
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Estimating the curvature by the corona decomposition

By an approximating argument, taking into account that the curvature and the Cauchy transform are bounded in Lipschitz graphs, we derive

$$\begin{aligned} c^2(\mu) &\lesssim \sum_{\mathcal{T}\in J} \Theta_\mu(\mathrm{Root}(\mathcal{T}))^2\, \mu(\mathrm{Root}(\mathcal{T})) \\ &\leq \sup_Q \Theta_\mu(Q) \sum_{\mathcal{T}\in J} \Theta_\mu(\mathrm{Root}(\mathcal{T}))\, \mu(\mathrm{Root}(\mathcal{T})) \\ &\lesssim \mu(E) + \iint_{x-y\in K_I} \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y). \end{aligned}$$

Remark: To estimate $c^2(\mu)$, the sharp power in $\Theta_\mu(\cdot)$ is 2.
Can the preceding result be improved?

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  Remark: There are sets $E \subset \mathbb{R}^2$ such that
  1. $\mathcal{H}^1(P_\theta E) > 0$ for a.e. $\theta \in [0, \pi]$.
  2. For all Borel measures μ such that $\mu(E) > 0$ and all intervals $I \subset [0, \pi)$, we have $\mathcal{H}^1(\{\theta \in I : P_\theta \mu \notin L^2\}) > 0$.

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- Characterize big pieces of Lipschitz graphs in terms of $\mathrm{Fav}(E)$, or more generally in terms of the projections of E.

Thank you!

Happy birthday, Steve.