Fatou Theorems in UR Domains

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Motivation

Let $\mathbb{D}$ be the unit disk in the complex plane. Given $u : \mathbb{D} \to \mathbb{C}$, for some fixed $\kappa > 0$ define its κ-nontangential boundary trace as

$$\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \Big)(z) := \lim_{\substack{|\zeta - z| < (1+\kappa)(1-|\zeta|) \\ \zeta \longrightarrow z}} u(\zeta) \quad \text{for} \quad z \in \partial\mathbb{D}.$$

Here is a classical result originating in Fatou's 1906 work:

> if $u : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded then for each $\kappa > 0$ the trace $\Big(u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \Big)(e^{i\theta})$ exists at $\mathcal{L}^1$-a.e. $\theta \in [0, 2\pi)$,

In general, one cannot hope for a better conclusion since Lusin has proved (in 1919) that

> for any Lebesgue measurable set $E \subseteq [0, 2\pi)$ with $\mathcal{L}^1(E) = 0$ there exists a bounded holomorphic function $u : \mathbb{D} \to \mathbb{C}$ whose radial limit $\lim\limits_{r \to 1^-} u(re^{i\theta})$ fails to exist for each angle $\theta \in E$.

Also, insisting that the limit is taken from within nontangential approach regions is both natural and optimal in the context of Fatou's theorem.

Motivation

Indeed, Littlewood has given an example of a bounded holomorphic function in $\mathbb{D}$ which diverges almost everywhere along rotated copies of any fixed, given curve in the unit disk, which ends tangentially to $\partial\mathbb{D}$. One thing one can do is to relax the boundedness demand on the holomorphic function u by the membership

$$\mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \mathcal{H}^1), \quad 0 < p \leq \infty,$$

for some, or all, $\kappa \in (0, \infty)$, where the nontangential maximal function $\mathcal{N}_\kappa u$ is defined as

$$(\mathcal{N}_\kappa u)(z) := \sup_{|\zeta - z| < (1+\kappa)(1-|\zeta|)} |u(\zeta)|, \qquad \forall\, z \in \partial\mathbb{D}.$$

This leads to the consideration of Hardy spaces

$$\mathscr{H}^p(\mathbb{D}) := \big\{ u \in \mathscr{O}(\mathbb{D}) : \mathcal{N}_\kappa u \in L^p(\partial\mathbb{D}, \mathcal{H}^1) \big\}, \quad 0 < p \leq \infty$$

Motivation/Goals

Thus, the nontangential boundary trace takes you

$$\mathscr{H}^p(\mathbb{D}) \ni u \mapsto u\Big|_{\partial\mathbb{D}}^{\kappa-\mathrm{n.t.}} \in L^p(\partial\mathbb{D}, \mathcal{H}^1)$$

and, at least if $p > 1$, the Cauchy integral operator goes the other way

$$\mathscr{C} : L^p(\partial\mathbb{D}, \mathcal{H}^1) \to \mathscr{H}^p(\mathbb{D})$$

Classical work:

J.Garcia-Cuerva, "Weighted H^p spaces" 1979, Dissertationes Math.,
C.Kenig, "Weighted H^p spaces on Lip domains" 1980, Amer. J. Math.

Present Goals:

Develop a theory which retains the aforementioned features, which can accommodate classes of domains and operators (generalizing the unit disk and Cauchy-Riemann operator) in the nature of best possible.
Want to work with subdomains of manifolds and differential operators acting between vectors bundles, so it's all about real-variable techniques.

Class of Operators

Given an arbitrary $N \times M$ homogeneous first-order system with constant complex coefficients in $\mathbb{R}^n$

$$D = \sum_{j=1}^{n} A_j \partial_j, \quad A_j \in \mathbb{C}^{N \times M},$$

recall that its `principal symbol` is defined as the $N \times M$ matrix

$$\mathrm{Sym}\,(D\,;\xi) := i \sum_{j=1}^{n} \xi_j A_j, \quad \forall\, \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

Call D `injectively elliptic` if

$$\mathrm{Sym}\,(D\,;\xi) : \mathbb{C}^M \to \mathbb{C}^N \;\text{ is injective for each }\; \xi \in \mathbb{R}^n \setminus \{0\}.$$

Class of Domains

Call $\Sigma \subset \mathbb{R}^n$ a UR `set` provided Σ is closed, upper ADR, and has BPLI. Call an open set $\Omega \subseteq \mathbb{R}^n$ a `UR domain` provided $\partial\Omega$ is a UR set and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.

