Boundary value problems for divergence form complex p-elliptic operators

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Workshop on Real Harmonic Analysis and its Applications to Partial Differential Equations and Geometric Measure Theory: on the occasion of the 60th birthday of Steve Hofmann

Overview

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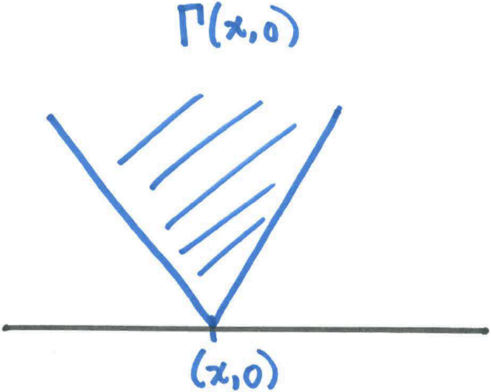
Dirichlet, Regularity, and a perturbation theory: in the context of a regularity theory for these operators, called *p-ellipticity.*

Nontangential convergence

Let $\Gamma(x,0) = \{(x',y) : |x - x'| < cy\}$ denote the cone at $(x,0)$, and $Nu(x) = \sup\{(x',t) \in \Gamma(x,0) : u(x',t)\}$:



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$$\lambda|\xi|^2 \leq \langle A(X)\xi,\xi\rangle := \sum_{i,j=1}^{n+1} A_{ij}(x)\xi_j\xi_i, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq \lambda^{-1}, \quad (1)$$

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Since the coefficients of A are not differentiable, as usual we interpret the solvability of $Lu = 0$ in the weak sense:

$$\int_{\mathbb{R}^n_+} A(X)\nabla u.\nabla\phi dX = 0$$

for all appropriate test functions ϕ and for all $u \in W^{1,2}$.

Motivation for studying regularity of solutions, and sharp boundary value problems

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- Geometry of boundary of a domain connected to properties of the harmonic/elliptic measure (Kenig-Toro, Milakis-Pipher-Toro, G. David, S. Hofmann, I., M. Mitrea, ...)

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- Study of elliptic systems, higher order elliptic and parabolic equations, and complex coefficient equations, theory is not as well developed. (Lack of: Positivity, maximum principles, and the existence of a boundary measure)

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- Study of elliptic systems, higher order elliptic and parabolic equations, and complex coefficient equations, theory is not as well developed. (Lack of: Positivity, maximum principles, and the existence of a boundary measure)
- When matrix A has complex coefficients: some milestones, and some partial progress: Kato square root problem is a Regularity/Neumann boundary value problem (Auscher - Hofmann - Lacey - McIntosh - Tchamitchian); perturbations of operators .

Properties of solutions to real and complex coefficient operators

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- Maximum principle, Interior Hölder continuity, Harnack property: thus one can study the Dirichlet problem via a mutually absolutely continuous family of representing doubling measures associated with L.
- The solvability of the Dirichlet problem with data in L^p is equivalent to a real-variable property of the measure, which in turn depends on the smoothness of the coefficients and the geometry of the domain.
- None of this applies in the bounded, measurable complex-coefficient setting.

Complex valued elliptic operators

Program: The study of solutions to operators of the form

$$L := -\operatorname{div} A(x,t)\nabla, \quad (x,t) \in \mathbb{R}^n_+$$

where A may be complex valued, and the natural boundary value problems associated with them.

Some results in the complex setting takes place under the assumption that solutions to L satisfy DeG-N-M bounds. Other work focuses on structural assumptions on these operators. In [1612.01568, 1805.08614, and a forthcoming preprint] we take the latter approach to develop a theory of regularity of solutions to complex coefficient operators and use this to solve certain boundary value problems.

The Kato square root problem

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The Kato square root problem (as re-formulated by McIntosh) asks about the domain of $\sqrt{L}$, namely whether one has the estimate $\|\sqrt{L}(f)\|_{L^2} \lesssim \|\nabla_x f)\|_{L^2}$.

Structural assumptions

The estimate on $\sqrt{L}$ is equivalent to solving an L^2 Regularity problem for the operator, $\tilde{L}$ below, or a Neumann problem for $\tilde{L}^*$, where the matrix for $\tilde{L}$ in dimension $n+1$ is

$$\tilde{A} = \left[\begin{array}{c|c} A & \vec{0} \\ \hline \vec{0} & 1 \end{array}\right]$$

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For $\tilde{L}$ as above in block form, the family of operators $\{e^{-t\sqrt{L}}\}$ is the Poisson semigroup: solutions to $\tilde{L}u = 0$ in $\mathbb{R}^{n+1}_+$ with data $f(x) \in \mathbb{R}^n$ are given by $\{e^{-t\sqrt{L}}f(x)\}$, and are uniformly bounded in L^2 for all t by the L^2 of the norm of the data. (The Dirichlet problem is solvable in a larger range of p [Mayboroda, 2010].

