A two weight local *Tb* theorem

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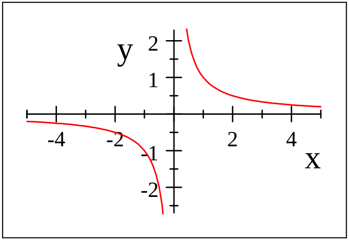
The Hilbert transform Hf arose in 1905 in connection with Hilbert's twenty-first problem, and for $f \in L^2(\mathbb{R})$ is defined almost everywhere by the *principal value* singular integral

$$\begin{aligned} Hf(x) &= p.v. \int \frac{1}{y-x} f(y)\, dy \\ &\equiv \lim_{\varepsilon \to 0} \int_{|y-x|>\varepsilon} \frac{1}{y-x} f(y)\, dy, \qquad a.e. x \in \mathbb{R}. \end{aligned}$$



The convolution kernel of H

The two weight problem

- **Problem**: Given two locally finite positive Borel measures σ and ω on $\mathbb{R}$, characterize the boundedness of H_σ from $L^2(\sigma)$ to $L^2(\omega)$:

$$\left(\int_{\mathbb{R}} |H_\sigma f|^2 \, d\omega\right)^{\frac{1}{2}} \leq \mathfrak{N} \left(\int_{\mathbb{R}} |f|^2 \, d\sigma\right)^{\frac{1}{2}}, \qquad f \in L^2(\sigma),$$

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  uniformly over all appropriate truncations of the operator T.

- Here $H_\sigma f \equiv H(f\sigma)$, and the appropriate truncations

$$H_{\sigma,\delta,R} f(x) \equiv \int_{\mathbb{R}} K_{\delta,R}(x,y) f(y) \, d\sigma(y), \qquad x \in \mathbb{R},$$

  are given by a family $\{\eta_{\delta,R}\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K_{\delta,R}(x,y) = \eta_{\delta,R}(|x-y|) \frac{1}{y-x}$ are bounded with compact support for fixed x or y.

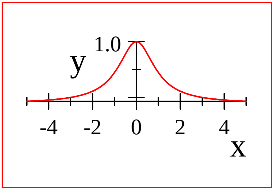
Toward a geometric characterization

The pivotal condition of NTV

- In 2004 Nazarov, Treil and Volberg showed that if a weight pair (ω, σ) satisfies the pivotal condition

$$\sum_{r=1}^{\infty} |I_r|_\omega \, \mathrm{P}(I_r, \chi_{I_0}\sigma)^2 \leq \mathcal{P}_*^2 \, |I_0|_\sigma ; \quad \mathrm{P}(I, \nu) = \int \frac{|I|}{|I|^2 + |x - c_I|^2} d\nu(x),$$

and its dual for all decompositions of an interval I_0 into subintervals I_r,



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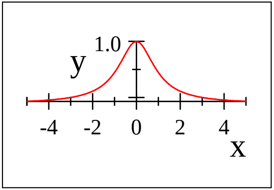

- then the Hilbert transform H satisfies the two weight L^2 inequality

$$\int |H(f\sigma)|^2 d\omega \leq \mathfrak{N} \int |f|^2 d\sigma,$$

uniformly for all smooth truncations of the Hilbert transform,

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$$\int_I |H(\chi_I \sigma)|^2 \, d\omega \leq \mathfrak{T}^2 \, |I|_\sigma \ \text{ and } \ \int_I |H(\chi_I \omega)|^2 \, d\sigma \leq (\mathfrak{T}^*)^2 \, |I|_\omega \,.$$

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- A key innovation of NTV was the use of random grids that were 'good' with large probability - good in the sense that small intervals too close to the boundary of a large grandparent could be safely ignored.

Energy and functional energy

- Lacey, Sawyer and Uriarte-Tuero introduced another key innovation in the energy condition (a refinement of NTV's pivotal condition):

$$\sum_{r=1}^{\infty} |I_r|_{\omega} \mathrm{E}(I_r,\omega)^2 \mathrm{P}(I_r,\chi_I \sigma)^2 \leq \mathcal{E}^2 |I|_{\sigma}, \qquad I = \dot{\bigcup}_{r=1}^{\infty} I_r ,$$

  a consequence of the testing conditions and the Muckenhoupt condition, and where

$$\mathrm{E}(J,\omega) \equiv \left(\mathbb{E}_J^{\omega(dx)} \mathbb{E}_J^{\omega(dx')} \left(\frac{|x-x'|}{|J|} \right)^2 \right)^{1/2} .$$

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- A related functional energy condition replaced the Poisson term $\mathrm{P}\left(I_r, \chi_I \sigma\right)$ with $\mathrm{P}\left(I_r, h\sigma\right)$, and played a crucial role in handling the 'far' forms, which led to an indicator/interval characterization:

The indicator/interval characterization

Unpublished

Theorem (Lacey, Sawyer, Shen and Uriarte-Tuero (2012))

The best constant $\mathfrak{N}$ *in the two weight inequality (1) for the Hilbert transform satisfies*

$$\mathfrak{N} \approx \sqrt{\mathcal{A}_2} + \mathfrak{I} + \mathfrak{I}^*,$$

where $\mathfrak{I}, \mathfrak{I}^*$ *are the best constants in the indicator/interval testing conditions,*

$$\int_I \left| H \left(\mathbf{1}_E \sigma \right) \right|^2 \omega \leq \mathfrak{I} \left| I \right|_\sigma , \quad \int_I \left| H \left(\mathbf{1}_E \omega \right) \right|^2 \sigma \leq \mathfrak{I}^* \left| I \right|_\omega ,$$

for all intervals I *and closed subsets* E *of* I.

The Nazarov Treil Volberg conjecture

A question raised in Volberg's 2003 CBMS book, known as the *NTV conjecture*, was whether or not

$$\int_{\mathbb{R}} |H(f\sigma)|^2 \omega \leq \mathfrak{N} \int_{\mathbb{R}} |f|^2 \sigma, \qquad f \in L^2(\sigma), \tag{1}$$

is equivalent to the $\mathcal{A}_2$ condition and the two interval testing conditions.

The NTV conjecture solved

- In 2013 a third key innovation was provided by M. Lacey who found a brilliant bottom/up stopping time and recursion argument needed to control what was then a mysterious local term. The proof is in a two part paper in Duke J. Math.: Part I (M.L., E.S., C.-Y.S., I.U.-T.) and Part II (M.L.) with Lacey's local argument in the second part.

Theorem

The best constant $\mathfrak{N}$ in the two weight inequality (1) for the Hilbert transform, with no common point masses, satisfies

$$\mathfrak{N} \approx \sqrt{\mathcal{A}_2} + \mathfrak{T} + \mathfrak{T}^*,$$

i.e. H_σ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $\mathcal{A}_2$ and interval testing conditions hold.

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- T. Hytönen included common point masses using 'holes' in the Muckenhoupt conditions.

Positive derivative of the kernel

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- Indeed, this property underlies the necessity of the energy condition for testing and Muckenhoupt, upon observing that for a positive measure μ supported outside the double $2J$,

$$\frac{H\mu(x) - H\mu(x')}{x - x'} = \frac{1}{x - x'} \int_{\mathbb{R}\setminus 2J} \left\{ \frac{1}{y - x} - \frac{1}{y - x'} \right\} d\mu(y)$$

$$= \frac{1}{|J|} \int_{\mathbb{R}\setminus 2J} \frac{|J|}{(y - x)(y - x')} d\mu(y) \approx \frac{\mathrm{P}(J, \mathbf{1}_{\mathbb{R}\setminus 2J}\mu)}{|J|},$$

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- and then using a 'create/plug the hole' argument,

$$LHS_{\text{energy}} \sim \sum_{r=1}^{\infty} \mathbb{E}_{I_r}^{\omega(dx)} \mathbb{E}_{I_r}^{\omega(dx')} \left| H\mathbf{1}_{I\setminus I_r}\sigma(x) - H\mathbf{1}_{I\setminus I_r}\sigma\left(x'\right)\right|^2 + \mathcal{A}_2 |I|_{\sigma},$$

with $H\mathbf{1}_{I\setminus I_r}\sigma = H\mathbf{1}_I\sigma - H\mathbf{1}_{I_r}\sigma$, and finally using testing on I and all the I_r.

