The Dirichlet problem for second order parabolic operators in divergence form

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Second order parabolic equations

$$\mathcal{H}u = (\partial_t + \mathcal{L})u := \partial_t u - \mathrm{div}_{\lambda,x} A(x,t)\nabla_{\lambda,x} u = 0 \tag{0.1}$$

in $\mathbb{R}^{n+2}_+ = \{(\lambda, x, t) = (\lambda, x_1, .., x_n, t) \in \mathbb{R}^{n+1} \times \mathbb{R} : \ \lambda > 0\}$.

$$\kappa|\xi|^2 \leq \sum_{i,j=0}^{n} A_{i,j}(x,t)\xi_i\xi_j, \qquad |A(x,t)\xi \cdot \zeta| \leq C|\xi||\zeta|. \tag{0.2}$$

A is real but no assumptions on symmetry of *A*.

Parabolic measure

Given $f \in C_0(\mathbb{R}^{n+1})$ there exists a unique weak solution to the continuous Dirichlet problem

$$\mathcal{H}u = 0 \text{ in } \mathbb{R}^{n+2}_+,$$
$$u \in C([0,\infty) \times \mathbb{R}^{n+1}),$$
$$u(0,x,t) = f(x,t) \text{ on } \mathbb{R}^{n+1}.$$

$$u(\lambda,x,t) = \iint_{\mathbb{R}^{n+1}} f(y,s)\, d\omega(\lambda,x,t,y,s).$$

$\omega(\lambda,x,t,\cdot)$: *parabolic measure* (at (λ,x,t)).

Doubling property of parabolic measure

$$Q = Q_r(x) := B(x,r) \subset \mathbb{R}^n,\ I = I_r(t) := (t - r^2, t + r^2),$$
$$\Delta = \Delta_r(x,t) = Q_r(x) \times I_r(t)\ ,\ \ell(\Delta) := r,\ c\Delta := cQ \times c^2 I.$$

Given $r > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$,

$$A_r^+(x_0, t_0) := (4r, x_0, t_0 + 16r^2).$$

Theorem

Assume that A is real and satisfies (0.2). *If* $(x_0, t_0) \in \mathbb{R}^{n+1}$, $0 < r_0 < \infty$, $\Delta_0 := \Delta_{r_0}(x_0, t_0)$, *then*

$$\omega(A^+_{4r_0}(x_0,t_0), 2\Delta) \lesssim \omega(A^+_{4r_0}(x_0,t_0), \Delta)$$

whenever $\Delta \subset 2\Delta_0$.

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Assume that A is real and satisfies (0.2)*. If $(x_0, t_0) \in \mathbb{R}^{n+1}$, $0 < r_0 < \infty$, $\Delta_0 := \Delta_{r_0}(x_0, t_0)$, then*

$$\omega(A_{4r_0}^+(x_0, t_0), 2\Delta) \lesssim \omega(A_{4r_0}^+(x_0, t_0), \Delta)$$

whenever $\Delta \subset 2\Delta_0$.

Definition: A_∞-property of parabolic measure

Definition

Let $(x_0, t_0) \in \mathbb{R}^{n+1}$, $0 < r_0 < \infty$, $\Delta_0 := \Delta_{r_0}(x_0, t_0)$. We say $\omega(A^+_{4r_0}(x_0, t_0), \cdot) \in A_\infty(\Delta_0, \mathrm{d}x\mathrm{d}t)$ if $\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0$ such that if $E \subset \Delta$ for some $\Delta \subset \Delta_0$, then

$$\frac{\omega(A^+_{4r_0}(x_0, t_0), E)}{\omega(A^+_{4r_0}(x_0, t_0), \Delta)} < \delta \quad \Longrightarrow \quad \frac{|E|}{|\Delta|} < \varepsilon.$$

$\omega \in A_\infty(\mathrm{d}x\mathrm{d}t)$ if $\omega(A^+_{4r_0}(x_0, t_0), \cdot) \in A_\infty(\Delta_0, \mathrm{d}x\mathrm{d}t)$ for all Δ_0 as above and with uniform constants.