The latter condition amounts to having the outward unit normal ν defined $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$.

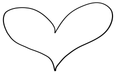

Figure: A UR domain which does *not* satisfy the corkscrew condition (int/ext)

Main Theorem

Theorem (Fatou in UR Domains)

Let $\Omega \subset \mathbb{R}^n$ be a UR domain. Denote $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and fix $\kappa > 0$. Let D be an injectively elliptic homogeneous first-order $N \times M$ system with constant complex coefficients in $\mathbb{R}^n$. Finally, let $u : \Omega \to \mathbb{C}^M$ satisfy

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \ \ \text{with} \ \ p \in \left(\tfrac{n-1}{n}, \infty\right),$$
$$\text{and} \ \ Du = 0 \ \ \text{in} \ \ [\mathcal{D}'(\Omega)]^N.$$

Then the nontangential boundary trace $u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists (in $\mathbb{C}^M$) at σ-a.e. point on $\partial\Omega$ and, if $p > 1$, there exists a "Cauchy operator"

$$\mathscr{C} : L^p(\partial\Omega, \sigma) \to \left\{ w : \ D^* D w = 0 \ \text{and} \ \mathcal{N}_\kappa w \in L^p(\partial\Omega, \sigma) \right\}$$

allowing us to recover u from the said trace, i.e.,

$$u = \mathscr{C}\left(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right) \ \ \text{in} \ \ \Omega.$$

Comments

- The theorem is sharp (having Ω a UR domain and D injectively elliptic are necessary).
- When $p \in (1,\infty)$, the "Cauchy reproducing formula" yields

$$\big\|\mathcal{N}_\kappa u\big\|_{L^p(\partial\Omega,\sigma)} \approx \Big\|u\Big|^{\mathrm{n.t.}}_{\partial\Omega}\Big\|_{[L^p(\partial\Omega,\sigma)]^M}$$

  which may be interpreted as an L^p-`Maximum Principle` for null-solutions u of D in Ω.
- Specializing our Fatou theorem to the case when $n=2$ and $D := \overline{\partial}$, the Cauchy-Riemann operator (hence, $M=N=1$), yields:

  > any holomorphic function u in a UR domain $\Omega \subseteq \mathbb{C}$ satisfying $\int_{\partial\Omega}\big(\mathcal{N}_\kappa u\big)^p\,d\mathcal{H}^1 < \infty$, for some $\kappa > 0$ and $p > \frac{1}{2}$, has the property that $u\Big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}$ exists in $\mathbb{C}$ at $\mathcal{H}^1$-a.e. point on $\partial\Omega$.

- Same result holds for the Clifford-Dirac operator $D := \sum_{j=1}^n e_j \odot \partial_j$.

Comments

- Holomorphic functions of several complex variables also fit into this framework by considering the first-order injectively elliptic operator $D := \left(\partial_{\bar{z}_j}\right)_{1\leq j\leq n}$. Specifically, the following Fatou-type result holds:

  any holomorphic function u in a UR domain $\Omega \subseteq \mathbb{C}^n$ with $\int_{\partial\Omega} \left(\mathcal{N}_\kappa u\right)^p \, d\mathcal{H}^{2n-1} < \infty$ for some $\kappa > 0$ and $p \in \left(\frac{2n-1}{2n}, \infty\right)$, has the property that $u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at $\mathcal{H}^{2n-1}$-a.e. point on $\partial\Omega$.

- Similar results are valid when D is the Hodge-Dirac operator $d+\delta$ (where d, δ are, respectively, the exterior derivative operator and its formal adjoint, acting on differential forms), and its complex counterpart $D = \bar{\partial} + \vartheta$ where $\overline{\partial}$ is the d-bar operator in the several complex variable theory and ϑ is its Hermitian adjoint.