Improving the T1 theorem I

- G. David, J.-L. Journé and S. Semmes improved the $T1$ theorem for Lebesgue measure by replacing the testing function 1 with a bounded accretive function b, i.e. $\mathrm{Re}\, b \geq c > 0$. Applications include an 'easy' proof of the boundedness of the Cauchy operator on Lipschitz curves.

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- M. Christ then further improved this *Tb* theorem to a *local Tb* theorem for a single doubling weight on a homogeneous space in which the testing functions are now a family $\mathbf{b} = \{b_Q\}$ of bounded accretive functions indexed by 'cubes' Q.

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- M. Christ then further improved this Tb theorem to a *local* Tb theorem for a single doubling weight on a homogeneous space in which the testing functions are now a family $\mathbf{b} = \{b_Q\}$ of bounded accretive functions indexed by 'cubes' Q.
- Further improvements, such as relaxing the integrability of the testing functions $\mathbf{b}$, and extending the weight to upper doubling, were then made by many authors, including Auscher, David, Hytönen, Hofmann, Lacey, Martikainen, Muscalu, Nazarov, Tao, Thiele, Treil, and Volberg, with applications to the solution of Painlevé's problem, the Kato problem, and layer potentials.

NTV advances

- Building on the b-adapted Haar functions of David, Journé and Semmes, and Coifman, Jones and Semmes, NTV used measure-adapted Haar functions $h_Q^{\mu,\mathbf{b}}$ with bounded testing functions $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ and $\mathbf{b}$-martingale differences $\triangle_I^{\sigma,\mathbf{b}} f$ (where in the setting of $T1$, $\triangle_I^{\sigma,1} f = \left\langle f, h_I^{\sigma,1} \right\rangle_\sigma h_I^{\sigma,1}$ is an orthogonal projection), together with a key new technique of random grids supporting the Haar functions, to reduce matters in the inner product

$$\langle Tf, g\rangle_\omega = \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{G}} \int \left(T_\sigma \triangle_I^{\sigma,\mathbf{b}} f \right) \triangle_J^{\sigma,\mathbf{b}^*} g d\omega$$

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- NTV then established frame inequalities for the martingale differences $\triangle_I^{\sigma,\mathbf{b}} f$ and their adjoints $\left(\triangle_I^{\sigma,\mathbf{b}}\right)^* f$, and
- they established surgery to handle the difficult nearby inner products $\int \left(T_\sigma \triangle_I^{\sigma,\mathbf{b}} f \right) \triangle_J^{\sigma,\mathbf{b}^*} g d\omega$ when I and J are close in scale and position.

Hytönen-Martikainen and Lacey-Martikainen advances

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- However, this argument uses methods of interpolation not immediately available in the two weight setting.

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Improving the T1 theorem II

- This raises the question of finding a Tb theorem for the Hilbert transform involving *two weights* instead of one weight.
- However, an immediate difficulty is the control of the energy condition by the Muckenhoupt and **b**-testing conditions when **b** 'breaks'. As a consequence we include both the Muckenhoupt and energy conditions in our characterization of the norm inequality.
- There is an example of Lacey, Sawyer and Uriarte-Tuero to show that the Muckenhoupt and energy conditions alone do not suffice for the norm inequality, but we do not know whether the Muckenhoupt and **b**-testing conditions alone suffice.

- A complex-valued function b on $\mathbb{R}$ is said to be *accretive* if

$$0 < c \leq \operatorname{Re} b\left(x\right) \leq \left|b\left(x\right)\right| \leq C < \infty, \qquad x \in \mathbb{R}.$$

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- Let $p \geq 2$ and let μ be a locally finite positive Borel measure on $\mathbb{R}$. We say that a family $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}}$ of functions indexed by $\mathcal{P}$ is a *p-weakly μ-accretive* family of functions on $\mathbb{R}$ if for all $Q \in \mathcal{P}$,

$$\text{support } b_Q \subset Q \ ,$$

$$0 \quad < \quad c_{\mathbf{b}} \leq \left| \frac{1}{|Q|_\mu} \int_Q b_Q \, d\mu \right| \leq \left(\frac{1}{|Q|_\mu} \int_Q |b_Q|^p \, d\mu \right)^{\frac{1}{p}} \leq C_{\mathbf{b}} < \infty \ .$$

Definitions I

Weakly accretive complex-valued functions

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- Without loss of generality we may take b_Q real-valued.

Definitions II

b-testing conditions

- Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}$. The **b**-testing and **b***-testing conditions for H are given by

$$\int_Q \left|H_\sigma b_Q\right|^2 d\omega \leq \mathfrak{T}^{\mathbf{b}} \left|Q\right|_\sigma , \qquad \text{for all intervals } Q,$$

$$\int_Q \left|H_\omega b_Q^*\right|^2 d\sigma \leq \mathfrak{T}^{\mathbf{b}^*,*} \left|Q\right|_\omega , \qquad \text{for all intervals } Q.$$

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- T. Hytönen show that the *full* **b**-testing conditions for H,

$$\int_{\mathbb{R}} \left|H_\sigma b_Q\right|^2 d\omega \leq \mathfrak{F}\mathfrak{T}^{\mathbf{b}} \left|Q\right|_\sigma , \qquad \text{for all intervals } Q,$$

are controlled by the **b**-testing and Muckenhoupt conditions.

A local two weight Tb theorem

Theorem (Sawyer, Shen and Uriarte-Tuero)

Suppose that σ and ω are locally finite positive Borel measures on the real line $\mathbb{R}$. Set $H_\sigma f = H(f\sigma)$ for any smooth truncation of T^α_σ, let $p > 2$ and let $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ and $\mathbf{b}^* = \{b^*_Q\}_{Q\in\mathcal{P}}$ be p-weakly σ-accretive families of functions on $\mathbb{R}$. Then the Hilbert transform H_σ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ with operator norm $\mathfrak{N}_H$ **uniformly** *in smooth truncations of H_σ , i.e.*

$$\left\| H_{\sigma,\delta,R} f \right\|_{L^2(\omega)} \leq \mathfrak{N}_H \left\| f \right\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma),\ 0<\delta<R<\infty,$$

if and only if the Muckenhoupt and energy conditions hold, and the **b**-*testing and* **b***-*testing conditions for H both hold. Moreover, we have the equivalence,*

$$\mathfrak{N}_H \approx \mathfrak{T}^{\mathbf{b}}_H + \mathfrak{T}^{\mathbf{b}^*}_H + \sqrt{\mathfrak{A}^\alpha_2} + \mathfrak{E}^\alpha_2 .$$

Generalization to alpha-fractional singular integrals

- Let $0 \leq \alpha < 1$. A standard α-fractional CZ kernel $K^{\alpha}(x,y)$ is a real-valued function defined on $\mathbb{R} \times \mathbb{R}$ satisfying the following for some $\delta > 0$: For $x \neq y$,

$$\left|K^{\alpha}(x,y)\right| \leq C_{CZ}\left|x-y\right|^{\alpha-1}, \ \left|\nabla K^{\alpha}(x,y)\right| \leq C_{CZ}\left|x-y\right|^{\alpha-2},$$

$$\left|\nabla K^{\alpha}(x,y) - \nabla K^{\alpha}(x',y)\right| \leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}\left|x-y\right|^{\alpha-2}.$$

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$$\left|\nabla K^{\alpha}(x,y) - \nabla K^{\alpha}(x',y)\right| \leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}\left|x-y\right|^{\alpha-2}.$$

- An α-fractional singular integral T^{α} with kernel K^{α} is *elliptic* if $\left|K^{\alpha}(x,y)\right| \geq c\left|x-y\right|^{\alpha-1}$, and *gradient elliptic* if

$$\frac{d}{dx}K^{\alpha}(x,y), -\frac{d}{dy}K^{\alpha}(x,y) \geq c\left|x-y\right|^{\alpha-2}. \tag{2}$$

The Hilbert transform kernel $K(x,y) = \frac{1}{y-x}$ is elliptic and (2) holds with $\alpha = 0$.