If $\omega \in A_\infty(\mathrm{d}x\mathrm{d}t)$, then

$$\mathrm{d}\omega(A^+_{4r_0}(x_0, t_0), x, t) = K(A^+_{4r_0}(x_0, t_0), x, t)\,\mathrm{d}x\mathrm{d}t.$$

Main Result

Theorem

Assume that A is real and satisfies (0.2). *Then parabolic measure ω belongs to $A_\infty(\mathrm{d}x\mathrm{d}t)$ with constants depending only n and the ellipticity constants.*

The result holds under no assumptions on $A = A(x, t)$ besides (0.2): t-dependent coefficients are allowed!

The result is new even in the case when $A(x, t)$ is symmetric.

L^2 results hold under stronger structural assumptions.

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Proof of the main result - references

- S. Hofmann, C. Kenig, S. Mayboroda and J. Pipher. Square function/non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators. J. Amer. Math. Soc. 28 (2015), 483–529.
- C. Kenig, B. Kirchheim, J. Pipher and T. Toro. Square functions and absolute continuity of elliptic measure. J. Geom. Anal. 26 (2016), no. 3, 2383–2410.

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Main result: reduction to a Carleson measure estimate

To conclude that $\omega \in A_\infty(\mathrm{d}x\mathrm{d}t)$ it suffices to prove

Theorem

Let $u(\lambda, x, t) = \omega(\lambda, x, t, S)$ for some Borel set $S \subset \mathbb{R}^{n+1}$. Then u satisfies the following Carleson measure estimate for all parabolic cubes $\Delta \subset \mathbb{R}^{n+1}$:

$$\int_0^{\ell(\Delta)} \iint_\Delta |\nabla_{\lambda,x} u|^2 \, \lambda \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \lesssim |\Delta|. \tag{0.3}$$

'Proof': Given $E \subset \Delta$, $\delta > 0$, there exists $K(\delta)$, such that if $\omega(A^+_{4r_0}(x_0, t_0), E) < \delta\omega(A^+_{4r_0}(x_0, t_0), \Delta)$, then there exists a set S, $E \subset S \subseteq \Delta$, such that if $u(\lambda, x, t) := \omega(\lambda, x, t, S)$, then

$$K(\delta)|E| \lesssim \int_0^{\ell(\Delta)} \iint_\Delta |\nabla_{\lambda,x} u|^2 \, \lambda \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda.$$

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It suffices to prove that

$$\int_0^{\ell(\Delta)} \iint_\Delta |\partial_\lambda u|^2 \, \lambda \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \lesssim |\Delta|, \tag{0.4}$$

for all parabolic cubes Δ.

To prove (0.4) it is enough to prove that the following holds: for each parabolic cube $\Delta \subset \mathbb{R}^{n+1}$, $r := \ell(\Delta)$, there is a Borel set $F \subset 16\Delta$ with $|\Delta| \lesssim |F|$, such that

$$\int_0^{r} \iint_F |\partial_\lambda u|^2 \, \lambda \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \lesssim |\Delta|. \tag{0.5}$$

Reduction: need to construct F and verify (0.5).

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Construction of the set $F \subset 16\Delta$

Definition

Let Δ be fixed. Given $\kappa_0 \gg 1$, we let $F \subset 16\Delta$ be the set of all $(x,t) \in 16\Delta$ such that

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\begin{aligned}
&\text{(i)} && \mathcal{M}(|\nabla_x \varphi|^2)(x,t) + \mathcal{M}(|\nabla_x \tilde{\varphi}|^2)(x,t) \leq \kappa_0^2,\\
&\text{(ii)} && \mathcal{M}_x\,\mathcal{M}_t(|H_t D_t^{1/2}\varphi|)(x,t) + \mathcal{M}_x\,\mathcal{M}_t(|H_t D_t^{1/2}\tilde{\varphi}|)(x,t) \leq \kappa_0,\\
&\text{(iii)} && \mathbb{D}\varphi(x,t) + \mathbb{D}\tilde{\varphi}(x,t) \leq \kappa_0,\\
&\text{(iv)} && N_*(\partial_\lambda P_\lambda^* \varphi)(x,t) + N_*(\partial_\lambda P_\lambda \tilde{\varphi})(x,t) \leq \kappa_0,\\
&\text{(v)} && \widetilde{N}_*(\nabla_x P_\lambda^* \varphi)(x,t) + \widetilde{N}_*(\nabla_x P_\lambda \tilde{\varphi})(x,t) \leq \kappa_0.
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We can choose κ_0, depending only on n and the ellipticity constants, so that

$$\frac{|16\Delta \setminus F|}{|16\Delta|} \leq 1/1000.$$

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Proof of the Carleson measure estimate