Structural assumptions and p-ellipticity

In a series of papers, Cialdea and Maz'ya define a notion they term L^p-dissipativity, motivated by understanding when the semigroups $\exp(-tL_A),\ \ t > 0$, generated by second order elliptic operators are contractive in L^p. (Always true for real second order elliptic operators.)

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Our condition, termed *p-ellipticity* in a very recent paper [Carbonaro, Dragičević], is a slight strengthening of L^p-dissipativity.

The matrix A is *p-elliptic* if

$$|1 - 2/p| < \mu(A)$$

where

$$\mu(A) = \operatorname*{ess\,inf}_{(x,\xi)\in\Omega\times\mathbb{C}^n\setminus\{0\}} Re \frac{\langle A(x), \xi, \xi\rangle}{|\langle A(x), \xi, \overline{\xi}\rangle|}.$$

For $p > 1$ define the $\mathbb{R}$-linear map $\mathcal{J}_p : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\mathcal{J}_p(\alpha + i\beta) = \frac{\alpha}{p} + i\frac{\beta}{p'}$$

where $p' = p/(p-1)$ and $\alpha, \beta \in \mathbb{R}^n$. [CD] shows that the matrix A is p-elliptic iff for a.e. $x \in \Omega$

$$\mathcal{R}e \, \langle A(x)\xi, \mathcal{J}_p\xi \rangle \geq \lambda_p |\xi|^2, \qquad \forall \xi \in \mathbb{C}^n \tag{2}$$

for some $\lambda_p > 0$.

The advantage of the expression in terms of $\mu(A)$ is the separation of A from p. It shows that if A is elliptic then there exists $p_0 \in [1, 2)$ such that A is p-elliptic if and only if $p \in (p_0, p_0')$, where $p_0 = 2/(1 + \mu(A))$.

Moreover, $p_0 = 1$ if and only if the matrix A is real.

Carbonaro-Dragicevic

Let $P_t^A = \exp(-tL_A)$, $t > 0$,. Then, [CD] show:

Theorem

Let $p > 1$. Suppose $A, B \in \mathcal{A}_{\lambda,\Lambda}(\mathbb{R}^n)$ are both p-elliptic. Then for all $f, g \in C_c^\infty(\mathbb{R}^n)$ we have

$$\int_0^\infty \int_{\mathrm{Re}^n} |\nabla P_t^A f(x)|\,|\nabla P_t^B g(x)|\; dx\, dt \lesssim \|f\|_p \|g\|_q\,, \tag{3}$$

with constants depending on ellipticity parameters, but not dimension.

When $A = B$, such bounds follow from square function estimates (Auscher, Hofmann, Martell for real coeffs, and complex coefficients in certain range of p).

Carbonaro-Dragicevic

(Auscher): if $|1/2 - 1/p| \leq 1/n$ then $(e^{-tL_A})_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$. Hofmann, Mayboroda and McIntosh proved that this condition is sharp in terms of n, in the sense that if $|1/2 - 1/p| > 1/n$ then $(e^{-tL_A})_{t>0}$ is not bounded on $L^p(\mathbb{R}^n)$ for all elliptic A.

- (Auscher) if $|1 - 2/p| \leq 2/n$ then $(P_t^A)_{t>0}$ is bounded on $L^p(\mathbb{R}^n)$;
- Theorem [CD]: if $|1 - 2/p| \leq \mu(A)$ then $(P_t^A)_{t>0}$ is contractive on $L^p(\mathbb{R}^n)$.

Regularity of solutions

In [DP], we use p-ellipticity to establish the following.