Generalization to alpha-fractional singular integrals

- Let $0 \leq \alpha < 1$. A standard α-fractional CZ kernel $K^\alpha(x,y)$ is a real-valued function defined on $\mathbb{R} \times \mathbb{R}$ satisfying the following for some $\delta > 0$: For $x \neq y$,

$$\left|K^\alpha(x,y)\right| \leq C_{CZ} \left|x-y\right|^{\alpha-1}, \ \left|\nabla K^\alpha(x,y)\right| \leq C_{CZ} \left|x-y\right|^{\alpha-2},$$

$$\left|\nabla K^\alpha(x,y) - \nabla K^\alpha(x',y)\right| \leq C_{CZ} \left(\frac{|x-x'|}{|x-y|}\right)^\delta \left|x-y\right|^{\alpha-2}.$$

- An α-fractional singular integral T^α with kernel K^α is *elliptic* if $\left|K^\alpha(x,y)\right| \geq c\left|x-y\right|^{\alpha-1}$, and *gradient elliptic* if

$$\frac{d}{dx}K^\alpha(x,y), -\frac{d}{dy}K^\alpha(x,y) \geq c\left|x-y\right|^{\alpha-2}. \tag{2}$$

The Hilbert transform kernel $K(x,y) = \frac{1}{y-x}$ is elliptic and (2) holds with $\alpha = 0$.

- If T^α is elliptic and gradient elliptic then

$$\mathfrak{N}_{T^\alpha} \approx \mathfrak{T}^{\mathbf{b}}_{\mathbf{R}^\alpha} + \mathfrak{T}^{\mathbf{b},*}_{\mathbf{R}^\alpha} + \sqrt{\mathfrak{A}^\alpha_2} + \mathfrak{E}^\alpha_2 .$$

The one weight case

- In the special case that $\sigma = \omega = \mu$ and $0 < \alpha < 1$, the classical Muckenhoupt A_2^α condition is

$$\sup_{Q \in \mathcal{P}} \frac{|Q|_\mu}{|Q|^{1-\alpha}} \frac{|Q|_\mu}{|Q|^{1-\alpha}} < \infty,$$

  which is precisely the upper doubling measure condition with exponent $1 - \alpha$, i.e.

$$|Q|_\mu \leq C\ell(Q)^{1-\alpha}, \qquad \text{for all intervals } Q.$$

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- Both Poisson integrals are then bounded,

$$\mathrm{P}^\alpha(Q, \mu) \lesssim \sum_{k=0}^\infty \frac{|Q|}{(2^k|Q|)^{2-\alpha}} \left|2^k Q\right|_\mu \lesssim C_\alpha < \infty,$$

$$\mathcal{P}^\alpha(Q, \mu) \lesssim \sum_{k=0}^\infty \left(\frac{|Q|}{(2^k|Q|)^2}\right)^{1-\alpha} \left|2^k Q\right|_\mu \lesssim C_\alpha < \infty.$$

- The equal weight pair (μ, μ) satisfies not only the Muckenhoupt $\mathcal{A}_2^\alpha$ condition, but also the strong energy condition $\mathcal{E}_2^\alpha$:

$$\sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^\alpha \left(I_r, \mathbf{1}_I \sigma \right)}{|I_r|} \right)^2 \left\| x - E_{I_r}^\omega x \right\|_{L^2(\mathbf{1}_{I_r}\omega)}^2 \leq C \sum_{r=1}^{\infty} \left\| \frac{x - E_{I_r}^\omega x}{|I_r|} \right\|_{L^2(\mathbf{1}_{I_r}\omega)}^2$$

$$\leq C \sum_{r=1}^{\infty} |I_r|_\omega \leq C\, |I|_\omega = C\, |I|_\sigma \ ,$$

since $\omega = \sigma = \mu$.

One weight Tb theorems

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$$\leq C \sum_{r=1}^{\infty} |I_r|_\omega \leq C\, |I|_\omega = C\, |I|_\sigma \ ,$$

  since $\omega = \sigma = \mu$.

- Thus our two weight *Tb* theorem, when restricted to a single weight $\sigma = \omega$, recovers a weaker version of the one weight theorem of Lacey and Martikainen for dimension $n = 1$ - weaker due to our assumption that $p > 2$.

- In order to control the dual martingale differences for 'breaking' children, i.e. when the testing function corresponding to a child is **not** the restriction of the testing function of the parent, we need to follow NTV in constructing coronas in which the restrictions don't change, and for which the 'breaking' intervals satisfy a Carleson condition.

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- This makes the so-called 'nearby' inner products $\left\langle T_\sigma^\alpha b_I, b_J^* \right\rangle_\omega$, i.e. those in which the intervals I and J are close in both position and scale, difficult to estimate due to the fact that the testing conditions are lost in the corona, except at the tops of coronas, and are replaced with just a **weak** testing condition.

Four difficulties with two weight Tb

Weak testing and the nearby terms

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- In the one weight setting, special considerations such as boundedness of Poisson integrals, are taken into account in handling nearby inner products with random surgery, and are unavailable to us here.

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- This makes the so-called 'nearby' inner products $\langle T^{\alpha}_{\sigma} b_I, b^*_J \rangle_{\omega}$, i.e. those in which the intervals I and J are close in both position and scale, difficult to estimate due to the fact that the testing conditions are lost in the corona, except at the tops of coronas, and are replaced with just a **weak** testing condition.
- In the one weight setting, special considerations such as boundedness of Poisson integrals, are taken into account in handling nearby inner products with random surgery, and are unavailable to us here.
- We develop a recursive method for controlling the nearby form with *energy conditions* and testing at the tops of the coronas - resurrecting the original testing functions discarded during the corona construction.

Four difficulties with two weight Tb

Paraproducts and weak Riesz inequalities

- As shown by Hytönen and Martikainen, martingale differences fail to satisfy two-sided frame-like and Riesz-like inequalities in the setting of a Tb theorem when $p = 2$, complicating the treatment of paraproducts.

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- We assume $p > 2$ in the upper L^p control of testing functions, and then reduce this case to that of *bounded* testing functions using an absorption and recursion argument. We then further reduce to the case where the testing functions b_Q are reverse Hölder on children Q'.
- For such families of testing functions, we prove two-sided weak frame and *Riesz* inequalities for martingale and dual martingale differences (**except** for lower *Riesz* inequalities for martingale differences, which remain open but not needed), and that enable many of the $T1$ two weight techniques to carry over here in the Tb setting. In particular these are key to controlling paraproducts here.