After the delicate construction of $F \subset 16\Delta$:

$$J_{\eta,\epsilon} := \iiint_{\mathbb{R}^{n+2}_+} A\nabla_{\lambda,x} u \cdot \nabla_{\lambda,x} u \, \Psi^2 \lambda \, \mathrm{d}x \mathrm{d}t \mathrm{d}\lambda,$$

$\Psi = \Psi_{\eta,\epsilon}$ is a smooth cut-off for a sawtooth above F. Then

$$\int_{2\epsilon}^{r} \iint_F |\partial_\lambda u|^2 \lambda \, \mathrm{d}x \mathrm{d}t \mathrm{d}\lambda \lesssim J_{\eta,\epsilon}.$$

Lemma (Key Lemma)

Let $\sigma, \eta \in (0,1)$ be given degrees of freedom. Then there exists a finite constant c depending only on n and the ellipticity constants, and a finite constant $\tilde{c}$ depending additionally on σ and η, such that

$$J_{\eta,\epsilon} \leq (\sigma + c\eta) J_{\eta,\epsilon} + \tilde{c}|\Delta|.$$

Weak solutions

u is a weak solution on $\mathbb{R}^{n+1}_+ \times \mathbb{R}$ if $u \in L^2_{\mathrm{loc}}(\mathbb{R}; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^{n+1}_+))$ and for all $\phi \in C_0^\infty(\mathbb{R}^{n+2}_+)$,

$$\int_{\mathbb{R}} \iint_{\mathbb{R}^{n+1}_+} \left(A\nabla_{\lambda,x} u \cdot \nabla_{\lambda,x}\phi - u \cdot \partial_t \phi \right) dx d\lambda dt = 0. \tag{0.6}$$

A problem: if we somehow want to control

$$||\nabla_{\lambda,x} u||_2 + ||H_t D^t_{1/2} u||_2$$

we notice a lack of coercivity in the form in (0.6).

Discovering hidden coercivity

$$\partial_t = D_t^{1/2} H_t D_t^{1/2} (= |\tau|^{1/2} i\text{sign}(\tau)|\tau|^{1/2}).$$

Consider the modified form

$$\begin{aligned} a_\delta(u,v) &= \iiint_{\mathbb{R}_+^{n+2}} A\nabla_{\lambda,x} u \cdot \nabla_{\lambda,x}(1+\delta H_t) v \, d\lambda dx dt \\ &+ \iiint_{\mathbb{R}_+^{n+2}} H_t D_t^{1/2} u \cdot D_t^{1/2}(1+\delta H_t) v \, d\lambda dx dt, \end{aligned}$$

where $\delta > 0$ is a (real) degree of freedom.

If we fix $\delta > 0$ small enough, then

$$a_\delta(u,u) \geq (\kappa - C\delta)\|\nabla_{\lambda,x} u\|_2^2 + \delta \|H_t D_t^{1/2} u\|_2^2$$

where κ, C are the ellipticity constants for A.

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Notation

$$A(x,t) = \begin{bmatrix} A_{\perp\perp}(x,t) & A_{\perp\|}(x,t) \\ A_{\|\perp}(x,t) & A_{\|\|}(x,t) \end{bmatrix}.$$

$\mathcal{H}_{\|} := \partial_t - \operatorname{div}_x A_{\|\|}\nabla_x.$

$\mathcal{H}_{\|}^* := \partial_t - \operatorname{div}_x A_{\|\|}^*\nabla_x.$

$\partial_t = D_t^{1/2} H_t D_t^{1/2} (= |\tau|^{1/2} i\operatorname{sign}(\tau)|\tau|^{1/2}).$

$\dot{\mathrm{E}} = \dot{\mathrm{E}}(\mathbb{R}^{n+1})$: the closure of $v \in C_0^\infty(\mathbb{R}^{n+1})$ w.r.t.

$$\|v\|_{\dot{\mathrm{E}}} := \left(\|\nabla_x v\|_{L^2(\mathbb{R}^{n+1})}^2 + \|H_t D_t^{1/2} v\|_{L^2(\mathbb{R}^{n+1})}^2\right)^{1/2}.$$