Traces: Warm-Up

Given an $\mathcal{L}^n$-measurable function $u:\Omega\to\mathbb{C}$ define at each $x\in\partial\Omega$

$$(\mathfrak{P}u)(x) := \sup_{0<r<2\,\mathrm{diam}\,(\partial\Omega)}\left\{\frac{1}{\sigma\big(\partial\Omega\cap B(x,r)\big)}\int_{\Omega\cap B(x,r)}|u|\,d\mathcal{L}^n\right\}\in[0,\infty].$$

Theorem (A)

Let $\Omega\subseteq\mathbb{R}^n$ be open, ADR boundary, and $\mathcal{H}^{n-1}(\partial\Omega\setminus\partial_*\Omega)=0$. Denote by ν its GMT outward unit normal, and fix $\kappa>0$. Let D be an $N\times M$ homogeneous first-order system D with constant complex coefficients in $\mathbb{R}^n$. Let $u:\Omega\to\mathbb{C}^M$ be $\mathcal{L}^n$-measurable such that

$$\mathcal{N}_\kappa u\in L^1(\partial\Omega,\sigma)\quad\textit{and}\quad u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\quad\textit{exists } \sigma\textit{-a.e. on } \partial\Omega.$$

Also assume that $Du\in[L^1_{\mathrm{loc}}(\Omega)]^N$ and satisfies $\mathfrak{P}(Du)\in L^1(\partial\Omega,\sigma)$. Then, in a quantitative sense,

$$\mathrm{Sym}\,(D\,;\nu)\big(u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\big)\in\big[H^1(\partial\Omega,\sigma)\big]^N$$

An Example

As a special case of Theorem A, consider the scenario in which $u := \nabla w$ for some scalar-valued function

$w \in \mathscr{C}^\infty(\Omega)$ which is harmonic in Ω, and for some $\kappa > 0$ satisfies $\mathcal{N}_\kappa(\nabla w) \in L^1(\partial\Omega, \sigma)$ and has the property that the nontangential trace $(\nabla w)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists σ-a.e. on $\partial\Omega$.

Then our theorem, used with $D := \mathrm{div}$ (which annihilates u and whose symbol is the dot product), gives that the normal derivative

$$\partial_\nu w := \nu \cdot \Big((\nabla w)\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\Big) \text{ belongs to the Hardy space } H^1(\partial\Omega, \sigma).$$

The case when Ω is a Lipschitz domain has been treated by B. Dahlberg and C. Kenig (1987), using a conceptually different approach (based on duality and Varopoulos' extension theorem).

Intermission

The next step is to extend this trace result to the case when $p < 1$. Since, in this scenario, the Hardy space $H^p(\partial\Omega, \sigma)$ consists of "distributions" (i.e., linear continuous functionals on $\mathrm{Lip}_c(\partial\Omega)$), we need to interpret $\mathrm{Sym}\,(D\,;\nu)$ acting on u as a distribution on the boundary (rather than the pointwise sense considered earlier).

The "bullet" product

Fix an arbitrary first order $N \times M$ system D. Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set and suppose $u \in \left[L^1_{\rm bdd}(\Omega)\right]^M$ is such that $Du \in [L^1_{\rm bdd}(\Omega)]^N$. In this setting, define a functional, denoted by $\mathrm{Sym}\,(D\,;\nu) \bullet u$, acting on each $\psi \in \left[\mathrm{Lip}_c(\partial\Omega)\right]^N$ according to

$$\left\langle (-i)\mathrm{Sym}\,(D\,;\nu) \bullet u, \psi \right\rangle := \int_\Omega \langle Du, \Psi \rangle \, d\mathcal{L}^n - \int_\Omega \langle u, D^\top \Psi \rangle \, d\mathcal{L}^n,$$

where $\Psi \in \left[\mathrm{Lip}(\,\overline{\Omega}\,)\right]^N$ satisfies $\Psi\big|_{\partial\Omega} = \psi$, and $\Psi \equiv 0$ outside of some compact subset of $\overline{\Omega}$.

Then the functional $\mathrm{Sym}\,(D\,;\nu) \bullet u$ is meaningfully and unambiguously defined and, in fact, belongs to the space $\left[\left(\mathrm{Lip}_c(\partial\Omega)\right)'\right]^M$.