- Only a weaker form of goodness due to Hytönen and Martikainen is available for use in two weight *Tb* theorems. Indeed, as emphasized by Hytönen-Martikainen, we can no longer simply add back in bad intervals whenever we want telescoping identities to hold.

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Weak goodness

- Only a weaker form of goodness due to Hytönen and Martikainen is available for use in two weight Tb theorems. Indeed, as emphasized by Hytönen-Martikainen, we can no longer simply add back in bad intervals whenever we want telescoping identities to hold.
- In fact, in the analysis of the form with $\ell(J) \leq \ell(I)$, it is necessary to have goodness for the intervals J and telescoping for the intervals I; and in the analysis of the form with $\ell(J) > \ell(I)$, it is necessary to have just the opposite. Thus goodness can only be introduced *after* we have restricted the sum to intervals J that have smaller side length than I.

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- We accommodate *weak goodness* in controlling functional energy with a different decomposition of the stopping intervals into 'Whitney' intervals, and two independent families of grids, and in bounding the stopping form by Lacey's size functional on admissible collections using the bottom/up corona construction of Lacey together with an additional top/down 'indented' corona construction

Four difficulties with two weight Tb

Montonicity and Energy Lemmas

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- We use the Lacey-Wick formulation from higher dimensions and introduce an additional square function bound on the right hand side involving an infimum of averages,

$$\inf_{z\in\mathbb{R}} \sum_{J'\in\mathfrak{C}_{\mathrm{broken}}(J)} \left|J'\right|_{\omega} \left(E_{J'}^{\omega}\left|x-z\right|\right)^{2},$$

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  summed over broken children.
- We also use the fact that the corresponding 'unbroken' dual martingale differences form projections, but then we also need to modify the testing function at the top of a corona, and also refine the triple corona construction, so that dual martingale differences have reverse Hölder controlled averages on children (automatic for doubling measures).

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- Then we define broken dual martingale differences in terms of the corona construction of testing functions. We assume wlog that the family $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}}$ of testing functions indexed by $\mathcal{P}$ is an ∞-strongly σ-controlled accretive family.

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- Then we control the nearby form with a new recursive argument using the energy stopping times and the 'original' testing functions.
- Finally we discuss control of functional energy and the stopping form argument of M. Lacey, now adding a top/down indented corona.

Three reductions

- For use in estimating the nearby terms, we first reduce to testing functions $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{D}}$ that satisfy the pointwise lower bound property *PLBP*:

$$|b_Q(x)| \geq c_1 > 0 \qquad \text{for } Q \in \mathcal{D} \text{ and } \mu\text{-a.e. } x \in \mathbb{R},$$

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- Then in order to obtain frame and Riesz inequalities and control paraproducts, we further reduce to the case of bounded weakly accretive testing functions:

$$0 < c_{\mathbf{b}} \leq \left| \frac{1}{|Q|_\mu} \int_Q b_Q \, d\mu \right| \leq \|b_Q\|_{L^\infty(\mu)} \leq C_{\mathbf{b}} < \infty \ ,$$

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- and finally, we use a corona construction to reduce to the case of testing functions with reverse Hölder control on children:

$$\left| \frac{1}{|Q'|_\sigma} \int_{Q'} b_Q \, d\mu \right| \geq c \left\| \mathbf{1}_{Q'} b_Q \right\|_{L^\infty(\mu)} > 0, \, Q' \in \mathfrak{C}(Q) \, , \left|Q'\right|_\mu > 0, \, Q \in \mathcal{P}$$

Stopping conditions

- Given S_0, define $\mathcal{S}(S_0)$ to be the maximal subintervals $I \subset S_0$ so that

$$\frac{1}{|I|_\sigma} \int_I |f| \, d\sigma > C_0 \frac{1}{|S_0|_\sigma} \int_{S_0} |f| \, d\sigma \ ,$$

$$\text{or } \left| \frac{1}{|I|_\sigma} \int_I b_{S_0} d\sigma \right| < \gamma c_{\mathbf{b}} \text{ or } \left(\frac{1}{|I|_\sigma} \int_I |b_{S_0}|^p \, d\sigma \right)^{\frac{1}{p}} > \Gamma C_{\mathbf{b}}$$

$$\text{or } \int_I \left| T_\sigma^\alpha (b_{S_0}) \right|^2 d\omega > \Gamma \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} \right)^2 |I|_\sigma \ ,$$

$$\text{or } \sup_{I \supset \dot{\cup} J_r} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^\alpha (J_r, |b_{S_0}| \, \sigma)}{|J_r|} \right)^2 \left\| \mathsf{P}_{J_r}^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^{\bigstar 2}$$

$$\geq \quad C_{\text{energy}} \left[\left(\mathfrak{E}_2^\alpha \right)^2 + \mathfrak{A}_2^\alpha \right] |I|_\sigma \ .$$

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& \text{or } \left| \frac{1}{|I|_\sigma} \int_I b_{S_0} d\sigma \right| < \gamma c_{\mathbf{b}} \text{ or } \left(\frac{1}{|I|_\sigma} \int_I |b_{S_0}|^p \, d\sigma \right)^{\frac{1}{p}} > \Gamma C_{\mathbf{b}} \\
& \text{or } \int_I \left| T_\sigma^\alpha \left(b_{S_0} \right) \right|^2 d\omega > \Gamma \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} \right)^2 |I|_\sigma \,, \\
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\geq \; & C_{\text{energy}} \left[\left(\mathfrak{E}_2^\alpha \right)^2 + \mathfrak{A}_2^\alpha \right] |I|_\sigma \,.
\end{aligned}
$$

- Set $\mathcal{S} = \{S_0\} \cup \bigcup_{n=0}^{\infty} \mathcal{S}_n$ where $\mathcal{S}_0 = \mathcal{S}(S_0)$ and $\mathcal{S}_{n+1} = \bigcup_{S \in \mathcal{S}_n} \mathcal{S}(S)$, with a twist to obtain the reverse Hölder condition on children.

Martingale averages and differences of testing functions

- Define the **b**-expectation operator $\mathbb{E}_Q^{\mu,\mathbf{b}}$ and the dual **b**-expectation operator $\mathbb{F}_Q^{\mu,\mathbf{b}}$ using the test function b_A in the corona $\mathcal{C}_A$:

$$\begin{aligned} \mathbb{E}_Q^{\mu,\mathbf{b}} f(x) &\equiv \mathbf{1}_Q(x) \frac{1}{\int_Q b_A d\mu} \int_Q f b_A d\mu, \qquad Q \in \mathcal{C}_A , \\ \mathbb{F}_Q^{\mu,\mathbf{b}} f(x) &\equiv \mathbf{1}_Q(x)\, b_A(x) \frac{1}{\int_Q b_A d\mu} \int_Q f d\mu, \qquad Q \in \mathcal{C}_A . \end{aligned}$$

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\end{aligned}$$

- Then define the corresponding martingale and dual martingale differences by

$$\begin{aligned}
\triangle_Q^{\mu,\mathbf{b}} f(x) &\equiv \left(\sum_{Q' \in \mathfrak{C}(Q)} \mathbb{E}_{Q'}^{\mu,\mathbf{b}} f(x) \right) - \mathbb{E}_Q^{\mu,\mathbf{b}} f(x) , \\
\square_Q^{\mu,\mathbf{b}} f(x) &\equiv \left(\sum_{Q' \in \mathfrak{C}(Q)} \mathbb{F}_{Q'}^{\mu,\mathbf{b}} f(x) \right) - \mathbb{F}_Q^{\mu,\mathbf{b}} f(x) .
\end{aligned}$$

Convergence of martingale differences

Both of the following identities hold pointwise μ-almost everywhere, as well as in the sense of strong convergence in $L^2(\mu)$:

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}_N} \mathbb{F}_I^{\mu,\mathbf{b}} f + \sum_{I\in\mathcal{D}:\ \ell(I)\geq N+1} \square_I^{\mu,\mathbf{b}} f \ , \\ f &= \sum_{I\in\mathcal{D}_N} \mathbb{E}_I^{\mu,\mathbf{b}} f + \sum_{I\in\mathcal{D}:\ \ell(I)\geq N+1} \triangle_I^{\mu,\mathbf{b}} f \ , \end{aligned}$$

provided that $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ is an ∞-weakly μ-controlled accretive family.