Theorem

Assume that the matrix A is p-elliptic. Then there exists $\lambda'_p = \lambda'_p(\lambda, \Lambda, \lambda_p) > 0$ such that for any nonnegative, bounded and measurable function χ and any u such that $|u|^{(p-2)/2} u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$, we have

$$\mathcal{R}e \int_\Omega \langle A(x)\nabla u, \nabla(|u|^{p-2}u)\rangle \chi(x)\, dx \geq \lambda'_p \int_\Omega |u|^{p-2}|\nabla u|^2 \chi(x)\, dx. \tag{4}$$

We also observe:
For all $p > 1$, and for all x for which $u(x) \neq 0$

$$|\nabla(|u(x)|^{p/2-1}u(x))|^2 \approx |u(x)|^{p-2}|\nabla u(x)|^2.$$

Regularity result

Suppose that $u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$ is the weak solution to the operator $\mathcal{L}u := \mathrm{div} A(x)\nabla u + B(x)\cdot\nabla u = 0$ in Ω. Let $p_0 = \inf\{p > 1 : A \text{ is } p\text{-elliptic}\}$, and suppose that B has measurable coefficients $B_i \in L^\infty_{loc}(\Omega)$ satisfying the condition

$$|B_i(x)| \le K(\delta(x))^{-1}, \quad \forall x \in \Omega \tag{5}$$

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where K is uniform, and $\delta(x)$ is distance of x to $\partial\Omega$.
Then we have the following improvement in the regularity of u.
For any $B_{4r}(x) \subset \Omega$ and $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\left(\int_{B_r(x)} |u|^p\, dy\right)^{1/p} \leq C_\varepsilon \left(\int_{B_{2r}(x)} |u|^q\, dy\right)^{1/q} + \varepsilon \left(\int_{B_{2r}(x)} |u|^2\, dy\right)^{1/2} \tag{6}$$

for all $p, q \in (p_0, \frac{p_0' n}{n-2})$. (Here $p_0' = p_0/(p_0 - 1)$ and when $n = 2$ one can take $p, q \in (p_0, \infty)$.)

Regularity, continued

The constant in the estimate depends on the dimension, the p-ellipticity constants, Λ, K and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or u.

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The constant in the estimate depends on the dimension, the p-ellipticity constants, Λ, K and $\varepsilon > 0$ but not on $x \in \Omega$, $r > 0$ or u. Moreover, for all $p \in (p_0, p_0')$ and any $\varepsilon > 0$

$$r^2 \int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \leq C_\varepsilon \iint_{B_{2r}(x)} |u(y)|^p dy + \varepsilon \left(\int_{B_{2r}(x)} |u(y)|^2 dy \right)^{p/2}$$

where the constants depend only on the dimension, p, Λ, K and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega; \mathbb{C})$.

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where the constants depend only on the dimension, p, Λ, K and $\varepsilon > 0$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega; \mathbb{C})$. The range in the reverse Hölder is sharp: Mayboroda gives a counterexample when $q = 2$ for any $p > \frac{2n}{n-2}$ under the assumption of 2-ellipticity.

Background

A method for proving A_∞ for non-symmetric real matrices appeared in [KKPT], based on equivalence of N and S in norm.

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A method for proving A_∞ for non-symmetric real matrices appeared in [KKPT], based on equivalence of N and S in norm. Applied (in [KP]) to: $\mathcal{L}u = \partial_i (A_{ij}(x)\partial_j u)$ with A real and $d\mu = \delta(X)|\nabla A_{i,j}(X)|^2 dX$ a Carleson measure. This condition on coefficients arises in a change of variables; answered a question of Dahlberg. In [DPP]: small Carleson condition implies solvability of D_p in $1 < p < \infty$. In [DR], the Neumann and Regularity problems solved in $1 < p < \infty$ in two dimensions; in [DPR] we have solvability in $1 < p < \infty$ in all dimensions.

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Advancing the understanding of ϵ-approximability, A_∞, Carleson measure conditions and connections to the geometry of domains, uniform rectifiability, NTA domains has been well studied by Steve and friends (Akman, Auscher, Azzam, Barton, Bortz, Cavero, Le, Kenig, Mitrea(s), Martell, Mayboroda, Morris, Mourgoglu, Nystrom, Tapiola, Tolsa, Toro, Uriarte-Tuero, Volberg, Zhao))!

The Dirichlet problem: the Carleson measure condition

Let $1 < p < \infty$, let $\Omega := \mathbb{R}^n_+ = \{(x_0, x') : x_0 > 0 \text{ and } x' \in \mathbb{R}^{n-1}\}$. Consider the operator

$$\mathcal{L}u = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$$

and assume that the matrix A is p-elliptic with constants λ_p, Λ and $\mathcal{I}m\, A_{0j} = 0$ for all $1 \leq j \leq n-1$ and $A_{00} = 1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} \left[|\nabla A(x)|^2 + |B(x)|^2\right] \delta(x)\, dx \tag{7}$$

is a Carleson measure in Ω. Let us also denote

$$d\mu'(x) = \sup_{B_{\delta(x)/2}(x)} \left[\sum_j |\partial_0 A_{0j}|^2 + \left|\sum_j \partial_j A_{0j}\right|^2 + |B(x)|^2\right] \delta(x)\, dx. \tag{8}$$

Then there exist $C, K = C, K(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ such that if $\|\mu'\|_{\mathcal{C}} < K$, then the L^p-Dirichlet problem is solvable for $\mathcal{L}$.

