Poisson Integral Representation Formulas for weakly elliptic systems in domains with Ahlfors-David regular boundaries

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joint work with Irina Mitrea and Marius Mitrea

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May 28, 2018

The classical Poisson integral representation formula for Δ

Let $\Omega = B(0,1) \subset \mathbb{R}^n$. Then if $k(x,y) := \dfrac{1-|x|^2}{\omega_{n-1}|x-y|^n}$ for $x \neq y$,

$$\left.\begin{array}{r} u \in C^2(\overline{\Omega}) \\ \Delta u = 0 \ \text{ in } \ \Omega \end{array}\right\} \Longrightarrow u(x) = \int_{\partial\Omega} k(x,y)\big(u\big|_{\partial\Omega}\big)(y)\, d\sigma(y) \quad \forall\, x \in \Omega$$

$k(x,y)$ is the Poisson kernel for the Laplacian for the unit ball.

Comments:

• Regarding the nature of k, we have $k(x,y) = -\partial_{\nu(y)}[G(x,y)]$, where G is the Green function for the Laplacian in Ω; i.e., for each $x \in \Omega$:

$$\begin{cases} G(x,\cdot) \in C^\infty(\overline{\Omega} \setminus \{x\}) \cap L^1_{loc}(\Omega) \\ \Delta_y G(x,y) = -\delta_x(y), \quad G(x,\cdot)\big|_{\partial\Omega} = 0 \end{cases}$$

Alternatively, we may define $k := \dfrac{d\omega}{d\sigma}$ but then the question becomes when is $k(x,y) = -\partial_{\nu(y)}[G(x,y)]$ (e.g., issue explicitly raised in Garnett & Marshall *Harmonic Measure* [Question 2, page 49]).

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• In the proof of the Poisson formula, use the classical Divergence Theorem in the bounded C^1 domain $\Omega_\varepsilon := \Omega \setminus \overline{B(x,\varepsilon)}$, $\varepsilon > 0$ small, where $x \in \Omega$ is an arbitrary fixed point, for the divergence-free vector field

$$\vec{F} := u\nabla G - G\nabla u \in C^1(\overline{\Omega_\varepsilon})$$

and then take the limit as $\varepsilon \to 0^+$. The assumption $u \in C^2(\overline{\Omega})$ is needed in the proof to ensure the regularity of $\vec{F}$, but seems like an overkill as far as the conclusion

$$u(x) = \int_{\partial\Omega} \partial_{\nu(y)}[G(x,y)]\big(u\big|_{\partial\Omega}\big)(y)\, d\sigma(y)$$

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Present Goal: Find geometric and analytic assumptions, in the nature of "best possible", ensuring the validity of the Poisson integral representation formula

$$u = -\int_{\partial\Omega} \partial_{\nu(y)}[G(\cdot, y)]\big(u\big|_{\partial\Omega}\big)(y)\, d\sigma(y)$$

Specifically:

- the nature of Ω is best described in the language of geometric measure theory; from now on, $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ and the outward unit normal ν is the De Giorgi-Federer normal for sets of locally finite perimeter ($\mathcal{H}^{n-1}$ is the $(n-1)$-dim. Hausdorff measure in $\mathbb{R}^n$).

- boundary traces taken in the nontangential approach sense

- replace the Laplacian by general weakly elliptic homogeneous constant complex coefficient second-order systems

- impose minimal size and smoothness assumptions on the solution u and Green function G

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The domain

Suppose Ω is an open subset of $\mathbb{R}^n$ satisfying the following properties:

• $\partial\Omega$ is lower Ahlfors-David regular, i.e., there exists $c \in (0, \infty)$ such that

$$c\, r^{n-1} \leq \mathcal{H}^{n-1}\big(B(x,r) \cap \Sigma\big) \text{ for each } x \in \Sigma \text{ and } r \in \big(0, 2\,\mathrm{diam}\,(\Sigma)\big).$$

• $\sigma = \mathcal{H}^{n-1} \lfloor \partial\Omega$ is a doubling measure on $\partial\Omega$, i.e., there exists some $C \geq 1$ such that $0 < \sigma\big(B(x,2r) \cap \partial\Omega\big) \leq C\sigma\big(B(x,r) \cap \partial\Omega\big) < +\infty$ for all $x \in \partial\Omega$ and $r \in (0, \infty)$.

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Fact: If σ is locally finite then Ω is a set of locally finite perimeter. As such, the De Giorgi-Federer unit normal ν to Ω exists and is defined σ-a.e. on the geometric measure theoretic boundary $\partial_*\Omega$

$$\partial_*\Omega := \Big\{x \in \mathbb{R}^n : \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\cap\Omega)}{r^n} > 0 \ \text{ and } \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B(x,r)\setminus\Omega)}{r^n} > 0\Big\},$$

where $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$. Fix $\kappa > 0$ playing the role of aperture parameter. For each $x \in \partial\Omega$ define the nontangential approach region

$$\Gamma_\kappa(x) := \big\{y \in \Omega : |y - x| < (1+\kappa)\text{dist}\,(y, \partial\Omega)\big\}$$

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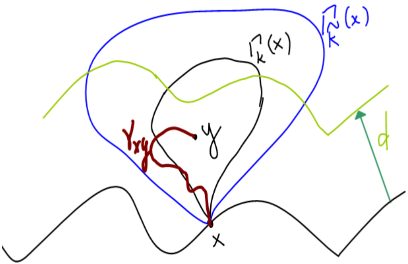
- Ω is locally pathwise nontangentially accessible if Ω is open and:

  given any $\kappa > 0$ there exist $\widetilde{\kappa} \geq \kappa$ along with $c \in [1, \infty)$ and $d > 0$ such that σ-a.e. point $x \in \partial\Omega$ has the property that any $y \in \Gamma_\kappa(x)$ with $\mathrm{dist}\,(y, \partial\Omega) < d$ may be joined by a rectifiable curve $\gamma_{x,y}$ satisfying $\gamma_{x,y} \setminus \{x\} \subset \Gamma_{\widetilde{\kappa}}(x)$ and whose length is $\leq c|x - y|$.