Weak Riesz inequalities for controlled accretive families

- We have ' weak upper Riesz' inequalities for pseudoprojections $\Psi_{\mathcal{B}}^{\mu,\mathbf{b}} \equiv \sum_{I\in\mathcal{B}} \square_I^{\mu,\mathbf{b}} f$:

$$\left\|\Psi_{\mathcal{B}}^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 \leq C \sum_{I\in\mathcal{B}} \left\|\square_I^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 + \sum_{I\in\mathcal{B}} \left\|\nabla_I^{\mu} f\right\|_{L^2(\mu)}^2, \qquad (3)$$

for all $f \in L^2(\mu)$ and all subsets $\mathcal{B}$ of the grid $\mathcal{D}$.

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- We have 'weak lower Riesz' inequalities:

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  for all $f \in L^2(\mu)$ and all subsets $\mathcal{B}$ of the grid $\mathcal{D}$.

- For martingale differences $\triangle_I^{\mu,\mathbf{b}}$, weak upper Riesz inequalities hold, but are open for weak lower Riesz inequalities.

- First we decompose the bilinear form $\int (T_\sigma f)\, g d\omega$ by interval side length size:

$$\begin{aligned}
\int (T_\sigma f)\, g d\omega &= \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{G}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f \right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&= \left\{ \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G} \\ \ell(J)\leq\ell(I)}} + \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G} \\ \ell(J)>\ell(I)}} \right\} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f \right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\equiv \Theta (f,g) + \Theta^* (f,g) .
\end{aligned}$$

Side length size decomposition

- First we decompose the bilinear form $\int (T_\sigma f)\, g d\omega$ by interval side length size:

$$\begin{aligned}\int (T_\sigma f)\, g d\omega &= \sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{G}}\int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\ &= \left\{ \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G}\\ \ell(J)\leq\ell(I)}} + \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G}\\ \ell(J)>\ell(I)}} \right\} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\ &\equiv \Theta(f,g) + \Theta^*(f,g)\,.\end{aligned}$$

- By symmetry it suffices to estimate the first form $\Theta(f,g)$ (that includes the diagonal).

The Hytönen-Martikainen decomposition

- Before introducing goodness into the sum, we follow Hytönen and Martikainen and split the form $\Theta(f,g)$ into 3 pieces:

$$\Theta(f,g) \equiv \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G} \\ \ell(J)\leq\ell(I)}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega$$

$$= \sum_{I\in\mathcal{D}} \left\{ \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I) \\ d(J,I)>2\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}}} + \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq 2^{-\rho}\ell(I) \\ d(J,I)\leq 2\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}}} + \sum_{\substack{J\in\mathcal{G}:\ 2^{-\rho}\ell(I)<\ell(J)\leq\ell(I) \\ d(J,I)\leq 2\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}}} \right\}$$

$$\equiv \Theta_1(f,g) + \Theta_2(f,g) + \Theta_3(f,g) ,$$

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$$\Theta(f,g) \equiv \sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G} \\ \ell(J)\leq\ell(I)}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega$$

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$$\equiv \Theta_1(f,g) + \Theta_2(f,g) + \Theta_3(f,g) ,$$

- The disjoint form $\Theta_1(f,g)$ can be handled by 'long-range' and 'mid-range' arguments, and the nearby form $\Theta_3(f,g)$ will be handled using surgery methods and a new recursive argument involving energy conditions and the 'original' testing functions discarded in the corona construction.

Lemma (Monotonicity Lemma)

Suppose that I and J are intervals in $\mathbb{R}$ such that $J \subset \gamma J \subset I$ for some $\gamma > 1$, and that μ is a signed measure on $\mathbb{R}$ supported outside I. Let $\Psi \in L^2(\omega)$, that T^α is a standard fractional singular integral on $\mathbb{R}$ with $0 < \alpha < 1$, and that $\mathbf{b}^*$ is an ∞-weakly μ-controlled accretive family on $\mathbb{R}$. Then

$$\left| \left\langle T^\alpha \mu, \square_J^{\omega,\mathbf{b}^*} \Psi \right\rangle_\omega \right| \lesssim C_{\mathbf{b}^*} C_{CZ} \, \Phi^\alpha(J, |\mu|) \, \left\| \square_J^{\omega,\mathbf{b}^*} \Psi \right\|_{L^2(\omega)}^{\bigstar}, \tag{5}$$

where

$$\Phi^\alpha(J, |\mu|) \equiv \frac{\mathrm{P}^\alpha(J, |\mu|)}{|J|} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^\alpha(J, |\mu|)}{|J|} \left\| x - m_J \right\|_{L^2(\mathbf{1}_J \omega)}$$

$$\left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \equiv \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{2} + \inf_{z \in \mathbb{R}} \sum_{J' \in \mathfrak{C}_{\mathrm{broken}}(J)} \left| J' \right|_\omega \left(E_{J'}^\omega |x - z| \right)^2,$$

$$\left\| \square_J^{\omega,\mathbf{b}^*} \Psi \right\|_{L^2(\mu)}^{\bigstar 2} \equiv \left\| \square_J^{\omega,\mathbf{b}^*} \Psi \right\|_{L^2(\mu)}^{2} + \sum_{J' \in \mathfrak{C}_{\mathrm{broken}}(J)} \left| J' \right|_\omega \left[E_{J'}^\omega |\Psi| \right]^2.$$

Lemma (**Energy Lemma**)

Let J be an interval in $\mathcal{G}$. Let Ψ_J be an $L^2(\omega)$ function supported in J with vanishing ω-mean, and let $\mathcal{H} \subset \mathcal{G}$ be such that $J' \subset J$ for every $J' \in \mathcal{H}$. Let ν be a positive measure supported in $\mathbb{R} \setminus \gamma J$ with $\gamma > 1$, and for each $J' \in \mathcal{H}$, let $d\nu_{J'} = \varphi_{J'} d\nu$ with $|\varphi_{J'}| \leq 1$. Suppose that $\mathbf{b}^*$ is an ∞-weakly μ-controlled accretive family on $\mathbb{R}$. Let T^α be a standard α-fractional singular integral operator with $0 \leq \alpha < 1$. Then we have

$$\begin{aligned}
& \left| \sum_{J' \in \mathcal{H}} \left\langle T^\alpha (\nu_{J'}), \square_{J'}^{\omega,\mathbf{b}^*} \Psi_J \right\rangle_\omega \right| \\
\lesssim \quad & C_\gamma \sqrt{\sum_{J' \in \mathcal{H}} \Phi^\alpha (J', \nu)^2} \sqrt{\sum_{J' \in \mathcal{H}} \left\| \square_{J'}^{\omega,\mathbf{b}^*} \Psi_J \right\|_{L^2(\mu)}^{\bigstar 2}} \\
\leq \quad & C_\gamma \left(\frac{\mathrm{P}^\alpha (J, \nu)}{|J|} \left\| \mathrm{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^\alpha (J, \nu)}{|J|} \left\| x - m_J z \right\|_{L^2(\mathbf{1}_J \omega)} \right) \\
& \qquad \times \left\| \mathrm{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} \Psi_J \right\|_{L^2(\mu)}^{\bigstar} .
\end{aligned}$$

The goodness problem

- The traditional method of introducing goodness is flawed here in the setting of b-dual martingale differences, since these differences are no longer orthogonal projections, and as emphasized by Hytönen and Martikainen, we cannot simply add back in bad intervals whenever we want telescoping identities to hold.