Given $\Delta = \Delta_r \subset \mathbb{R}^{n+1}$, there exist $\varphi, \tilde{\varphi} \in \dot{\mathsf{E}}(\mathbb{R}^{n+1})$ such that

$$\mathcal{H}_{\|}^{*}\varphi = \mathrm{div}_x(A_{\perp\|}\chi_{8\Delta}), \qquad \mathcal{H}_{\|}\tilde{\varphi} = \mathrm{div}_x(A_{\|\perp}\chi_{8\Delta}),$$

and satisfying the *a priori* estimates

$$\iint_{\mathbb{R}^{n+1}} |\nabla_x\varphi|^2 + |H_t D_t^{1/2}\varphi|^2 \, \mathrm{d}x\mathrm{d}t \lesssim \iint_{16\Delta} |A_{\perp\|}|^2 \, \mathrm{d}x\mathrm{d}t \lesssim |\Delta|,$$
$$\iint_{\mathbb{R}^{n+1}} |\nabla_x\tilde{\varphi}|^2 + |H_t D_t^{1/2}\tilde{\varphi}|^2 \, \mathrm{d}x\mathrm{d}t \lesssim \iint_{16\Delta} |A_{\|\perp}|^2 \, \mathrm{d}x\mathrm{d}t \lesssim |\Delta|.$$

The Kato square root estimate

$$\mathrm{E} = \mathrm{E}(\mathbb{R}^{n+1}) = \dot{\mathrm{E}}(\mathbb{R}^{n+1}) \cap L^2(\mathbb{R}^{n+1}).$$

Theorem

The operator $\mathcal{H}_{||} = \partial_t - \mathrm{div}_x\, A_{||||}(x,t)\nabla_x$ arises from an accretive form, it is maximal-accretive in $L^2(\mathbb{R}^{n+1})$ and

$$\|\sqrt{\mathcal{H}_{||}}\, u\|_2 \sim \|\nabla_x u\|_2 + \|H_t D_t^{1/2} u\|_2 \qquad (u \in \mathrm{E}).$$

No assumptions on $A_{||||} = A_{||||}(x,t)$ besides measurability and uniform ellipticity: t-dependent coefficients are allowed!

The case $A_{||||}^* = A_{||||}$ does not follow from abstract functional analysis as $\mathcal{H}_{||}$ not self-adjoint.

Extending φ and $\tilde{\varphi}$: introducing $P_\lambda^*\varphi$ and $P_\lambda\tilde{\varphi}$

Given $m \in \mathbb{Z}_+$, $\lambda > 0$ we introduce

$$P_\lambda^*\varphi := (1+\lambda^2\mathcal{H}_{\|}^*)^{-m}\varphi, \qquad P_\lambda\tilde{\varphi} := (1+\lambda^2\mathcal{H}_{\|})^{-m}\tilde{\varphi},$$

within the homogeneous energy space $\dot{\mathrm{E}}(\mathbb{R}^{n+1})$.

Lemma

There exists c, $1 \leq c < \infty$, depending only on n, the ellipticity constants and $m \geq 1$ such that

$$\text{(i)} \qquad \iiint_{\mathbb{R}^{n+2}_+} |\partial_\lambda P_\lambda^*\varphi|^2 + |\partial_\lambda P_\lambda\tilde{\varphi}|^2 \, \frac{\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda}{\lambda} \leq c|\Delta|.$$

Proof: $\partial_\lambda P_\lambda\tilde{\varphi} = -2m\sqrt{\lambda^2\mathcal{H}_{\|}}(1+\lambda^2\mathcal{H}_{\|})^{-m-1}\sqrt{\mathcal{H}_{\|}}\tilde{\varphi}$.

Square function estimates

Lemma

There exists c, $1 \leq c < \infty$, depending only on n, the ellipticity constants and $m \geq 1$ such that

$$\text{(i)} \quad \iiint_{\mathbb{R}^{n+2}_+} |\partial_\lambda P^*_\lambda \varphi|^2 + |\partial_\lambda P_\lambda \tilde{\varphi}|^2 \, \frac{\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda}{\lambda} \leq c|\Delta|,$$

$$\text{(ii)} \quad \iiint_{\mathbb{R}^{n+2}_+} |\lambda \nabla_x \partial_\lambda P^*_\lambda \varphi|^2 + |\lambda \nabla_x \partial_\lambda P_\lambda \tilde{\varphi}|^2 \, \frac{\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda}{\lambda} \leq c|\Delta|,$$