By solvability of the L^p-Dirichlet problem, we mean

$$\|\tilde{N}_{p,a}u\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega;\mathbb{C})}$$

where

$$\tilde{N}_{p,a}(u)(Q) := \sup_{x\in\Gamma_a(Q)} w(x)$$

with

$$w(x) := \left(\int_{B_{\delta(x)/2}(x)} |u(z)|^p \, dz\right)^{1/p}.$$

The complex coefficient equation $\mathcal{L}u = \partial_i \left(A_{ij}(x)\partial_j u\right) + B_i(x)\partial_i u$ can be formulated as a skey-symmetric system of equations for the real and imaginary parts of the solution u.

For this reason, the $p = 2$ case of the solvability of the Dirichlet problem is a consequence of the the L^2 solvability of the analogous Dirichlet problem for (non-symmetric) elliptic systems in Dindos-Hwang-M.Mitrea (https://arxiv.org/abs/1708.02289)

Corollary

Suppose the operator $\mathcal{L}$ on $\mathbb{R}^n_+$ has the form

$$\mathcal{L}u = \partial_0^2 u + \sum_{i,j=1}^{n-1} \partial_i (A_{ij} \partial_j u)$$

where the matrix A has coefficients satisfying the Carleson condition.
Then for all $1 < p < \infty$ for which A is p-elliptic, the L^p-Dirichlet problem is solvable for $\mathcal{L}$ and the estimate

$$\|\tilde{N}_{p,a} u\|_{L^p(\partial\Omega)} \leq C \|f\|_{L^p(\partial\Omega;\mathbb{C})} \qquad (9)$$

holds for all energy solutions u with datum f.

Perturbation Theory

We also show that Dahlberg's perturbation theory applies to the class of complex coefficient elliptic operators $\mathcal{L}_0$ satisfying the assumptions of the previous theorem, under only the algebraic assumption of p-ellipticity.

The perturbation criteria is the same as that for real coefficient operators, although the "smallness" of the Carleson measure will also be a function of the p in p-ellipticity.

Let $\mathcal{L}_0$ satisfy Carleson measure conditions. Let $\mathcal{L}_1 = \mathrm{div} A^1(x)\nabla + B^1(x)\cdot\nabla$ be a perturbation of $\mathcal{L}_0$ in the following sense:

$$dm(x) = \sup_{B_{\delta(x)/2}(x)} \left[|A^0 - A^1|^2 \delta^{-1}(x) + |B^0 - B^1|^2 \delta(x) \right] dx \quad (10)$$

is a Carleson measure in Ω.

Perturbation Theory

Theorem

Then there exist $K = K(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ *and* $C(\lambda_p, \Lambda, \|\mu\|_{\mathcal{C}}, n, p) > 0$ *such that if*

$$\|m\|_{\mathcal{C}} < K \tag{11}$$

then the matrix A^1 *is p-elliptic and the* L^p*-Dirichlet problem is also solvable for the operator* $\mathcal{L}_1$*.*

As a corollary: the Carleson measure condition on the gradients of coefficients can be replaced by an oscillation condition:

$$d\mu = \left(\delta(x)^{-1} \left(\operatorname{osc}_{B_{\delta(x)/2}(x)} A \right)^2 + \sup_{B_{\delta(x)/2}(x)} |B|^2 \delta(x) \right) dx \tag{12}$$

Further results

Now consider the Regularity problem for $\mathcal{L}u = \partial_i\left(A_{ij}(x)\partial_j u\right)$,

$$\begin{cases} \mathcal{L}u = 0 & \text{in } \Omega, \\ u = f & \text{for } \sigma\text{-a.e. } x \in \partial\Omega, \\ \tilde{N}_{p,a}(\nabla u) \in L^p(\partial\Omega), \end{cases} \tag{13}$$

with Dirichlet datum f belonging to the Sobolev space $W^{1,p}(\mathbb{R}^{n-1})$.