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- In fact, in the analysis of the form $\Theta(f,g)$, it is necessary to have goodness for the intervals J and telescoping for the intervals I. On the other hand, in the analysis of the form $\Theta^*(f,g)$, it is necessary to have just the opposite - namely goodness for the intervals I and telescoping for the intervals J. This unfortunate set of circumstances prevents us from introducing goodness in the *full* sum over all I and J, prior to splitting according to side lengths of I and J.

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- However, one must work harder to introduce goodness directly into the form $\Theta(f, g)$ *after* we have restricted the sum to intervals J that have smaller side length than I. This is accomplished using the *weaker form of goodness* introduced by Hytönen and Martikainen.

Weak goodness

- For intervals $R \in \mathcal{G}$ and $Q \in \mathcal{D}$ let $\kappa(Q, R) = \log_2 \frac{\ell(Q)}{\ell(R)}$. For $R \in \mathcal{G}$, let $\kappa(R) \equiv \kappa\left(R^{\maltese}, R\right)$ denote the smallest integer k, if it exists, such that R is good with respect to all $Q \in \mathcal{D}$ with $\ell(Q) \geq 2^k \ell(R)$.

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- We define for $\kappa(R) < \infty$

$$R^{\maltese} = \pi_{\mathcal{D}}^{\kappa(R)} R,$$

where $\pi_{\mathcal{D}}^{k} R$ denotes the interval $Q \in \mathcal{D}$ that contains R and has side length $\ell(Q) = 2^k \ell(R)$, provided that such an interval Q exists (in particular such Q exists for $k \geq \kappa(R)$ if $\kappa(R) < \infty$).

Weak goodness decomposition

- We decompose

$$\begin{aligned}
\Theta_2(f,g) &= \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\rho}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\quad + \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\rho}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\equiv \Theta_2^{\mathrm{bad}}(f,g) + \Theta_2^{\mathrm{good}}(f,g) \ .
\end{aligned}$$

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$$\begin{aligned}\Theta_2(f,g) &= \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\rho}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\ &\quad + \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\rho}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\ &\equiv \Theta_2^{\mathrm{bad}}(f,g) + \Theta_2^{\mathrm{good}}(f,g) .\end{aligned}$$

- The bad form $\Theta_2^{\mathrm{bad}}(f,g)$ satisfies

$$\mathbf{E}_\Omega^{\mathcal{D}} \mathbf{E}_\Omega^{\mathcal{G}} \Theta_2^{\mathrm{bad}}(f,g) \leq C_{\mathrm{good}} 2^{-(\varepsilon-\varepsilon')\mathbf{r}} \mathfrak{N}_{T^\alpha} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} ,$$

by the arguments in [HyMa], and so can be absorbed.

The nearby form I

- We prove the following lemma that controls the expectation, over two independent grids, of the nearby form $\Theta_3(f,g)$.

Lemma

Suppose T^α *is a standard fractional singular integral with* $0 \leq \alpha < 1$. *Let* $0 < \delta < 1$. *For* $f \in L^2(\sigma)$ *and* $g \in L^2(\omega)$ *we have*

$$\mathbf{E}_\Omega^{\mathcal{D}} \mathbf{E}_\Omega^{\mathcal{G}} \sum_{I \in \mathcal{D}} \sum_{\substack{J \in \mathcal{G}:\ 2^{-\rho}\ell(I) < \ell(J) \leq \ell(I) \\ d(J,I) \leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \left| \left\langle T_\sigma^\alpha \left(\square_I^{\sigma,\mathbf{b}} f \right), \square_J^{\omega,\mathbf{b}^*} g \right\rangle_\omega \right| \tag{6}$$

$$\lesssim \left(\mathfrak{T}_\alpha^{\mathbf{b}} + \mathfrak{T}_\alpha^{\mathbf{b}^*,*} + \sqrt{"\mathcal{A}_2^{\alpha}"} + \frac{1}{\sqrt{\delta}} "\mathcal{E}_2^{\alpha}" + \sqrt{\delta} \mathfrak{N}_{T^\alpha} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} .$$

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- Since Poisson integrals are no longer bounded, a new idea is needed.
- We use the *original* testing functions b_I^{orig} for I, discarded when constructing the corona C_A, as well as $b_I = \mathbf{1}_I b_A$.

The nearby form II

The original testing function trick

- For subsets $E, F \subset A \cap B$ and intervals $K \subset A \cap B$ we define

$$\begin{aligned}
\{E, F\} &\equiv \left\langle T_{\sigma}^{\alpha}\left(b_A \mathbf{1}_E\right), b_B^* \mathbf{1}_F \right\rangle_{\omega} , \\
K_{\mathrm{in}} &\equiv K \setminus \partial_{\delta} K \text{ and } K_{\mathrm{out}} \equiv K \cap \partial_{\delta} K , \\
\{K, K\} &= \{A, K_{\mathrm{in}}\} - \{A \setminus K, K_{\mathrm{in}}\} + \{K_{\mathrm{out}}, K_{\mathrm{out}}\} + \{K_{\mathrm{in}}, K_{\mathrm{out}}\} .
\end{aligned}$$

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\{K,K\} &= \{A,K_{\mathrm{in}}\}-\{A\setminus K,K_{\mathrm{in}}\}+\{K_{\mathrm{out}},K_{\mathrm{out}}\}+\{K_{\mathrm{in}},K_{\mathrm{out}}\}.
\end{aligned}
$$

- The first two terms on the right side satisfy

$$
\begin{aligned}
\left|\{A,K_{\mathrm{in}}\}\right| &= \left|\int_{K_{\mathrm{in}}}\left(T_\sigma^\alpha b_A\right)b_B^*d\omega\right|\leq\left\|\mathbf{1}_{K_{\mathrm{in}}}T_\sigma^\alpha b_A\right\|_{L^2(\omega)}\left\|\mathbf{1}_{K_{\mathrm{in}}}b_B^*\right\|_{L^2}\\
\left|\{A\setminus K,K_{\mathrm{in}}\}\right| &\lesssim \frac{\mathrm{P}^\alpha\left(K_{\mathrm{in}},\left|b_A\right|\mathbf{1}_{A\setminus K}\right)}{\left|K_{\mathrm{in}}\right|}\left\|\mathrm{P}_{K_{\mathrm{in}}}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\bigstar}\sqrt{\int_{K_{\mathrm{in}}}\left|b_B^*\right|^2d\omega},
\end{aligned}
$$

upon using the trick with the **original** testing function $b_K^{*,\mathrm{orig}}$.