The nontangential maximal operator with aperture κ acts on any measurable function $u : \Omega \to \mathbb{C}$ according to

$$\big(\mathcal{N}_\kappa u\big)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x))}, \quad x \in \partial\Omega,$$

and the nontangential boundary trace of u is defined as

$$\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x) := \lim_{\Gamma_\kappa(x)\ni y\to x} u(y),$$

$$\text{whenever } \ x \in \partial\Omega \ \text{ is such that } \ x \in \overline{\Gamma_\kappa(x)}.$$

For $\rho > 0$ define the truncated nontangential maximal operator

$$\big(\mathcal{N}^\rho_\kappa u\big)(x) := \|u\|_{L^\infty(\Gamma_\kappa(x)\cap\mathcal{O}_\rho)}, \qquad x \in \partial\Omega,$$

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Nontangential maximal operator and nontangential traces

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The operator

Fix $n, M \in \mathbb{N}$, with $n \geq 2$. We work with a homogeneous $M \times M$ second-order complex constant coefficient system in $\mathbb{R}^n$ (with the summation convention over repeated indices)

$$L = \left(a_{rs}^{\alpha\beta} \partial_r \partial_s\right)_{1 \leq \alpha, \beta \leq M}$$

which is weakly elliptic, i.e., its $M \times M$ symbol matrix

$$L(\xi) := \left(a_{rs}^{\alpha\beta} \xi_r \xi_s\right)_{1 \leq \alpha, \beta \leq M}, \qquad \forall\, \xi = (\xi_r)_{1 \leq r \leq n} \in \mathbb{R}^n,$$

satisfies

$$\det\left[L(\xi)\right] \neq 0, \qquad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}.$$

Examples to keep in mind.
Scalar operators: $L = a_{jk} \partial_j \partial_k$ with $a_{jk} \in \mathbb{C}$ (e.g., the Laplacian).
Genuine systems: $L = \mu\Delta + (\lambda + \mu)\nabla \mathrm{div}$ with $\mu, \lambda \in \mathbb{C}$ (Lamé-like).

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Coefficient tensors

Consider the coefficient tensor

$$A = \left(a^{\alpha\beta}_{rs}\right)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$$

where $a^{\alpha\beta}_{rs} \in \mathbb{C}$. Its transposed is given by

$$A^\top := \left(a^{\beta\alpha}_{sr}\right)_{\substack{1\leq s,r\leq n\\ 1\leq\beta,\alpha\leq M}}.$$

With each such A we may canonically associate a homogeneous constant (complex) coefficient second-order $M \times M$ system L_A in $\mathbb{R}^n$ which is expressed as

$$L_A := \left(a^{\alpha\beta}_{rs}\partial_r\partial_s\right)_{\substack{1\leq\alpha\leq M\\ 1\leq\beta\leq N.}}$$

In particular, $(L_A)^\top = L_{A^\top}$.

Note: Given a homogeneous second-order system L, there exist *infinitely many* coefficient tensors A such that $L = L_A$.

Conormal derivative

Let Ω be a set of locally finite perimeter in $\mathbb{R}^n$. Denote by $\nu = (\nu_r)_{1\leq r\leq n}$ the De Giorgi-Federer outward unit normal to Ω (defined σ-a.e. on $\partial_*\Omega$). Let $A = \left(a_{rs}^{\alpha\beta}\right)_{\substack{1\leq r,s\leq n\\ 1\leq\alpha,\beta\leq M}}$ be a coefficient tensor with complex entries. Also fix an aperture parameter $\kappa > 0$.

If $u \in \left[W^{1,1}_{\mathrm{loc}}(\Omega)\right]^M$ then the conormal derivative of u with respect to the coefficient tensor A and the set Ω is the $\mathbb{C}^M$-valued function

$$\partial_\nu^A u := \left(\nu_r a_{rs}^{\alpha\beta}(\partial_s u_\beta)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}\right)_{1\leq\alpha\leq M} \quad \text{at } \sigma\text{-a.e. point on } \partial_*\Omega,$$

whenever meaningful.

Note: Starting with a homogeneous second-order system L, for each writing $L = L_A$ there corresponds a typically distinct conormal derivative ∂_ν^A.

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Main Theorem

Theorem (A Sharp Poisson formula [MMM2018])

Let $\Omega \subset \mathbb{R}^n$ be a bounded locally pathwise nontangentially accessible set with a lower Ahlfors-David regular boundary and such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$.

Suppose L is a weakly elliptic, homogenous, constant complex coefficient, second-order, $M \times M$ system in $\mathbb{R}^n$.

Fix an aperture parameter $\kappa > 0$, along with an arbitrary point $x_0 \in \Omega$, and choose a truncation $0 < \rho < \frac{1}{4}\,\mathrm{dist}\,(x_0, \partial\Omega)$.

Then there exists some $\widetilde{\kappa} > 0$, which depends only on Ω and κ, with the following significance.