Traces: The rest of the story

Theorem (B)

Let $\Omega \subseteq \mathbb{R}^n$ be an open set with an ADR *boundary. Abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$, and fix $\kappa > 0$.*
Consider an arbitrary $N \times M$ homogeneous first-order system D with constant complex coefficients in $\mathbb{R}^n$, along with some $\mathcal{L}^n$-measurable function $u : \Omega \to \mathbb{C}^M$ with the property that

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \ \text{ for some } \ p \in \left(\tfrac{n-1}{n}, \infty\right).$$

Also, assume that $Du \in [L^1_{\mathrm{loc}}(\Omega)]^N$ and $\mathfrak{P}(Du) \in L^p(\partial\Omega, \sigma)$.
Then, in a quantitative sense,

$$\mathrm{Sym}\,(D\,;\nu) \bullet u \in \big[H^p(\partial\Omega, \sigma)\big]^N.$$

Moreover, this is compatible with the trace result with nontangential pointwise traces (formulated earlier for $p = 1$).

Integral Representation Formula

The theorem below is central to the present considerations.

Theorem (C)

Let $\Omega \subseteq \mathbb{R}^n$ be open, ADR boundary, and $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$. Denote by ν its GMT outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1} \lfloor \partial\Omega$.
Let D be an injectively elliptic, homogeneous, first-order $N \times M$ system with constant complex coefficients in $\mathbb{R}^n$. Hence, $L := D^* D$ is an elliptic second-order $M \times M$ system in $\mathbb{R}^n$. In particular, $L^\top$ has a decent fundamental solution $E_{L^\top}$. Then, with $\overline{D}$ acting on the columns of $E_{L^\top}$, consider the fundamental solution for $D^\top$ given by

$$\widetilde{E} := \overline{D} E_{L^\top}$$

Next, let $u : \Omega \to \mathbb{C}^M$ be a measurable function satisfying, for some $\kappa > 0$,

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \quad \text{for some} \quad p \in \left(\tfrac{n-1}{n}, \infty\right).$$

Integral Representation Formula

Theorem (Continuation)

In addition, assume that

$$Du \in \left[L^1_{\mathrm{loc}}(\Omega)\right]^N \ \ \textit{and}\ \ \mathfrak{P}(Du) \in L^p(\partial\Omega, \sigma).$$

Then $\mathrm{Sym}\,(D\,;\nu) \bullet u$ *belongs to the Hardy space* $\left[H^p(\partial\Omega, \sigma)\right]^N$. *Moreover, if* $p \in \left(\frac{n-1}{n}\,,\,1\right]$ *then for* $\mathcal{L}^n$*-a.e. Lebesgue point* $x \in \Omega$ *for the function* u *with the property that*

$$\int_\Omega \frac{|(Du)(y)|}{|x-y|^{n-1}}\,dy < +\infty$$

one has

$$u(x) = \Big\langle \widetilde{E}^\top(x-\cdot)\big|_{\partial\Omega}\,,\,(-i)\mathrm{Sym}\,(D\,;\nu) \bullet u \Big\rangle$$

$$- \int_\Omega \big\langle \widetilde{E}^\top(x-y)\,,\,(Du)(y)\big\rangle\,dy$$

Integral Representation Formula

Theorem (Continuation)

The bracket $\langle\cdot,\cdot\rangle$ *in the first line above is viewed as the duality pairing between the rows of the* $M\times N$ *matrix* $\widetilde{E}^{\top}(x-\cdot)\big|_{\partial\Omega}$, *each of which belonging to*

$$\Big(\big[H^p(\partial\Omega,\sigma)\big]^N\Big)^* = \begin{cases} \big[\mathscr{C}^{(n-1)(\frac{1}{p}-1)}(\partial\Omega)\big]^N & \text{if } p<1,\\ \big[\mathrm{BMO}(\partial\Omega,\sigma)\big]^N & \text{if } p=1, \end{cases}$$

and $(-i)\mathrm{Sym}\,(D\,;\nu)\bullet u\in\big[H^p(\partial\Omega,\sigma)\big]^N$.