- For K an interval, we write $K_{\mathrm{out}} = K_{\mathrm{left}} \cup K_{\mathrm{right}}$ where K_{left} and K_{right} are the two small subintervals on the left and right hand sides of K respectively, and then we have

$$\{K_{\mathrm{out}}, K_{\mathrm{out}}\} = \{K_{\mathrm{left}}, K_{\mathrm{left}}\} + \{K_{\mathrm{right}}, K_{\mathrm{right}}\} + \{K_{\mathrm{left}}, K_{\mathrm{right}}\} + \{K_{\mathrm{right}},$$

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- We define a collection of intervals $\mathcal{M} = \mathcal{M}(K)$ by recursion,

$$\begin{aligned} \mathcal{M}_0 &\equiv \{K\}, \\ \mathcal{M}_{k+1} &\equiv \bigcup \{M_{\mathrm{left}}, M_{\mathrm{right}} : M \in \mathcal{M}_k\}, \qquad k \geq 0, \\ \mathcal{M} &= \mathcal{M}(K) \equiv \bigcup_{k=0}^{\infty} \mathcal{M}_k , \end{aligned}$$

so that

$$\mathcal{M} = \left\{K, K_{\mathrm{left}}, K_{\mathrm{right}}, \left(K_{\mathrm{left}}\right)_{\mathrm{left}}, \left(K_{\mathrm{left}}\right)_{\mathrm{right}}, \left(K_{\mathrm{right}}\right)_{\mathrm{left}}, \left(K_{\mathrm{right}}\right)_{\mathrm{right}}, \dots\right.$$

$$\left| \{K,K\} - \sum_{M\in\mathcal{M}(K)} \left[\{M_{\mathrm{in}}, M_{\mathrm{out}}\} - \{M_{\mathrm{out}}, M_{\mathrm{in}}\}^{\mathrm{orig}} \right] \right.$$

$$\left. - \sum_{M\in\mathcal{M}_n} \left[\{M_{\mathrm{left}}, M_{\mathrm{right}}\} + \{M_{\mathrm{right}}, M_{\mathrm{left}}\} \right] \right|$$

$$\leq \sqrt{\sum_{M\in\mathcal{M}(K)} \left\| \mathbf{1}_{M_{\mathrm{in}}} T_\sigma^\alpha b_A \right\|_{L^2(\omega)}^2} \sqrt{\sum_{M\in\mathcal{M}(K)} \left\| \mathbf{1}_{M_{\mathrm{in}}} b_B^* \right\|_{L^2(\omega)}^2}$$

$$+ \sqrt{\sum_{M\in\mathcal{M}} \left[\frac{\mathrm{P}^\alpha \left(M_{\mathrm{in}}, \left| b_A \right| \sigma \right)}{\left| M_{\mathrm{in}} \right|} \right]^2 \left\| \mathrm{P}_{M_{\mathrm{in}}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\bigstar 2}} \sqrt{\sum_{M\in\mathcal{M}} \int_{M_{\mathrm{in}}} \left| b_B^* \right|^2 d\omega}$$

$$\lesssim \left(\mathfrak{T}_{T^\alpha} + \mathcal{E}_2^\alpha + \delta^{\alpha-1} \sqrt{{}''\mathcal{A}_2^{\alpha}{}''} \right) \sqrt{\left| I' \right|_\sigma \left| J' \right|_\omega}.$$

Control of functional energy

- This argument remains essentially the same with only two changes:

1. Weak goodness is used in place of usual goodness via consideration of pairs $(I, J) \in \mathcal{D} \times \mathcal{G}$ with $J^{\maltese} \subsetneqq I$. Here $J^{\maltese}$ is the smallest interval K in $\mathcal{G}$ such that J is good in K and beyond.
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2. Broken martingale differences of testing functions are used in place of the usual Haar differences.

- However, in the proof that functional energy is controlled by the Muckenhoupt and energy conditions, it can now happen that an interval $J \in \mathcal{G}$ can 'cut across' an interval $I \in \mathcal{D}$, resulting in additional terms to be treated.

Control of the stopping form

- As in Lacey we construct $\mathcal{L}$-coronas from the 'bottom up' with stopping times involving the energies $\left\|\square_J^{\omega,\mathbf{b}^*}\right\|_{L^2(\omega)}^2$, but then overlay this with an additional top/down 'indented' corona construction $\mathcal{H}$ in order to accommodate the weaker goodness of Hytönen and Martikainen.

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- by absorbing the case when both I and $J^{\maltese}$ belong to the same $\mathcal{L}$-corona, and
- by using the Straddling and Substraddling Lemmas and the Orthogonality Lemma to control the case when I and $J^{\maltese}$ lie in different coronas, with a geometric gain coming from the separation of the indented $\mathcal{H}$-coronas.

- For an A-admissible collection $\mathcal{P}$ of pairs, define an atomic measure $\omega_{\mathcal{P}}$ in the upper half space $\mathbb{R}^2_+$ by

$$\omega_{\mathcal{P}} \equiv \sum_{J \in \Pi_2 \mathcal{P}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \delta_{\left(c_{J^{\maltese}}, \ell\left(J^{\maltese}\right)\right)}.$$

Lacey's bottom/up corona I

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- Define the tent $\mathbf{T}(K)$ over an interval $K = L$ to be $\mathbf{T}(L)$ where $\mathbf{T}(L)$ is the convex hull of the interval $L \times \{0\}$ and the point $(c_L, \ell(L)) \in \mathbb{R}^2_+$, and the size functional of $\mathcal{P}$ by

$$\mathcal{S}_{\text{size}}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi_1^{\text{below}} \mathcal{P}} \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K}\sigma\right)}{|K|} \right)^2 \omega_{\mathcal{P}}(\mathbf{T}(K)).$$

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$$\mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi_1^{\mathrm{below}} \mathcal{P}} \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K} \sigma\right)}{|K|} \right)^2 \omega_{\mathcal{P}}(\mathbf{T}(K)).$$

- The generation $\mathcal{L}_0$ consists of the *minimal* dyadic intervals K in $\Pi_1^{\mathrm{below}} \mathcal{P}$ such that

$$\frac{\Psi^\alpha(K;\mathcal{P})^2}{|K|_\sigma} \equiv \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K} \sigma\right)}{|K|} \right)^2 \omega_{\mathcal{P}}(\mathbf{T}(K)) \geq \varepsilon \mathcal{S}_{\mathrm{size}}^{\alpha,A}(\mathcal{P})^2.$$

Lacey's bottom/up corona II

- Choose $\rho = 1 + \varepsilon$ and define a sequence of generations $\{\mathcal{L}_m\}_{m=0}^{\infty}$ and coronas by letting $\mathcal{L}_m$ consist of the *minimal* dyadic intervals L in $\Pi_1^{\text{below}}\mathcal{P}$ that contain an interval from some previous level $\mathcal{L}_\ell$, $\ell < m$, such that

$$\omega_{\mathcal{P}}\left(\mathbf{T}\left(L\right)\right) \geq \rho\omega_{\mathcal{P}}\left(\bigcup \mathbf{T}\left(L'\right) : L' \in \bigcup_{\ell=0}^{m-1} \mathcal{L}_\ell \text{ and } L' \subset L\right).$$

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- For $L \in \mathcal{L}$, denote by $\mathcal{C}_L$ the *corona* associated with L in the tree $\mathcal{L}$,

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- The parameter m in $\mathcal{L}_m$ refers to the level at which the stopping construction was performed, but for $L \in \mathcal{L}_m$, the corona children L' of L are *not* all necessarily in $\mathcal{L}_{m-1}$, but may be in $\mathcal{L}_{m-t}$ for t large.

The indented corona

- To address the lack of goodness in $\Pi_1^{\mathrm{below}}\mathcal{P}$ we introduce an additional top/down stopping time over the collection $\mathcal{L}$. Given the initial generation

$$\mathcal{H}_0 \equiv \mathcal{L}_{M+1} = \{\text{maximal } L \in \mathcal{L}\} = \left\{\text{maximal } I \in \Pi_1^{\mathrm{below}}\mathcal{P}\right\},$$

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- We refer to the stopping intervals $L \in \mathcal{H}$ as *indented* stopping intervals since $3L \subset \pi_{\mathcal{H}} L$ for all L at indented generation one or more, i.e. each successive such L is 'indented' in its $\mathcal{H}$-parent. This property of indentation is precisely what is required in order to generate geometric decay from the straddling lemma in indented generations.