Theorem (Continuation)

Assume G is a matrix-valued function satisfying

$$\begin{cases} G \in \left[L^1_{\mathrm{loc}}(\Omega)\right]^{M\times M}, \\ L^\top G = -\delta_{x_0} I_{M\times M} \ \ \textit{in}\ \ \mathcal{D}'(\Omega), \\ \left(\nabla G\right)\Big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}} \ \ \textit{exists at } \sigma\textit{-a.e. point on } \ \partial\Omega, \\ G\Big|_{\partial\Omega}^{\widetilde{\kappa}-\mathrm{n.t.}} = 0 \ \ \textit{at } \sigma\textit{-a.e. point on } \ \partial\Omega, \end{cases}$$

and assume u is a $\mathbb{C}^M$-valued function satisfying

$$\begin{cases} u \in \left[C^\infty(\Omega)\right]^M, \quad Lu = 0 \ \ \textit{in} \ \ \Omega, \\ u\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}} \ \ \textit{exists at } \sigma\textit{-a.e. point on } \ \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \mathcal{N}^\rho_\kappa u \cdot \mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G)\, d\sigma < +\infty. \end{cases}$$

Theorem (Continuation)

Then for any choice of a coefficient tensor A which permits writing L as L_A, one has the Poisson integral representation formula

$$u(x_0) = -\int_{\partial_*\Omega} \big\langle u\big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}, \partial_\nu^{A^\top} G\big\rangle \, d\sigma$$

where ν denotes the De Giorgi-Federer outward unit normal to Ω and $\partial_\nu^{A^\top}$ stands for the conormal derivative associated with $A^\top$ acting on the columns of the matrix-valued function G.

A few examples when $\int_{\partial\Omega} \mathcal{N}_\kappa^\rho u \cdot \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)\, d\sigma < +\infty$ holds include, with $p, q, p', q' \in [1, \infty]$ satisfy $1/p + 1/p' = 1 = 1/q + 1/q'$,

- Ordinary Lebesgue spaces: $\mathcal{N}_\kappa^\rho u \in L^p(\partial\Omega, \sigma)$ and $\mathcal{N}_\kappa^\rho(\nabla G) \in L^{p'}(\partial\Omega, \sigma)$

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Proof

Fix $\beta \in \{1, \dots, M\}$ and define the vector field

$$\vec{F} := \Big(u_\alpha a^{\gamma\,\alpha}_{kj}\,\partial_k G_{\gamma\,\beta} - G_{\alpha\beta}\,a^{\alpha\gamma}_{jk}\,\partial_k u_\gamma \Big)_{1\leq j\leq n} \quad \text{a.e. in} \quad \Omega.$$

The strategy to prove the desired integral representation formula is to apply to this vector field a suitable version of the Divergence Theorem, much more potent than the classical one.

A word of caution: The classical Divergence Formula for bdd. C^1 domains and C^1 vector fields on the closure fails hopelessly short, and so does the De Giorgi-Federer version (involving sets of locally finite perimeters but requiring the vector field to be C^1 with compact support in the *entire* $\mathbb{R}^n$).

Step I. From $G \in \big[C^\infty(\Omega \setminus \{x_0\}) \cap W^{1,1}_{\mathrm{loc}}(\Omega)\big]^{M\times M}$ and $u \in \big[C^\infty(\Omega)\big]^M$ it follows that

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Step II. Show that $\operatorname{div}\vec{F} = -u_\beta(x_0)\,\delta_{x_0}$ in $\mathcal{D}'(\Omega)$.

In the sense of distributions in Ω, we have

$$\operatorname{div}\vec{F} = (\partial_j u_\alpha)\,a_{kj}^{\gamma\,\alpha}\,(\partial_k G_{\gamma\,\beta}) + u_\alpha\,a_{kj}^{\gamma\,\alpha}\,(\partial_j\partial_k G_{\gamma\,\beta})$$
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Changing variables $j\,' = k$, $k\,' = j$, $\alpha\,' = \gamma$, and $\gamma\,' = \alpha$ in I_3 yields

$$I_3 = -(\partial_{k\,'} G_{\gamma\,'\beta})\,a_{k\,'j\,'}^{\gamma\,'\alpha'}\,(\partial_{j\,'} u_{\alpha\,'}) = -I_1$$

while, $I_4 = -G_{\alpha\beta}\,(L_A u)_\alpha = -G_{\alpha\beta}\,(Lu)_\alpha = 0$. In addition,

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Step III. Show that $\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$.

Recall that $\vec{F} = (F_j)_{1\leq j\leq n}$ with

$$F_j = u_\alpha a_{kj}^{\gamma\,\alpha}\,\partial_k G_{\gamma\,\beta} - G_{\alpha\beta}\,a_{jk}^{\alpha\gamma}\,\partial_k u_\gamma, \quad j\in\{1,\dots,n\}$$

and that, by assumption,

$$(\nabla G)\Big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}} \text{ and } u\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \text{ exist at } \sigma\text{-a.e. point on } \partial\Omega.$$

Since $\widetilde{\kappa}\geq\kappa$, the first piece in F_j is OK. We are left with proving that

$$(G\,\nabla u)\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} \quad \text{exists at } \sigma\text{-a.e. point on } \partial\Omega.$$

Define the nontangentially accessible boundary of Ω by

$$\partial_{\rm nta}\Omega := \left\{x \in \partial\Omega : \, x \in \overline{\Gamma_\kappa(x)} \text{ for each } \kappa > 0\right\}.$$

Fact: Ω locally pathwise nontangentially accessible set and σ doubling measure on $\partial\Omega \Longrightarrow \mathcal{H}^{n-1}(\partial\Omega \setminus \partial_{\rm nta}\Omega) = 0$

Choose a suitable (dictated by geometry) $\widetilde{\kappa} > \kappa$ and set

$N_1 := \left\{x \in \partial\Omega : \mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G)(x) = +\infty \text{ or } \left(G\big|^{\widetilde{\kappa}-{\rm n.t.}}_{\partial\Omega}\right)(x) \neq 0\right\}$,

$N_2 := \left\{x \in \partial_{\rm nta}\Omega : \left(u\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}\right)(x) \text{ fails to exist}\right\}$,

$N_3 := \left\{x \in \partial\Omega \text{ excluded in the locally pathwise n.t.a. definition}\right\}$.

Let $N := N_1 \cup N_2 \cup N_3$.

Then the current assumptions ultimately imply $\sigma(N) = 0$.

Now fix $x \in \partial_{\rm nta}\Omega \setminus N$ and pick $y \in \Gamma_\kappa(x)$ with $\delta_{\partial\Omega}(y) := {\rm dist}\,(y, \partial\Omega)$ sufficiently small. Let γ_{xy} be a rectifiable curve joining x and y guaranteed to exist by the locally pathwise nontangential accessibility of Ω.