A similar integral representation formula holds if $p\in(1,\infty)$, *using ordinary integration over* $\partial\Omega$ *in place of the duality brackets above.*

Jump Formulas: Weighted Lebesgue Spaces

Theorem (D)

Let $\Omega \subset \mathbb{R}^n$ be an open set with a UR boundary. Denote by ν its GMT outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Consider a function

$$k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\}) \quad \textit{with} \quad k(-x) = -k(x) \quad \textit{and}$$
$$k(\lambda x) = \lambda^{1-n} k(x) \quad \forall\, \lambda > 0, \ \ \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

and for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ define the boundary to domain SIO

$$\mathcal{T}f(x) := \int_{\partial\Omega} k(x-y) f(y)\, d\sigma(y), \qquad x \in \Omega,$$

along with its boundary to boundary version

$$Tf(x) := \lim_{\varepsilon \to 0^+} \int_{y \in \partial\Omega,\, |x-y| > \varepsilon} k(x-y) f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

Jump Formulas: Weighted Lebesgue Spaces

Theorem (Continuation)

Then for each $f \in L^1\big(\partial\Omega, \frac{\sigma(x)}{1+|x|^{n-1}}\big)$ *the jump-formula*

$$\lim_{\Gamma_\kappa(x)\ni z\to x} \mathcal{T}f(z) = \frac{1}{2\sqrt{-1}}\,\widehat{k}(\nu(x))f(x) + Tf(x)$$

holds at σ*-a.e.* $x \in \partial_*\Omega$ *(with 'hat' denoting the Fourier transform in* $\mathbb{R}^n$*).*

The next step is to extend this result as to allow f to be a distribution in a Hardy space on $\partial\Omega$. In such a scenario, we can no longer speak of pointwise values of f, so a new point of view is required.

On a given closed ADR set $\Sigma \subset \mathbb{R}^n$, it turns out that the identity map between the Hardy scale H^p and the Lebesgue scale L^p when $p \in (1, \infty)$ may be further extended uniquely to a linear and bounded mapping in the range $p \in \left(\frac{n-1}{n}\,, 1\right]$.

The Filtering Operator

Theorem (E)

Let $\Sigma \subseteq \mathbb{R}^n$ *be a closed* ADR *set and abbreviate* $\sigma := \mathcal{H}^{n-1}\lfloor\Sigma$. *Also, consider an approximation to the identity* $S_t : \Sigma \times \Sigma \to \mathbb{R}$ *indexed by* $t \in \big(0\,,\,\mathrm{diam}\,\Sigma\big)$ *and satisfying, for all* $x, y, z \in \Sigma$,

$$0 \leq S_t(x,y) \leq Ct^{1-n}, \quad S_t(x,y) = 0 \;\; \textit{if} \;\; |x-y| \geq Ct,$$

$$|S_t(x,y) - S_t(z,y)| \leq Ct^{-n}|x-z|,$$

$$S_t(x,y) = S_t(y,x), \textit{ and } \int_\Sigma S_t(x,y)\,d\sigma(y) = 1.$$

Then for each $f \in H^p(\Sigma,\sigma)$ *with* $\frac{n-1}{n} < p < \infty$, *the limit*

$$(\mathfrak{H}f)(x) := \lim_{t\to 0^+} {}_{(H^p(\Sigma,\sigma))^*}\big\langle S_t(x,\cdot)\,,\,f\big\rangle_{H^p(\Sigma,\sigma)}$$

exists for σ*-a.e. point* $x \in \Sigma$.

The Filtering Operator

Theorem (Continuation)

Moreover, the assignment $f \mapsto \mathfrak{H}f$ *induces a well-defined linear and bounded operator*

and $$\mathfrak{H} : H^p(\Sigma, \sigma) \to L^p(\Sigma, \sigma) \text{ for each } p \in \left(\tfrac{n-1}{n}, \infty\right),$$

$\mathfrak{H}f = f$ *whenever* $f \in H^p(\Sigma, \sigma) \cap L^1_{\rm loc}(\Sigma, \sigma)$ *with* $p \in \left(\tfrac{n-1}{n}, \infty\right)$, *hence in particular for each* $f \in H^p(\Sigma, \sigma)$ *with* $1 \leq p < \infty$.

Note: While $\mathfrak{H}$ becomes the identification of $H^p(\Sigma, \sigma)$ with $L^p(\Sigma, \sigma)$ when $1 < p < \infty$, the L^p-filtering operator fails to be injective when $p \in \left(\tfrac{n-1}{n}, 1\right)$.