Further results

Theorem

Let $1 < p < \infty$, and let $\mathcal{L}u = \partial_i\left(A_{ij}(x)\partial_j u\right)$ with A is p-elliptic, $A_{00} = 1$ and $\mathcal{I}m\, A_{0j} = 0$ for all $1 \le j \le n-1$. Assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} |\nabla A|^2 \delta(x)\, dx \tag{14}$$

is a Carleson measure in Ω.
Then there exist $K = K(\lambda_p, \Lambda, n, p) > 0$ and $C(\lambda_p, \Lambda, n, p) > 0$ such that if $\|\mu\|_{\mathcal{C}} < K$ then the L^p Regularity problem is solvable and the estimate

$$\|\tilde{N}_{p,a}(\nabla u)\|_{L^p(\partial\Omega)} \le C\|\nabla_T f\|_{L^p(\partial\Omega;\mathbb{C})} \tag{15}$$

holds for all energy solutions u with datum f.

Some ingredients in the proofs

For $\Omega \subset \mathbb{R}^n$ as above, the square function of some $u \in W^{1,2}_{loc}(\Omega; \mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_a(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 \delta(x)^{2-n} \, dx \right)^{1/2} \tag{16}$$

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Definition

[Dindos-Petermichl-P.] For $\Omega \subset \mathbb{R}^n$, the p-adapted square function of $u \in W^{1,2}_{loc}(\Omega;\mathbb{C})$ at $Q \in \partial\Omega$ relative to the cone $\Gamma_a(Q)$ is defined by

$$S_{p,a}(u)(Q) := \left(\int_{\Gamma_a(Q)} |\nabla u(x)|^2 |u(x)|^{p-2} \delta(x)^{2-n}\, dx \right)^{1/2} \tag{17}$$

Regularity of solutions: the case $p > 2$

Lemma

Let $p \geq 2$ and A be p-elliptic; and assume B has coefficients satisfying $|B_i(x)| \leq K(\delta(x))^{-1}$ Suppose u is a $W^{1,2}_{loc}(\Omega;\mathbb{C})$ solution to $\mathcal{L}$ in Ω. Then if $r < \delta(x)/4$,

$$\int_{B_r(x)} |\nabla u(y)|^2 |u(y)|^{p-2} dy \lesssim r^{-2} \int_{B_{2r}(x))} |u(y)|^p dy \qquad (18)$$

$$\left(\iint_{B_r(x))} |u(y)|^q dy \right)^{1/q} \lesssim \left(\iint_{B_{2r}(x)} |u(y)|^2 dy \right)^{1/2} \qquad (19)$$

for all $q \in (2, \frac{np}{n-2}]$ when $n > 2$, and where the implied constants depend only p-ellipticity and K. When $n = 2$, q can be any number in $(2, \infty)$. In particular, $|u|^{(p-2)/2} u$ belongs to $W^{1,2}_{loc}(\Omega;\mathbb{C})$.

Sketch of proof

Let $v = u\varphi$ where φ is a cut-off function function associated to the ball $B_r(x)$, and compute

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla\varphi + A^*\nabla u \cdot \nabla\varphi.$$

Sketch of proof

Let $v = u\varphi$ where φ is a cut-off function function associated to the ball $B_r(x)$, and compute

$$\mathcal{L}v = u\mathcal{L}\varphi + A\nabla u \cdot \nabla\varphi + A^*\nabla u \cdot \nabla\varphi.$$

Multiply both sides of this equation by $|v|^{p-2}\overline{v}$ and integrate by parts to obtain:

$$
\begin{aligned}
\int \nabla(|v|^{p-2}\overline{v}) \cdot A\nabla v \, dy = & \int (|v|^{p-2}\overline{v}) B \cdot \nabla v \, dy \\
& + \int \nabla(|v|^{p-2}\overline{v} u) \cdot A \nabla \varphi \, dy \\
& - \int |v|^{p-2}\overline{v} u \, B \cdot \nabla \varphi dy \\
& - \int |v|^{p-2}\overline{v} A \nabla u \cdot \nabla \varphi \, dy \\
& - \int |v|^{p-2}\overline{v} A^* \nabla u \cdot \nabla \varphi \, dy
\end{aligned}
$$

By p-ellipticity, the real part of the left hand side is bounded from below by $\lambda_p \int |v|^{p-2} |\nabla v|^2 \, dy$.