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$$G(y) = G\big(\gamma_{xy}(t)\big)\Big|_{t=0}^{t=1} = \int_0^1 \frac{d}{dt}\Big[G\big(\gamma_{xy}(t)\big)\Big]\,dt$$

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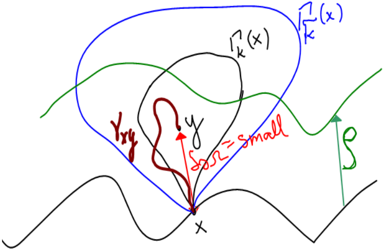

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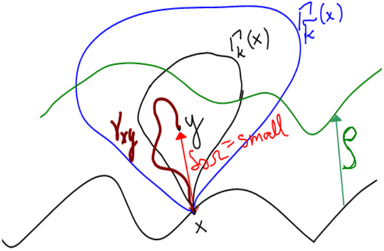


The choice of $\widetilde{\kappa}$ implies $\gamma_{xy}((0,1]) \subset \Gamma_{\widetilde{\kappa}}(x)$ and the smallness of $\delta_{\partial\Omega}(y)$ is tailored to ensure $\mathrm{dist}\,(\gamma_{xy}, \partial\Omega) < \rho$.

Recall $\gamma_{xy}((0,1]) \subset \Gamma_{\widetilde{\kappa}}(x)$ and $\mathrm{dist}\,(\gamma_{xy}, \partial\Omega) < \rho$. In addition,

$$\mathrm{length}(\gamma_{xy}([0,1])) \leq c|x-y| \leq c(1+\kappa)\mathrm{dist}(y,\partial\Omega) = C\delta_{\partial\Omega}(y).$$

As we have just seen, the Fundamental Theorem of Calculus gives

$$G(y) = \int_0^1 (\nabla G)\big(\gamma_{xy}(t)\big) \cdot \frac{d}{dt}\big[\gamma_{xy}(t)\big]\, dt$$

so we may further estimate

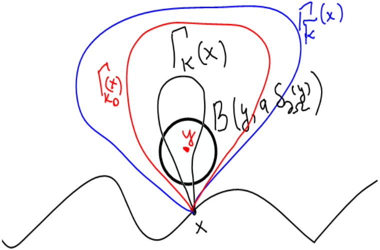
$$\begin{aligned} |G(y)| &\leq \mathcal{N}^{\rho}_{\widetilde{\kappa}}(\nabla G)(x) \cdot \mathrm{length}(\gamma_{xy}([0,1])) \\ &\leq \mathcal{N}^{\rho}_{\widetilde{\kappa}}(\nabla G)(x) \cdot C \cdot \underbrace{\delta_{\partial\Omega}(y)}_{\text{rate of vanishing}} \end{aligned}$$

Using interior estimates in $B\big(y\,,\,a\cdot\delta_{\partial\Omega}(y)\big)$ with $a>0$ small for $w(z):=u(z)-\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)$, $z\in\Omega$, which is a null-solution for L,

$$|(\nabla u)(y)|=|(\nabla w)(y)|\leq\frac{C}{\delta_{\partial\Omega}(y)}\fint_{B(y,a\cdot\delta_{\partial\Omega}(y))}\Big|u(z)-\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big|\,dz$$

$$\leq C\cdot\underbrace{\delta_{\partial\Omega}(y)^{-1}}_{\text{blow up rate}}\cdot\sup_{\substack{z\in\Gamma_{\kappa_o}(x)\\|x-z|<(1+c)\delta_{\partial\Omega}(y)}}\Big|u(z)-\big(u\big|^{\kappa-\mathrm{n.t.}}_{\partial\Omega}\big)(x)\Big|$$

for some $\kappa_o>0$ big. Unfortunately $\kappa_o>\kappa$, so we loose control!



Remedy:
start with $y\in\Gamma_{\kappa'}(x)$ for suitable $\kappa'<\kappa$ to end up with $z\in\Gamma_\kappa(x)$.

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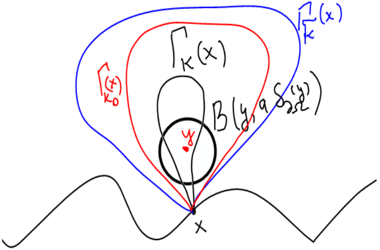

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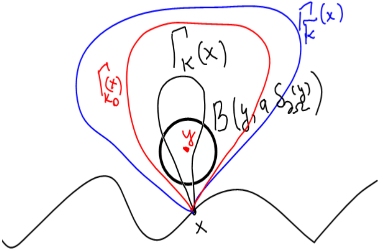

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Hence matters can be arranged so that

$$|(\nabla u)(y)| \leq C \cdot \underbrace{\delta_{\partial\Omega}(y)^{-1}}_{\text{blow up rate}} \cdot \sup_{\substack{z\in\Gamma_\kappa(x)\\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} \Big|u(z) - \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)\Big|.$$

When combined with the earlier estimate on G, namely

$$|G(y)| \leq C \cdot \underbrace{\delta_{\partial\Omega}(y)}_{\text{vanishing rate}} \cdot \mathcal{N}_{\widetilde{\kappa}}^{\rho}(\nabla G)(x),$$

this yields

$$|G(y)||(\nabla u)(y)| \leq C\mathcal{N}_{\widetilde{\kappa}}^{\rho}(\nabla G)(x) \cdot \underbrace{\sup_{\substack{z\in\Gamma_\kappa(x)\\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} \Big|u(z) - \big(u\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big)(x)\Big|}_{\text{qualitative vanishing rate}}$$

Consequently,

$$\lim_{\Gamma_\kappa(x)\ni y\to x} |G(y)||(\nabla u)(y)| = 0 \quad \text{for each} \quad x \in \partial_{\text{nta}}\Omega \setminus N.$$

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Consequently,

$$\lim_{\Gamma_\kappa(x)\ni y\to x} |G(y)||(\nabla u)(y)| = 0 \text{ for each } x\in\partial_{\text{nta}}\Omega\setminus N.$$

Hence $\vec{F}\big|_{\partial\Omega}^{\kappa-\text{n.t.}}$ exists at all points in $\partial_{\text{nta}}\Omega \setminus N$.