Endpoint towers

- For $L \in \mathcal{H}_k$ and $t \geq 0$ apply the straddling lemma to

$$\mathcal{P}_{L,t}^{\mathcal{H}} \equiv \left\{ (I,J) \in \mathcal{P} : I \in \mathcal{C}_L^{\mathcal{H}},\ J \in \mathcal{C}_{L'}^{\mathcal{H},\mathrm{shift}} \text{ for some } L' \in \mathcal{H}_{k+t},\ L' \subset L \right\}$$

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- Within the $\mathcal{H}$-corona $\mathcal{C}_L^{\mathcal{H}}$ there are further intervals $T \in \mathcal{L} \setminus \mathcal{H}$, but these are contained in the two endpoint towers

$$\begin{aligned} \mathcal{T}_{\text{left}}(L) &\equiv \left\{ L' \in \mathcal{L} : L' \subset L \text{ and } \text{left}(L') = \text{left}(L) \right\}, \\ \mathcal{T}_{\text{right}}(L) &\equiv \left\{ L' \in \mathcal{L} : L' \subset L \text{ and } \text{right}(L') = \text{right}(L) \right\}, \end{aligned}$$

where $\text{left}(I)$ and $\text{right}(I)$ denote the left and right hand endpoints of I respectively. Let $\mathcal{T}_{\text{left}}(L) = \left\{ L^k \right\}_{k=0}^{\infty}$. We ignore $\mathcal{T}_{\text{right}}(L)$ as it can be handled similarly,

Left/right decomposition

- For $L \in \mathcal{H}$ and $t = 0$ we decompose

$$\begin{aligned} \mathcal{P}_{L,0}^{\mathcal{H}} &\equiv \mathcal{P}_{L,0}^{\mathcal{H}-small} \dot{\cup} \mathcal{P}_{L,0}^{\mathcal{H}-big}; \\ \mathcal{P}_{L,0}^{\mathcal{H}-big} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\mathcal{H}} : \exists L' \in \mathcal{T}(L),\ J^{\maltese} \subset L' \subset I \right\}. \end{aligned}$$

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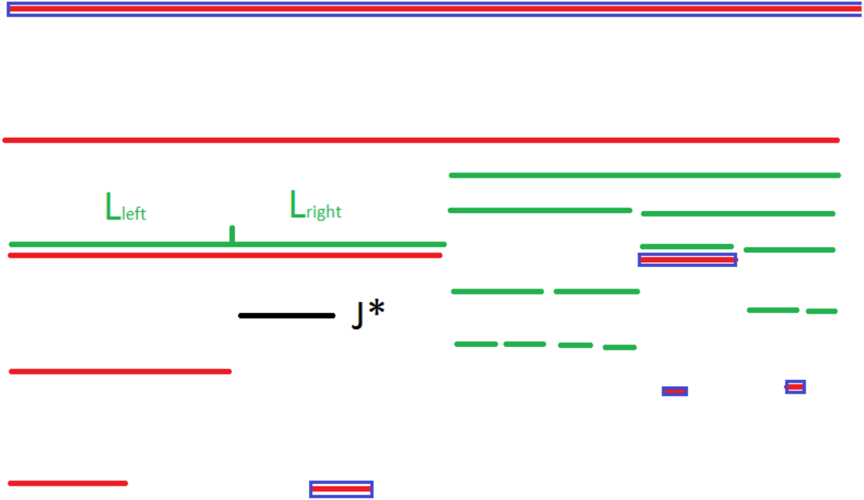
- Then we further decompose

$$\begin{aligned}
\mathcal{P}_{L,0}^{\mathcal{H}-big} &= \dot{\bigcup}_{k=1}^{\infty} \left\{ \mathcal{R}_{L_{\text{left}}^{k}}^{\mathcal{L}} \dot{\cup} \mathcal{R}_{L_{\text{right}}^{k}}^{\mathcal{L}} \right\} = \left(\dot{\bigcup}_{k=1}^{\infty} \mathcal{R}_{L_{\text{left}}^{k}}^{\mathcal{L}} \right) \dot{\cup} \left(\dot{\bigcup}_{k=1}^{\infty} \mathcal{R}_{L_{\text{right}}^{k}}^{\mathcal{L}} \right.\\
\mathcal{R}_{L_{\text{right}}^{k}}^{\mathcal{L}} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\mathcal{H}-big} : I \in \mathcal{C}_{L^{k-1}}^{\mathcal{L},\text{restrict}} \text{ and } J^{\maltese} \subset L_{\text{right}}^{k} \right\} ,\\
\mathcal{R}_{L_{\text{left}}^{k}}^{\mathcal{L}} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\mathcal{H}-big} : I \in \mathcal{C}_{L^{k-1}}^{\mathcal{L},\text{restrict}} \text{ and } \right.\\
&\qquad \left. J \subset L_{\text{left}}^{k} \text{ or "} J^{\maltese} = L^{k} \text{ and } J \subset L_{\text{right}}^{k} \text{"} \right\} .
\end{aligned}$$

Corona diagram

Lacey's bottom/up stopping times in red segments, and the indented stopping times in blue rectangles around red segments



Straddling on right and Substraddling on left

- Now apply the Straddling Lemma to the 'right' admissible collection $\mathcal{Q} \equiv \bigcup_{k=1}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\text{right}}}$ with $\mathcal{S} \equiv \left\{ L^k_{\text{right}} \right\}_{k=1}^{\infty}$ to obtain the estimate

$$\mathfrak{N}_{\text{stop},\square^{\omega}}^{A,\bigcup_{k=1}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\text{right}}}} \leq C\mathcal{S}^{\alpha,A}_{\text{size}}\left(\mathcal{P}^{\mathcal{H}-big}_{L,0}\right) .$$

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- As for the remaining 'left' form $\left| \mathrm{B} \right|_{\mathrm{stop},\square^{\omega}}^{A,\bigcup_{k=0}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\mathrm{left}}}} (f, g)$, we note that if the interval pair $(I, J) \in \mathcal{R}^{\mathcal{L}}_{L^k_{\mathrm{left}}}$, then there is a unique interval $K \in \mathcal{W}(L)$ that contains J, and moreover we have the crucial inclusion $3K \subset I$ (because $J^{\maltese} \subsetneqq I$).

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- Thus the admissible collection $\mathcal{Q} \equiv \bigcup_{k=0}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\mathrm{left}}}$ *substraddles* the interval L, and the Substraddling Lemma yields the bound

$$\mathfrak{N}_{\mathrm{stop},\square^{\omega}}^{A,\bigcup_{k=0}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\mathrm{left}}}} \leq C\mathcal{S}^{\alpha,A}_{\mathrm{size}} \left(\bigcup_{k=0}^{\infty} \mathcal{R}^{\mathcal{L}}_{L^k_{\mathrm{left}}} \right) \leq C\mathcal{S}^{\alpha,A}_{\mathrm{size}} \left(\mathcal{P}^{\mathcal{H}-big}_{L,0} \right) .$$

Open problems

- Does the two weight Tb theorem remain true in the case $p = 2$, i.e. when $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}}$ is a 2-weakly σ-accretive family of functions, and $\mathbf{b}^* = \{b_Q^*\}_{Q \in \mathcal{P}}$ is a 2-weakly ω-accretive family of functions? (True when $p = 2$ for one weight by Lacey-Martikainen, suspect false for two weights when $p = 1$.)

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- Thanks to the organizers Chema, Svitlana and Simon!