$$\text{(iii)} \quad \iiint_{\mathbb{R}^{n+2}_+} |\lambda \mathcal{H}^*_{\|} P^*_\lambda \varphi|^2 + |\lambda \mathcal{H}_{\|} P_\lambda \tilde{\varphi}|^2 \, \frac{\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda}{\lambda} \leq c|\Delta|,$$

$$\text{(iv)} \quad \iiint_{\mathbb{R}^{n+2}_+} |\lambda^2 \mathcal{H}^*_{\|} \partial_\lambda P^*_\lambda \varphi|^2 + |\lambda^2 \mathcal{H}_{\|} \partial_\lambda P_\lambda \tilde{\varphi}|^2 \, \frac{\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda}{\lambda} \leq c|\Delta|.$$

A non-tangential maximal function estimate

$$N_*F(x,t) = \sup_{\lambda>0} \sup_{\Lambda\times Q\times I} |F(\mu,y,s)|.$$

Lemma

Fix $m = n+1$ in the definitions of P_λ^* and P_λ. Then

$$||N_*(\partial_\lambda P_\lambda^*\varphi)||_2^2 + ||N_*(\partial_\lambda P_\lambda\tilde{\varphi})||_2^2 \lesssim |\Delta|.$$

Lemma

For $\lambda > 0$ and $m \geq 1$, the resolvent $P_\lambda = (1+\lambda^2\mathcal{H}_{||})^{-m}$ can be represented by an integral kernel $K_{\lambda,m}$ with pointwise bounds

$$|K_{\lambda,m}(x,t,y,s)| \leq \frac{C1_{(0,\infty)}(t-s)}{\lambda^{2m}}(t-s)^{-n/2+m-1}\mathrm{e}^{-\frac{t-s}{\lambda^2}}\mathrm{e}^{-c\frac{|x-y|^2}{t-s}}.$$

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We can choose κ_0, depending only on n and the ellipticity constants, so that

$$\frac{|16\Delta \setminus F|}{|16\Delta|} \leq 1/1000.$$

Proof of the Key Lemma

$u\Psi^2\lambda$ is a test function for the weak formulation:

$$0 = \iiint_{\mathbb{R}^{n+2}_+} A\nabla_{\lambda,x}u \cdot \nabla_{\lambda,x}(u\Psi^2\lambda) + \partial_t u(u\Psi^2\lambda)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda.$$

$J := J_{\eta,\epsilon} = J_1 + J_2 + J_3$, where

$$\begin{aligned} J_1 &:= -\iiint_{\mathbb{R}^{n+2}_+} (A\nabla_{\lambda,x}u \cdot \nabla_{\lambda,x}\Psi^2)u\,\lambda\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda, \\ J_2 &:= -\iiint_{\mathbb{R}^{n+2}_+} (A\nabla_{\lambda,x}u \cdot \nabla_{\lambda,x}\lambda)u\Psi^2\,\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda, \\ J_3 &:= -\iiint_{\mathbb{R}^{n+2}_+} \partial_t u(u\Psi^2\lambda)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda. \end{aligned}$$

$$|J_1| + |J_3| \leq \sigma J + \tilde{c}|\Delta|.$$

Proof of the Key Lemma

$J_2 = J_{21} + J_{22}$, where

$$J_{21} := -\iiint_{\mathbb{R}^{n+2}_+} (A_{\perp\|} \cdot \nabla_x u) u \Psi^2 \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda,$$

$$J_{22} := -\iiint_{\mathbb{R}^{n+2}_+} (A_{\perp\perp} \partial_\lambda u) u \Psi^2 \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda.$$

To estimate J_{21} we use that

$$A_{\perp\|} \cdot \nabla_x \left(\frac{u^2 \Psi^2}{2} \right) = (A_{\perp\|} \cdot \nabla_x u) u \Psi^2 + (A_{\perp\|} \cdot \nabla_x \Psi) u^2 \Psi$$

and we write $J_{21} = J_{211} + J_{212}$, where

$$J_{211} := -\iiint_{\mathbb{R}^{n+2}_+} A_{\perp\|} \cdot \nabla_x \left(\frac{u^2 \Psi^2}{2} \right) \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda,$$

$$J_{212} := \iiint_{\mathbb{R}^{n+2}_+} (A_{\perp\|} \cdot \nabla_x \Psi) u^2 \Psi \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda.$$