Since $\sigma\big(\partial\Omega \setminus (\partial_{\text{nta}}\Omega \setminus N)\big) = 0$, this nontangential trace exists at σ-a.e. point on $\partial\Omega$ and, in fact

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-\text{n.t.}} = \Big(\big(u_\alpha\big|_{\partial\Omega}^{\kappa-\text{n.t.}}\big) a_{kj}^{\gamma\,\alpha} \big(\partial_k G_{\gamma\,\beta}\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-\text{n.t.}}\Big)_{1\leq j\leq n}.$$

Step IV. Show that there exists some $\varepsilon_0 > 0$ such that

$$\mathcal{N}_\kappa^{\varepsilon_0}\vec{F} \in L^1(\partial\Omega,\sigma).$$

• Choose $\varepsilon_0 < \rho$ sufficiently small and fix $x \in \partial_{\text{nta}}\Omega$. For each $y \in \Gamma_\kappa(x)$ with $\delta_{\partial\Omega}(y) < \varepsilon_0$ use interior estimates for u

$$|(\nabla u)(y)| \leq \frac{C}{\delta_{\partial\Omega}(y)} \fint_{B(y,a\cdot\delta_{\partial\Omega}(y))} |u(z)|\,dz$$

$$\leq C\delta_{\partial\Omega}(y)^{-1} \cdot \sup_{\substack{z\in\Gamma_\kappa(x)\\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} |u(z)| \leq C\,\delta_{\partial\Omega}(y)^{-1} \cdot \big(\mathcal{N}_\kappa^\rho u\big)(x).$$

• Recall the earlier estimate $|G(y)| \leq C\,\delta_{\partial\Omega}(y)\cdot\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)(x)$.

• Hence $\mathcal{N}_\kappa^{\varepsilon_0}\big(|G||\nabla u|\big) \leq C\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)\cdot\mathcal{N}_\kappa^\rho u$ at σ-a.e. point on $\partial\Omega$.

• Also, $\mathcal{N}_\kappa^{\varepsilon_0}\big(|\nabla G||u|\big) \leq \mathcal{N}_\kappa^{\varepsilon_0}(\nabla G)\cdot\mathcal{N}_\kappa^{\varepsilon_0} u \leq \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)\cdot\mathcal{N}_\kappa^\rho u$ at each point on $\partial\Omega$.

Since by assumption $\mathcal{N}_\kappa^\rho u\cdot\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \in L^1(\partial\Omega,\sigma)$, it follows that $\mathcal{N}_\kappa^{\varepsilon_0}\vec{F} \in L^1(\partial\Omega,\sigma)$.

- Choose $\varepsilon_0 < \rho$ sufficiently small and fix $x \in \partial_{\rm nta}\Omega$. For each $y \in \Gamma_\kappa(x)$ with $\delta_{\partial\Omega}(y) < \varepsilon_0$ use interior estimates for u

$$|(\nabla u)(y)| \leq \frac{C}{\delta_{\partial\Omega}(y)} \fint_{B(y,a\cdot\delta_{\partial\Omega}(y))} |u(z)|\,dz$$

$$\leq C\delta_{\partial\Omega}(y)^{-1} \cdot \sup_{\substack{z\in\Gamma_\kappa(x)\\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} |u(z)| \leq C\,\delta_{\partial\Omega}(y)^{-1} \cdot \big(\mathcal{N}_\kappa^\rho u\big)(x).$$

- Recall the earlier estimate $|G(y)| \leq C\,\delta_{\partial\Omega}(y)\,\cdot \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)(x)$.

- Hence $\mathcal{N}_\kappa^{\varepsilon_0}\big(|G||\nabla u|\big) \leq C\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \cdot \mathcal{N}_\kappa^\rho u$ at σ-a.e. point on $\partial\Omega$.

- Also, $\mathcal{N}_\kappa^{\varepsilon_0}\big(|\nabla G||u|\big) \leq \mathcal{N}_\kappa^{\varepsilon_0}(\nabla G) \cdot \mathcal{N}_\kappa^{\varepsilon_0} u \leq \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \cdot \mathcal{N}_\kappa^\rho u$ at each point on $\partial\Omega$.

Since by assumption $\mathcal{N}_\kappa^\rho u \cdot \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \in L^1(\partial\Omega,\sigma)$, it follows that $\mathcal{N}_\kappa^{\varepsilon_0}\vec{F} \in L^1(\partial\Omega,\sigma)$.

• Choose $\varepsilon_0 < \rho$ sufficiently small and fix $x \in \partial_{\text{nta}}\Omega$. For each $y \in \Gamma_\kappa(x)$ with $\delta_{\partial\Omega}(y) < \varepsilon_0$ use interior estimates for u

$$|(\nabla u)(y)| \leq \frac{C}{\delta_{\partial\Omega}(y)} \fint_{B(y, a\cdot\delta_{\partial\Omega}(y))} |u(z)|\, dz$$

$$\leq C\delta_{\partial\Omega}(y)^{-1} \cdot \sup_{\substack{z\in\Gamma_\kappa(x) \\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} |u(z)| \leq C\, \delta_{\partial\Omega}(y)^{-1} \cdot \big(\mathcal{N}^\rho_\kappa u\big)(x).$$

• Recall the earlier estimate $|G(y)| \leq C\, \delta_{\partial\Omega}(y)\, \cdot \mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G)(x)$.