Each term is treated separately. For example, the first of the five terms on the right hand side above has the bound

$$\left|\int (|v|^{p-2}\overline{v})\cdot B\nabla v\,dy\right| \lesssim Kr^{-1}\left(\int |v|^{p-2}|\nabla v|^2\,dy\right)^{1/2}\left(\int |v|^p\,dy\right)^{1/2}$$

which yields

$$\int_{B_r(x)} |\nabla u(y)|^2|u(y)|^{p-2}dy \lesssim r^{-2}\int_{B_{2r}(x))} |u(y)|^p dy$$

The Sobolev embedding gives

$$\left(\int_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{2r}(x)} |v|^{\tilde{p}}\, dy\right)^{1/\tilde{p}}$$

$$\lesssim \left(r^2 \int_{B_{2r}(x)} |\nabla(|v|^{p/2-1} v)|^2\, dy\right)^{1/p}$$

where $\tilde{p} = \frac{pn}{n-2}$.

This gives a reverse Hölder inequality for u. That is,

$$\left(\int_{B_r(x)} |u|^{\tilde{p}}\, dy\right)^{1/\tilde{p}} \lesssim \left(\int_{B_{\alpha r}(x)} |u|^p\, dy\right)^{1/p}$$

which can be iterated k times to give

$$\left(\int_{B_r(x)} |u|^{p_k}\, dy\right)^{1/p_k} \lesssim \left(\int_{B_{\alpha^k r}(x)} |u|^2\, dy\right)^{1/2}$$

for $p_k = 2(\frac{n}{n-2})^k$, as long as $p_{k-1} < p$.

The L^p Dirichlet problem

From now on, in addition to p-ellipticity, assume that

$$d\mu(x) = \sup_{B_{\delta(x)/2}(x)} [|\nabla A|^2 + |B|^2]\delta(x)\, dx$$

is a Carleson measure in Ω. Sometimes, and for certain coefficients of A, we will assume that their Carleson norm $\|\mu\|_{\mathcal{C}}$ is small.

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is a Carleson measure in Ω. Sometimes, and for certain coefficients of A, we will assume that their Carleson norm $\|\mu\|_{\mathcal{C}}$ is small. The Carleson measure conditions on the coefficients of $\mathcal{L}$, as well as p-ellipticity of A, are compatible with a useful change of variables that is a bijection from $\overline{\mathbb{R}^n_+}$ onto $\overline{\Omega}$.

Assumptions on the coefficients, explained

Some observations on the structural assumptions made for solvability of the Dirichlet problem. It suffices to formulate the result in the case $\Omega = \mathbb{R}^n_+$ by using the pull-back map alluded to above. Because the coefficients are required to have *small* Carleson norm this puts a restriction on the size of the Lipschitz constant of the map that defines the domain Ω.

For technical reasons we also required that all coefficients A_{0j}, $j = 0, 1, \dots, n-1$ are real. This can be ensured as follows. When $j > 0$:

$$\partial_0([\mathcal{I}m\, A_{0j}]\partial_i u) = \partial_j([\mathcal{I}m\, A_{0j}]\partial_0 u) + (\partial_0[\mathcal{I}m\, A_{0j}])\partial_i u - ([\partial_i \mathcal{I}m\, A_{0j}])\partial_0 u$$

which allows one to move the imaginary part of the coefficient A_{0j} onto the coefficient A_{j0} at the expense of two first order terms. However, this does not work for the coefficient A_{00}.

We will require that A_{00} is real, then a multiplication of A_{00} (and addition of a lower order term, to $\mathcal{L} = \partial_i (A_{ij}(x)\partial_j) + B_i(x)\partial_i)$ by $\alpha = A_{00}^{-1}$ reduces one to $A_{00} = 1$. When α is real (or when $\mathcal{I}m\,\alpha$ is sufficiently small) p-ellipticity of A is equivalent to p-ellipticity of the new operator.

if $\mathcal{I}m\,\alpha$ is not small, the p-ellipticity may not be preserved. Thus one must assume the p-ellipticity of the new matrix $\tilde{A}$ which has all coefficients $\tilde{A}_{0j}$, $j = 0, 1, \ldots, n-1$ real.

The proof proceeds by establishing, through an integration by parts and stopping time argument, the equivalence of the p-adapted square function and the p-averaged nontangential maximal function. The connection to p-ellipticity is made in the following estimate:

$$\lambda_p' \iint_{\mathbb{R}^n_+} |\nabla u|^2 |u|^{p-2} x_0 \, dx' \, dx_0 \leq \int_{\mathbb{R}^{n-1}} |u(0,x')|^p \, dx' + C\|\mu'\|_{\mathcal{C}} \int_{\mathbb{R}^{n-1}} \left[\tilde{N}_{p,a}(u)\right]^p \, dx'.$$

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