E.g., for each two distinct points $x_0, x_1 \in \Sigma$ we have $\delta_{x_0} - \delta_{x_1} \in H^p(\Sigma, \sigma)$ for every $p \in \left(\tfrac{n-1}{n}, 1\right)$ and (assuming $n \geq 2$) we have

$$\mathfrak{H}(\delta_{x_0} - \delta_{x_1}) = 0 \text{ at } \sigma\text{-a.e. point on } \Sigma.$$

Jump Formulas: Hardy Spaces

Theorem (F)

Let $\Omega \subseteq \mathbb{R}^n$ be open, with $\partial\Omega$ a UR set. Denote by ν its GMT outward unit normal and abbreviate $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$. Fix $k \in \mathscr{C}^N(\mathbb{R}^n \setminus \{0\})$ which is odd and positive homogeneous of degree $1-n$, and pick $p \in \big(\frac{n-1}{n}, 1\big]$. Then the principal-value SIO of formal convolution with the kernel k on $\partial\Omega$ induces a well-defined linear and bounded mapping

$$T : H^p(\partial\Omega, \sigma) \to L^p(\partial\Omega, \sigma).$$

Also, if we fix $\kappa > 0$ and for each $f \in H^p(\partial\Omega, \sigma)$ we define

$$(\mathcal{T}f)(x) := {}_{(H^p(\partial\Omega,\sigma))^*}\Big\langle k(x-\cdot)\Big|_{\partial\Omega}\,,\, f\,\Big\rangle_{H^p(\partial\Omega,\sigma)} \quad \text{for} \ \ x \in \Omega,$$

then for each $f \in H^p(\partial\Omega, \sigma)$ the following jump-formula holds:

$$(\mathcal{T}f)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}(x) = \frac{1}{2\sqrt{-1}}\,\widehat{k}\big(\nu(x)\big)(\mathfrak{H}f)(x) + (Tf)(x) \ \ \text{for } \sigma\text{-a.e.} \ \ x \in \partial_*\Omega.$$

The end-game in the proof of Fatou Theorem

Recall that $\Omega \subset \mathbb{R}^n$ is a UR domain, D is injectively elliptic, and $u : \Omega \to \mathbb{C}^M$ satisfies

$$\mathcal{N}_\kappa u \in L^p(\partial\Omega, \sigma) \ \text{ with } \ p \in \left(\tfrac{n-1}{n}, \infty\right),$$
$$\text{and } \ Du = 0 \ \text{ in } \ [\mathcal{D}'(\Omega)]^N.$$

Then Theorem B on p. 14 implies that

$$\mathrm{Sym}\,(D\,;\nu) \bullet u \in \big[H^p(\partial\Omega, \sigma)\big]^N.$$

Granted this, the integral representation formula from Theorem C on p. 16, with $k := (-i)\widetilde{E}^\top$, gives

$$u(x) = \Big\langle \widetilde{E}^\top(x - \cdot)\big|_{\partial\Omega}\,,\,(-i)\mathrm{Sym}\,\big(D\,;\nu\big) \bullet u \Big\rangle$$
$$= \Big(\mathcal{T}\big(\mathrm{Sym}\,(D\,;\nu) \bullet u\big)\Big)(x) \ \text{ for each } \ x \in \Omega.$$

The end-game in the proof of Fatou Theorem

Finally, the jump-formula from Theorem F on p. 22 applies and ensures that

$$\left(u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right)(x) = \left(\mathcal{T}\big(\mathrm{Sym}\,(D\,;\nu)\bullet u\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right)(x)$$

$$= \frac{1}{2\sqrt{-1}}\,\widehat{k}\big(\nu(x)\big)\Big(\mathfrak{H}\big(\mathrm{Sym}\,(D\,;\nu)\bullet u\big)\Big)(x)$$

$$+\Big(T\big(\mathrm{Sym}\,(D\,;\nu)\bullet u\big)\Big)(x)$$

for σ-a.e. $x \in \partial_*\Omega$, hence for σ-a.e. $x \in \partial\Omega$ since we are presently assuming that $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_*\Omega) = 0$.
QED