• Hence $\mathcal{N}^{\varepsilon_0}_\kappa\big(|G||\nabla u|\big) \leq C\mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G) \cdot \mathcal{N}^\rho_\kappa u$ at σ-a.e. point on $\partial\Omega$.

• Also, $\mathcal{N}^{\varepsilon_0}_\kappa\big(|\nabla G||u|\big) \leq \mathcal{N}^{\varepsilon_0}_\kappa(\nabla G) \cdot \mathcal{N}^{\varepsilon_0}_\kappa u \leq \mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G) \cdot \mathcal{N}^\rho_\kappa u$ at each point on $\partial\Omega$.

Since by assumption $\mathcal{N}^\rho_\kappa u \cdot \mathcal{N}^\rho_{\widetilde{\kappa}}(\nabla G) \in L^1(\partial\Omega, \sigma)$, it follows that $\mathcal{N}^{\varepsilon_0}_\kappa \vec{F} \in L^1(\partial\Omega, \sigma)$.

- Choose $\varepsilon_0 < \rho$ sufficiently small and fix $x \in \partial_{\rm nta}\Omega$. For each $y \in \Gamma_\kappa(x)$ with $\delta_{\partial\Omega}(y) < \varepsilon_0$ use interior estimates for u

$$|(\nabla u)(y)| \leq \frac{C}{\delta_{\partial\Omega}(y)} \fint_{B(y,a\cdot\delta_{\partial\Omega}(y))} |u(z)|\,dz$$

$$\leq C\delta_{\partial\Omega}(y)^{-1} \cdot \sup_{\substack{z\in\Gamma_\kappa(x)\\ |x-z|<(1+c)\delta_{\partial\Omega}(y)}} |u(z)| \leq C\,\delta_{\partial\Omega}(y)^{-1} \cdot \big(\mathcal{N}_\kappa^\rho u\big)(x).$$

- Recall the earlier estimate $|G(y)| \leq C\,\delta_{\partial\Omega}(y)\,\cdot \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)(x)$.
- Hence $\mathcal{N}_\kappa^{\varepsilon_0}\big(|G||\nabla u|\big) \leq C\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \cdot \mathcal{N}_\kappa^\rho u$ at σ-a.e. point on $\partial\Omega$.
- Also, $\mathcal{N}_\kappa^{\varepsilon_0}\big(|\nabla G||u|\big) \leq \mathcal{N}_\kappa^{\varepsilon_0}(\nabla G) \cdot \mathcal{N}_\kappa^{\varepsilon_0} u \leq \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \cdot \mathcal{N}_\kappa^\rho u$ at each point on $\partial\Omega$.

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$$
\begin{aligned}
|(\nabla u)(y)| &\leq \frac{C}{\delta_{\partial\Omega}(y)} \fint_{B(y,a\cdot\delta_{\partial\Omega}(y))} |u(z)|\,dz \\
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• Hence $\mathcal{N}_\kappa^{\varepsilon_0}\big(|G||\nabla u|\big) \leq C\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)\cdot\mathcal{N}_\kappa^\rho u$ at σ-a.e. point on $\partial\Omega$.

• Also, $\mathcal{N}_\kappa^{\varepsilon_0}\big(|\nabla G||u|\big) \leq \mathcal{N}_\kappa^{\varepsilon_0}(\nabla G)\cdot\mathcal{N}_\kappa^{\varepsilon_0} u \leq \mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G)\cdot\mathcal{N}_\kappa^\rho u$ at each point on $\partial\Omega$.

Since by assumption $\mathcal{N}_\kappa^\rho u\cdot\mathcal{N}_{\widetilde{\kappa}}^\rho(\nabla G) \in L^1(\partial\Omega,\sigma)$, it follows that $\mathcal{N}_\kappa^{\varepsilon_0}\vec{F} \in L^1(\partial\Omega,\sigma)$.

In summary, for the current choice of $\vec{F}$ we have proved

$$\vec{F} \in \big[L^1_{\rm loc}(\Omega)\big]^n, \quad {\rm div}\,\vec{F} = -u_\beta(x_0)\,\delta_{x_0} \in \mathcal{E}'(\Omega),$$

$$\mathcal{N}_\kappa^{\varepsilon_0}\vec{F} \in L^1(\partial\Omega,\sigma) \quad \text{for some } \varepsilon_0 > 0,$$

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} \quad \text{exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and}$$

$$\vec{F}\Big|_{\partial\Omega}^{\kappa-{\rm n.t.}} = \Big(\big(u_\alpha\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)a_{kj}^{\gamma\,\alpha}\big(\partial_k G_{\gamma\,\beta}\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-{\rm n.t.}}\Big)_{1\leq j\leq n}.$$

Step V. Apply the Divergence Theorem (to be stated next):

$$-u_\beta(x_0) = {}_{(C_b^\infty(\Omega))^*}\big({\rm div}\,\vec{F}\,,\,1\big)_{C_b^\infty(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)\,d\sigma$$

$$= \int_{\partial_*\Omega} \big(u_\alpha\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\big)\nu_j a_{kj}^{\gamma\,\alpha}\big(\partial_k G_{\gamma\,\beta}\big)\big|_{\partial\Omega}^{\widetilde{\kappa}-{\rm n.t.}}\,d\sigma$$

$$= \int_{\partial_*\Omega} \big\langle u\big|_{\partial\Omega}^{\kappa-{\rm n.t.}}\,,\,\partial_\nu^{A^\top}G_{\bullet\beta}\big\rangle\,d\sigma,$$

$\nu = (\nu_j)_j$ being the De Giorgi-Federer outward unit normal to Ω.