Proof of the Key Lemma

We introduced φ as the energy solution on $\mathbb{R}^{n+1}$ to the problem

$$\text{div}_x(A_{\perp\|}\chi_{8\Delta}) = -\partial_t\varphi - \text{div}_x(A^*_{\|\|}\nabla_x\varphi) = \mathcal{H}^*_{\|}\varphi.$$

Let $\theta_{\eta\lambda} = \varphi - P^*_{\eta\lambda}\varphi$. Then, splitting $\varphi = \theta_{\eta\lambda} + P^*_{\eta\lambda}\varphi$ and

$$J_{211} = J_{2111} + J_{2112} + J_{2113} + J_{2114},$$

where

$$J_{2111} := \iiint_{\mathbb{R}^{n+2}_+} (\theta_{\eta\lambda})\partial_t\left(\frac{u^2\Psi^2}{2}\right) \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda,$$

$$J_{2112} := \iiint_{\mathbb{R}^{n+2}_+} (P^*_{\eta\lambda}\varphi)\partial_t\left(\frac{u^2\Psi^2}{2}\right) \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda,$$

$$J_{2113} := \iiint_{\mathbb{R}^{n+2}_+} A^*_{\|\|}\nabla_x\theta_{\eta\lambda}\cdot\nabla_x\left(\frac{u^2\Psi^2}{2}\right) \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda,$$

$$J_{2114} := \iiint_{\mathbb{R}^{n+2}_+} A^*_{\|\|}\nabla_x P^*_{\eta\lambda}\varphi\cdot\nabla_x\left(\frac{u^2\Psi^2}{2}\right) \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda.$$

Proof of the Key Lemma

$J_{2112} + J_{2114}$ equals (by a sequence of manipulations)

$$
\begin{aligned}
&-\iiint_{\mathbb{R}^{n+2}_+} (A^*_{\|\|}\nabla_x\partial_\lambda P^*_{\eta\lambda}\varphi)\cdot\nabla_x\left(\frac{u^2\Psi^2}{2}\right)\lambda\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \\
&+\iiint_{\mathbb{R}^{n+2}_+} (\mathcal{H}^*_{\|}P^*_{\eta\lambda}\varphi)\partial_\lambda\left(\frac{u^2\Psi^2}{2}\right)\lambda\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \\
&+\iiint_{\mathbb{R}^{n+2}_+} (\partial_\lambda P^*_{\eta\lambda}\varphi)\nabla_x u\cdot(A\nabla_{\lambda,x}u)_{\|}\Psi^2\,\lambda\mathrm{d}x\mathrm{d}t\mathrm{d}\lambda \\
&+\ldots.
\end{aligned}
$$

The two first terms can be controlled with the square function estimates for $|\lambda\nabla_x\partial_\lambda P^*_{\eta\lambda}\varphi|^2\lambda^{-1}$, $|\lambda\mathcal{H}^*_{\|}P^*_{\eta\lambda}\varphi|^2\lambda^{-1}$.

To handle the third term we use the pointwise control of $\partial_\lambda P^*_{\eta\lambda}\varphi$ on the sawtooth.

Proof of the Key Lemma

All in all we derive

$$|J_{211}| \leq (\sigma + c\eta)J + \tilde{c}|\Delta| + III_1,$$

where

$$III_1 := \iiint_{\mathbb{R}^{n+2}_+} A_{\|\perp} \cdot \nabla_x (u^2 \Psi^2 \partial_\lambda P^*_{\eta\lambda} \varphi) \, \mathrm{d}x\mathrm{d}t\mathrm{d}\lambda$$

We introduced $\tilde{\varphi}$ as the energy solution on $\mathbb{R}^{n+1}$ to the problem

$$\mathrm{div}_x(A_{\|\perp}\chi_{8\Delta}) = \partial_t \tilde{\varphi} - \mathrm{div}_x(A_{\|\|}\nabla_x \tilde{\varphi}) = \mathcal{H}_\| \tilde{\varphi}.$$

Let $\tilde{\theta}_{\eta\lambda} = \tilde{\varphi} - P_{\eta\lambda}\tilde{\varphi}$. Then, splitting $\tilde{\varphi} = \tilde{\theta}_{\eta\lambda} + P_{\eta\lambda}\tilde{\varphi}$ and .... we can in the end control all terms !

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