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$$\vec{F} \in \big[L^1_{\text{loc}}(\Omega)\big]^n, \quad \operatorname{div} \vec{F} = -u_\beta(x_0)\,\delta_{x_0} \in \mathcal{E}'(\Omega),$$

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$$\vec{F}\Big|^{\kappa-\text{n.t.}}_{\partial\Omega} \quad \text{exists at } \sigma\text{-a.e. point on } \partial\Omega \text{ and}$$

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Step V. Apply the Divergence Theorem (to be stated next):

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$C_b^\infty(\Omega) := \{ f \in C^\infty(\Omega) : f \text{ bounded in } \Omega \}$

A sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_b^\infty(\Omega)$ converges to $f \in C_b^\infty(\Omega)$ provided

$$\sup_{j\in\mathbb{N}} \sup_{x\in\Omega} |f_j(x)| < +\infty$$

$$\forall \text{ compact } K \subset \Omega \;\; \exists\, j_K \in \mathbb{N} \text{ such that } f_j \equiv f \text{ on } K \text{ if } j \geq j_K.$$

Let $\big(C_b^\infty(\Omega)\big)^*$ denote the algebraic dual of this linear space, so that

$$\lim_{j\to\infty} {}_{(C_b^\infty(\Omega))^*}\big(\Lambda\,,\, f_j\big)_{C_b^\infty(\Omega)} = {}_{(C_b^\infty(\Omega))^*}\big(\Lambda\,,\, f\big)_{C_b^\infty(\Omega)}$$

$$\text{whenever } \Lambda \in \big(C_b^\infty(\Omega)\big)^* \text{ and } \lim_{j\to\infty} f_j = f \text{ in } C_b^\infty(\Omega)$$

- If $u \in \mathcal{D}'(\Omega)$ and exist $\Lambda_u \in \big(C_b^\infty(\Omega)\big)^*$ then this extension is unique.
- $\mathcal{E}'(\Omega) + L^1(\Omega) \subseteq \big(C_b^\infty(\Omega)\big)^*$ If $u = w + g$, $w \in \mathcal{E}'(\Omega)$, $g \in L^1(\Omega)$, then $\Lambda_u \in \big(C_b^\infty(\Omega)\big)^*$ where

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$$_{(C_b^\infty(\Omega))^*}\big(\Lambda_u, f\big)_{C_b^\infty(\Omega)} := {}_{\mathcal{E}'(\Omega)}\langle w, f\,\rangle_{\mathcal{E}(\Omega)} + \int_\Omega f g \, d\mathcal{L}^n, \quad \forall\, f \in C_b^\infty(\Omega)$$

$C_b^\infty(\Omega) := \big\{ f \in C^\infty(\Omega) :\ f \text{ bounded in } \Omega \big\}$

A sequence $\{f_j\}_{j\in\mathbb{N}} \subset C_b^\infty(\Omega)$ converges to $f \in C_b^\infty(\Omega)$ provided

$$\sup_{j\in\mathbb{N}} \sup_{x\in\Omega} |f_j(x)| < +\infty$$

$$\forall \text{ compact } K \subset \Omega \ \ \exists\, j_K \in \mathbb{N} \text{ such that } f_j \equiv f \text{ on } K \text{ if } j \geq j_K.$$

Let $\big(C_b^\infty(\Omega)\big)^*$ denote the algebraic dual of this linear space, so that

$$\lim_{j\to\infty} {}_{(C_b^\infty(\Omega))^*}\big(\Lambda\,,\ f_j\big)_{C_b^\infty(\Omega)} = {}_{(C_b^\infty(\Omega))^*}\big(\Lambda\,,\ f\big)_{C_b^\infty(\Omega)}$$

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Theorem (Divergence Theorem [MMM 2018])

Let $\Omega \subset \mathbb{R}^n$ be bounded, open, with a lower Ahlfors-David regular boundary, such that $\sigma := \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Let ν be the De Giorgi-Federer outward unit normal to Ω. Fix $\kappa > 0$ and assume

$$\vec{F} \in \big[\mathcal{E}'(\Omega) + L^1_{\rm loc}(\Omega)\big]^n \subset \big[\mathcal{D}'(\Omega)\big]^n$$

is a vector field satisfying (for some $0 < \varepsilon < \operatorname{dist}(\operatorname{regsupp}\vec{F}, \partial\Omega)$)

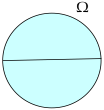
$$\mathcal{N}^{\varepsilon}_{\kappa}\vec{F} \in L^1(\partial\Omega,\sigma), \quad \vec{F}\big|^{\kappa-{\rm n.t.}}_{\partial\Omega} \text{ exists } \sigma\text{-a.e. on } \partial_{\rm nta}\Omega, \text{ and}$$

$$\operatorname{div}\vec{F} \in \mathcal{D}'(\Omega) \text{ extends to a continuous functional in } (C^\infty_b(\Omega))^*.$$

Then for any $\kappa' > 0$ the trace $\vec{F}\big|^{\kappa'-{\rm n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial_{\rm nta}\Omega$ and agrees with $\vec{F}\big|^{\kappa-{\rm n.t.}}_{\partial\Omega}$ and, with the dependence on aperture dropped,

$${}_{(C^\infty_b(\Omega))^*}\big(\operatorname{div}\vec{F}, 1\big)_{C^\infty_b(\Omega)} = \int_{\partial_*\Omega} \nu\cdot\big(\vec{F}\,\big|^{\rm n.t.}_{\partial\Omega}\big)\,d\sigma.$$

Sharpness aspect of our Divergence Theorem: Let Ω be the slit unit ball in $\mathbb{R}^n$



Then $\partial\Omega = S^{n-1} \cup \{(x',0) : |x'| < 1\}$, $\partial_*\Omega = S^{n-1}$,
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Also let

$$\vec{F} := \begin{cases} +\mathbf{e}_n & \text{in } \ \Omega \cap \mathbb{R}^n_+, \\ -\mathbf{e}_n & \text{in } \ \Omega \cap \mathbb{R}^n_-. \end{cases}$$

Observe that $\vec{F} \in \big[C^\infty(\Omega)\big]^n$, ${\rm div}\,\vec{F} = 0$ in Ω,
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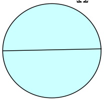

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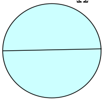

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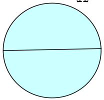

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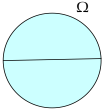

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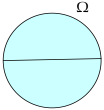

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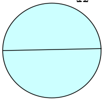

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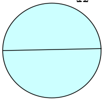

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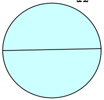

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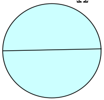

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Hence, on the one hand we have

$$\int_{\partial_*\Omega} \nu \cdot \Big(\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}\Big)\, d\sigma = \int_{S^{n-1}_+} \nu \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} - \int_{S^{n-1}_-} \nu \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1}$$

$$= 2\int_{|x'|<1} \mathbf{e}_n \cdot \mathbf{e}_n \, d\mathcal{H}^{n-1} = 2\mathcal{H}^{n-1}\big(\{|x'|<1\}\big) \neq 0,$$

while on the other hand, $\int_\Omega \operatorname{div} \vec{F}\, d\mathcal{L}^n = 0$.

Conclusion: The demand that $\vec{F}\big|^{\kappa-\text{n.t.}}_{\partial\Omega}$ exists σ-a.e. on $\partial_{\text{nta}}\Omega$ and not just on the (potentially smaller) set $\partial_*\Omega$ **is necessary**, even though it is $\partial_*\Omega$ which appears in the very formulation of the Divergence Formula.

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Our Poisson Integral Representation Formula also holds for Ω unbounded under appropriate decay conditions.

• If Ω is an exterior domain, i.e., Ω is the complement of a compact subset of $\mathbb{R}^n$, we also ask that

$$G(x) = o(1) \text{ and } u(x) = o(1) \text{ as } |x| \longrightarrow \infty.$$

• If $\partial\Omega$ is unbounded, we make the additional assumption

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Our theorem yields nontrivial, new results even in the case when $\Omega = \mathbb{R}^n_+$. Availing ourselves of estimates for the Green function for a system L in this setting (C.Martell/DM/I.Mitrea/M.Mitrea) our theorem gives that if u satisfies

$$\begin{cases} u \in \left[C^\infty(\mathbb{R}^n_+)\right]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x') \frac{dx'}{1+|x'|^{n-1}} < \infty, \end{cases}$$

then $u\Big|^{\kappa-\text{n.t.}}_{\mathbb{R}^{n-1}}$ exists at $\mathcal{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$ and u has the Poisson integral representation formula

$$u(x) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y')\big(u\big|^{\kappa-\text{n.t.}}_{\mathbb{R}^{n-1}}\big)(y')\,dy' \qquad \forall\, x = (x',t) \in \mathbb{R}^n_+,$$

where P^L is the Agmon-Douglis-Nirenberg Poisson kernel for the system L in $\mathbb{R}^n_+$ and $P^L_t(x') = t^{1-n}P^L(x'/t)$ for all $x' \in \mathbb{R}^{n-1}$, $t > 0$.

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Theorem ([MMM 2018])

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded regular domain for the Dirichlet problem for Δ. Suppose Ω is locally pathwise nontangentially accessible, has a lower Ahlfors regular boundary, and $\sigma = \mathcal{H}^{n-1}\lfloor\partial\Omega$ is a doubling measure on $\partial\Omega$. Fix $x_0 \in \Omega$ and $\kappa > 0$, and assume that G, the Green function for the Δ with pole at x_0, satisfies

$$\mathcal{N}_\kappa^\varepsilon(\nabla G) \in L^1(\partial\Omega, \sigma) \ \ \textit{for some} \ \ \varepsilon \in \big(0\,,\,\mathrm{dist}\,(x_0, \partial\Omega)\big),$$

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Then ω^{x_0}, the harmonic measure on $\partial\Omega$ with pole at x_0, is absolutely continuous with respect to σ and

$$\frac{d\omega^{x_0}}{d\sigma} = -\mathbf{1}_{\partial_*\Omega} \cdot \partial_\nu G \ \ \textit{at } \sigma\textit{-a.e. point on } \partial\Omega,$$

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Comments:

• Whenever $\omega^{x_0} \ll \sigma$, the Poisson kernel for Ω, defined as $k^{x_0} := \dfrac{d\omega^{x_0}}{d\sigma}$ belongs to $L^1(\partial\Omega, \sigma)$ (and satisfies $\int_{\partial\Omega} k^{x_0}\, d\sigma = 1$). As such, from the perspective of the conclusion we seek that $k^{x_0} = -\mathbf{1}_{\partial_*\Omega} \cdot \partial_\nu G$ at σ-a.e. point on $\partial\Omega$, the assumption $\mathcal{N}_\kappa^\varepsilon(\nabla G) \in L^1(\partial\Omega, \sigma)$ is natural.

• If Ω is a UR domain then $\big(\nabla G_\Omega(\cdot, x_0)\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists at σ-a.e. point on $\partial\Omega$. This is a consequence of a more general Fatou type theorem in UR domains [MMM2018]:

If Ω is a UR domain in $\mathbb{R}^n$, $u \in C^\infty(\Omega)$, $Lu = 0$ in Ω, $\mathcal{N}_\kappa(\nabla u) \in L^p(\partial\Omega, \sigma)$ for some $\kappa > 0$ and $p \in \Big(\dfrac{n-1}{n}, \infty\Big)$, then $\big(\nabla u\big)\Big|_{\partial\Omega}^{\kappa-\mathrm{n.t.}}$ exists σ-a.e. on $\partial\Omega$.

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Sketch of proof: Let $f \in C^0(\partial\Omega)$ and consider

$$u \in C^\infty(\Omega) \cap C^0(\overline{\Omega}), \quad \Delta u = 0 \text{ in } \Omega, \quad u\big|_{\partial\Omega} = f.$$

Then $u(x_0) = \int_{\partial\Omega} f \, d\omega^{x_0}$

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